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ORIGINAL RESEARCH

One Man, One Vote, One Price

László Á. Kóczy[1,2] · Balázs R. Sziklai[1,3]

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Abstract
Introduced initially to allocate parliamentary seats among states or provinces, apportionment methods can distribute any discrete scarce resource based on a series of claims or budgets such as the states' voting populations. Here, we focus on the price of representation: how one can turn the claims, such as the number of voters, into a share of the resource, such as parliamentary seats. The D'Hondt method naturally emerges as a competitive equilibrium. We show that the class of parametric divisor methods is fully characterised by a competitive equilibrium augmented by a uniform credit or debt with respect to the original claims. Based on the insight gained from the competitive equilibrium representation, we provide an alternative proof of majorization: divisor methods with smaller parameters favour players with smaller budgets and *vice versa*. While in a competitive equilibrium, all pay the same price, when the budget allows for the purchase of a few items, leftovers may constitute a substantial part of the original budget. In other words, not all are equally efficient at converting their budgets into items. Optimisation apportionment methods, such as the Leximin, assign individual prices; hence, each budget is fully spent, and focus on minimising price differences. We demonstrate how the D'Hondt and Adams methods can be formulated as optimisation methods and highlight their connection to the Leximin method.

Keywords apportionment problem · competitive equilibrium · divisor methods · majorization · Leximin method

László Á. Kóczy and Balázs R. Sziklai contributed equally to this work.

✉ Balázs R. Sziklai
sziklai.balazs@krtk.elte.hu

László Á. Kóczy
koczy.laszlo@krtk.elte.hu
https://kti.krtk.hu/en/researchers/laszlo-a-koczy/

1 Institute of Economics, ELTE Centre for Economic and Regional Studies, Tóth Kálmán u. 4., Budapest 1097, Hungary

2 Department of Finance, Faculty of Economic and Social Sciences, Budapest University of Technology and Economics, Műegyetem rkp. 3., Budapest 1111, Hungary

3 Department of Operations Research and Actuarial Sciences, Corvinus University of Budapest, Fővám tér 8., Budapest 1093, Hungary

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1 Introduction

The One Man, One Vote rule formalises the democratic principle that everyone should have an equal say. Originating from ancient Greece, it became particularly prominent in the United States during the 20th century. It was popularised by the U.S. Supreme Court in landmark cases such as Baker v. Carr (1962) (Supreme Court of the United States, 1962) and Reynolds v. Sims (1964) (Supreme Court of the United States, 1964).

In representative democracies, equal 'say' corresponds to fair representation. For a chamber of parliament that is elected directly and on a geographical basis, such as the House of Representatives in the United States, this implies equally-sized constituencies, and when the elections are organised in larger administrative or territorial units, such as states, the seats of the House must be distributed among them in proportion to their populations. Vast differences in constituency sizes mean a discrepancy in the voters' influence. Such differences can be substantial. In the 2020 congressional elections, we saw a difference of 82.5% in the average district size of Delaware (990,837) versus Montana (542,704). Historically, there have been even more serious discrepancies. The largest difference was 88.4% in 2010 (Montana vs. Rhode Island) and over 140% back in 1950 (Rhode Island vs. Nevada) (Eckman, 2021).

Meanwhile, in Europe, the Venice Commission, an advisory body of the Council of Europe, has emerged as a prominent authority on electoral matters. The Venice Commission provides valuable guidance on best practices for achieving fairness, transparency, and inclusiveness in electoral systems through its reports and recommendations. By synthesising comparative analyses and expert assessments, the Commission offers valuable insights into the strengths and limitations of various apportionment methodologies, contributing to advancing democratic principles worldwide. In its publication "Code of Good Practice in Electoral Matters" (Venice Commission, 2002), the Commission outlined a set of principles that voting districts should satisfy, including adherence to geographical and administrative boundaries and ensuring that district sizes do not deviate from the average by more than 10% (or 15% in extreme cases). While the first condition is no surprise and is the basis for non-trivial apportionment problems, the latter brings in the issue of (in)equality explicitly and sets strict requirements that are in stark contrast with US practices.

Why does the Old Continent seem to take the One Man, One Vote principle more seriously than the US, from where the modern interpretation of the concept originates? Firstly, this is not entirely true. The principle is enforced quite rigorously *within* states. In the Karcher v. Daggett (1983) (Supreme Court of the United States, 1983) case, the Supreme Court found a 0.69% discrepancy too large, as a better result could have been reached by transferring political entities between neighbouring contiguous districts. Equal-sized constituencies across states are much harder to achieve. Such transfers are politically not realistic. Biró et al. (2015) also uncovered that under the current House size, there will be at least a 50% difference between the smallest and the largest constituencies. Indeed, under the current census data, there is no allocation of seats that could decrease the gap between the smallest and largest constituencies. The 2010 apportionment could have been improved in this aspect, though. The reason why the gap was not contested is rooted in the history of apportionment.

The United States has witnessed a rich spectrum of apportionment methodologies throughout its history, marked by a perpetual quest for fairness and representational accuracy. From the nation's earliest days, the process of allocating congressional seats among states has been subject to intense scrutiny and debate. The infamous Three-Fifths Compromise of 1787, which counted enslaved individuals as three-fifths of a person for apportionment purposes, exemplifies the complexities inherent in early apportionment methods. Over the years, several apportionment procedures have been used, starting with the Jefferson method (known as D'Hondt in Europe, Balinski and Young, 1978), then the Webster (Balinski & Young, 1980), the Hamilton, once again the Webster and finally, since 1941, the Huntington-Hill or *Equal Proportions* method. These changes were triggered by oddities and paradoxes of the apportionment procedures, raising interest in their mathematical properties. Despite the expanding literature, the Equal Proportions method received no challengers and is widely considered fair. Luce (1930), Balinski and Young (1982, 1994), and Pukelsheim (2014) give thorough overviews of the need for apportionment as well as the existing methods.

The Equal Proportions method belongs to the same family of apportionment methods—known as divisor methods—as the Jefferson/D'Hondt method. While the Jefferson and D'Hondt methods are equivalent in the sense that they always yield the same apportionment, they are rooted in completely different views about the distribution process. The Jefferson procedure, while looking for the right divisor that distributes a number of seats equal to the House size, implicitly sets an equilibrium price for the seats. D'Hondt's algorithm is conducted in discrete steps, in contrast to the continuous nature of setting the divisor.

In this article, our objective is to understand how the claims are converted into resource items. Segal-Halevi (2020) showed that the D'Hondt method establishes a competitive equilibrium. We generalise this result and characterise parametric divisor methods as the only apportionment procedures that always correspond to some competitive equilibrium. We also discuss how price relates to voters' influence. In addition, we reestablish the Adams and Jefferson/D'Hondt methods as optimisation methods and link them to the new Leximin method. We demonstrate that the Leximin method combines the Adams and Jefferson/D'Hondt methods initialised at a different starting allotment. The results highlight how voters' influence relates to the concept of price and how different apportionment methods handle setting the equilibrium price or the price vector.

2 Literature

In this section, we review the relevant literature on apportionment, with an emphasis on both the development of the theory and its relation to other fair division problems and concepts.

2.1 Overview

The apportionment literature originated from the problem of proportional representation in the House of Representatives of the United States. Initially, constituencies had a fixed size, and the number of representatives for each state was directly determined. As the population grew, the number of constituencies, that is, the size of the House, increased dramatically, requiring another approach. First, the constituency size was increased repeatedly. Even though this offered only a temporary solution, the house size was fixed, and the seats were shared proportionally to the states' populations. Ever since, apportionment methods have been used to solve this problem.

The classic apportionment problem is about allocating parliamentary seats among states, counties or other administrative units. Today, apportionment methods are used from apportioning delegates in US presidential primaries Jones et al. (2023) to the efficient use of high-tech parts (Luss, 2012).

Indeed, already Ibaraki and Katoh (1988) and more recently Pretolani (2014) noticed the connection between integer resource allocation problems and apportionment. Resource allocation is a much more general problem, taking additional constraints and considerations into account. The solutions often emerge from nonlinear and/or integer optimisation problems without closed form or easy solutions.

Despite the broader class of resource allocation problems, there is a wide range of topics where apportionment methods can be used. Without being exhaustive, these include the allocation of servers to websites based on demand in a hosting service (Chase et al., 2001 offers a dynamic approach), the allocation of generators in a microgrid to maximise utility and reliability (Mojica-Nava et al., 2014), the optimal assignment of workers to subunits based on the number of standardised tasks (Kóczy et al., 2026), the allocation of scarce human resource or infrastructure among multiple uses in healthcare (Johannesson & Weinstein, 1993; Lin & Gen, 2008) and the allocation of emergency facilities (Luss, 2012). Gölz et al. (2025) suggest a randomized apportionment mechanism for determining which countries should nominate commissioners in the EU. They also suggests this model for allocating courses to faculty or shifts to workers.

A number of applications require more complex apportionment models. Balinski and Demange (1989) and Cembrano et al. (2021, 2022) generalised the problem to biproportional apportionment and to apportionment in multiple dimensions to allow for representation based not only on geography or political preferences but also on the grounds of nationality and gender.

In a number of countries, there are no constituencies, or rather, the entire country is a single constituency. Here, party lists receive votes, and the apportionment of the house determines the number of winners from each list based on the party votes. Brill et al. (2024) combine list and approval voting and allows voters to cast votes on multiple party lists. The combined method not only exhibits attractive properties from the perspective of representation but is computable in polynomial time.

2.2 Relation to other fair division problems

What distinguishes apportionment from other fair division problems is that the good being distributed is both discrete and homogeneous. By contrast, in cake cutting the good is heterogeneous and continuous in nature, while in the bankruptcy problem (also known as the claims problem) the distributed good is homogeneous but continuous such as money or land. Interestingly, there is a close parallel between certain bankruptcy rules and apportionment methods, which we briefly explore in Section 7.1.

Competitive equilibrium is a fundamental concept in economics describing a state of stability, but also—from a normative point of view—the fairness of a distribution Debreu (1982); Segal-Halevi (2020). The apportionment problem is analogous to the allocation of identical and indivisible goods to agents, each endowed with a budget. In this model, money has no intrinsic value, so an agent strictly prefers to acquire more items whenever this is affordable. A competitive equilibrium (CE) consists of a price and an allocation of items such that each agent can afford the bundle assigned to them, but cannot afford any strictly larger bundle.

3 Notation

We define the apportionment problem and methods. Let $N = \{1, 2, \ldots, n\}$ be the set of states of the country. Since we draw parallels between apportionment and indivisible object allocation, we also refer to the members of N as *players*. An *apportionment problem* $(\boldsymbol{p}, H)$ is a pair consisting of a vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ of state populations $p_i \in \mathbb{N}_+$, and a positive integer $H \in \mathbb{N}_+$ denoting the number of seats in the House. Throughout the paper, we assume that $H > n$. In the context of object allocation, $\boldsymbol{p}$ corresponds to the budget of the players, while H is the number of available identical and indivisible objects. Let $P = \sum_{i=1}^{n} p_i$ be the population of the country and let $\overline{a} = \frac{P}{H}$ denote the average size of a constituency measured in the number of voters—in various contexts, this value is also referred to sometimes in the literature as the standard divisor, or quota. The fraction $\frac{p_i}{P} H = \frac{p_i}{\overline{a}}$ is the *respective share* of state i.

An *apportionment method* determines the non-negative integers $s_1, s_2, \ldots, s_n$ with $\sum_{i=1}^{n} s_i = H$ specifying the number of constituencies (or seats) each of the states $1, 2, \ldots, n$ gets. Formally, it is a function M that assigns an allotment for each apportionment problem $(\boldsymbol{p}, H)$.

An apportionment method is a *divisor method*, if there exists a monotone increasing function $f : \mathbb{N} \to \mathbb{R}$, the *divisor criterion*, such that the seats are, in each round, allocated to the state with the highest $\frac{p_i}{f(s_i)}$ value. More precisely, suppose that $k-1$ seats are already allotted and the resulting apportionment is $\boldsymbol{s}$, then the k^{th} seat goes to the state $i(k, p, f) = \arg\max_j \frac{p_j}{f(s_j)}$ – or $i(k)$ for short if p and f are fixed – for which the fraction $\frac{p_i}{f(s_i)}$ is the highest[1]. The procedure continues until the last seat is allotted to $i(H)$. This process is called the *iterative largest quotient procedure* . In the following, we will refer to $\frac{p_i}{f(s_i)}$ as the current claim of State i, or simply the *claim*.

A divisor method with a linear $f(s)$ is called linear (or parametric, (Balinski & Ramirez, 2012)). Some divisor methods require special attention. The divisor method with $f(s) = s$ is known as the Adams method, $f(s) = s + \frac{1}{2}$ the Webster or Sainte-Laguë method, and $f(s) = s + 1$ the Jefferson or D'Hondt method.

In this paper, we consider arbitrary linear divisor methods with divisor function $f_x(s) = s + x$ with $x \in \mathbb{R}$. The iterative largest quotient procedure can be generalised with some modifications. For $x \leq 0$, the divisor may be negative or even zero. For the Adams method ($x = 0$), we overcome this by treating the initial claims as infinitely large, which is equivalent to allocating a seat to every player. After the first seat has been allotted, the procedure works as before, since all claims are finite and positive. It is easy to see that the same trick works for any $-1 < x < 0$. For $x = -1$, we allocate *two* seats. In general, allocating $\lfloor 1 - x \rfloor$ works, where $\lfloor \ \rfloor$ denotes the lower integer part operator. This is equivalent to treating every negative claim as infinitely large. Thus, if $x \leq 0$, the states are allotted some seats for free, irrespective of the size of their claims or population. Throughout this paper, we *assume* that $H > n \lfloor 1 - x \rfloor$, otherwise, not all players can get the $\lfloor 1 - x \rfloor$ seats.

Although the iterative largest quotient procedure does not break down for $x > 1$, we can give a new interpretation for this case. Now, instead of being allocated free seats, the states are burdened by an imposed *debt*. They have to purchase $\lfloor 1 - x \rfloor$ virtual seats as an entrance fee, before they are allotted real ones. This is equivalent to starting the iterative largest quotient procedure from $s_i = \lfloor 1 - x \rfloor < 0$ and only decreasing the available seats if the state is allocated a real one. If the distribution process ends before a state has obtained a real seat, it gets no seats. At this point, this description might seem a bit forced, as we can

[1] We assume distinct $\frac{p_i}{f(s_i)}$ values, moreover, dividing by zero is considered an infinitely large claim. In practice, ties are unlikely, and usually, no tie-breaking rules are specified in the literature. From a legal perspective, however, ties must be taken into account. In Hungary, ties are resolved according to the order in which parties appear on the ballot; this order is determined by a random draw conducted by the Election Committee.

interpret the $x > 1$ case without introducing the concept of debt. In the following section, we will highlight the aptness of this analogy.

4 Apportionment methods as competitive equilibria

The apportionment problem is analogous to the allocation of H identical and indivisible goods to n agents who each have a budget p_i. In this model, money has no intrinsic value, so an agent always strictly prefers to buy more items if she can afford them. Competitive equilibrium (CE) refers to a pair $(\pi, \boldsymbol{s})$ where π is the equilibrium price and $\boldsymbol{s}$ is an allotment of the indivisible items ($\sum s_i = H$), such that, each agent i can afford exactly s_i items and not more, that is, $\pi s_i \leq p_i < \pi(s_i + 1)$. In a classical setting, CE would entail that each agent can afford its bundle, and it is the best among the affordable ones. This second requirement is simplified in our case since the items are indivisible and homogeneous.

Segal-Halevi (2020) proved that the D'Hondt method results in a competitive equilibrium, where the equilibrium price is the claim of $i(H)$ that won the last seat. We extend this result for all linear divisor methods, $f_x(s) = s + x$, $x \in \mathbb{R}$. The key insight is that if $x \neq 1$, the price is not proportional: If $(x < 1)$, the player receives some free credit, while if $(x > 1)$, the players start with a debt, and first they have to cover a kind of an entry fee to the allocation process. Theoretically, this surcharge can be even higher than the cost of a seat, that is, the equilibrium price. Consequently, a competitive equilibrium (CE) is described by three parameters: a budget vector $\boldsymbol{p}$, a price π and a uniform credit/debt c. The solution induced by the CE is $\left(\left\lfloor \frac{p_1+c}{\pi} \right\rfloor^+, \left\lfloor \frac{p_2+c}{\pi} \right\rfloor^+, \ldots, \left\lfloor \frac{p_n+c}{\pi} \right\rfloor^+\right)$, where $^+$ denotes the positive part operator $(y^+ = \max(y, 0))$.

Example 1 Suppose we have three states, A, B and C with the following populations $\boldsymbol{p} =$ (600, 270, 140) and the Adams method is used to allocate the seats. The next table lists the possible claims of the states in decreasing order, with the last row highlighting the state to which the actual claim belongs.

House size (H)	1	2	3	4	5	6	7	8	9	10
Claims in decreasing order	∞	∞	∞	600	300	270	200	140	135	120
State receiving the seat	A	B	C	A	A	B	A	C	B	A

Thus, if the house size is 6, then the resulting apportionment is (3,2,1) and state B receives the last seat.

It is clear from this example that if the House size increases, no state loses seats—that is, methods that can be obtained by the iterative largest quotient procedure are always House-monotonic (as opposed to e.g. largest remainder methods, like the Hamilton-method).

When US President Thomas Jefferson first outlined the apportionment method named subsequently after him, he did not describe the iterative process that distributed the seats one by one based on the claim function. Instead, he proposed the following procedure. Calculate the standard divisor ($\overline{a}$), divide the state populations by the standard divisor and take the lower integer part of the obtained values (the *lower quota*). Decrease the standard divisor until the lower quotas add up to the House size. If we consider the divisor as the price of a seat, then this process is akin to finding a market-clearing price. Likely, Jefferson did not overlook this aspect; the mathematical process most likely stemmed from considerations of this nature.

In Theorem 2, we prove that this idea can be extended to all linear divisor methods. First, we need the following lemma.

Lemma 1 *Let $(\boldsymbol{p}, H)$ be an apportionment problem and F_x a divisor method with claim function $f_x(s) = s + x$. The following two statements are equivalent.*

i) $F_x(\boldsymbol{p}, H) = \boldsymbol{s}$ and $j = i(H)$ in the iterative largest quotient procedure corresponding to $f_x(s)$;
ii) There exists $\boldsymbol{s}$ such that $\sum s_i = H$, $s_i \geq \lfloor 1 - x \rfloor^+$ for all i, and a j with $s_j > \lfloor 1 - x \rfloor^+$ and

$$\frac{p_i}{s_i + x} < \frac{p_j}{s_j - 1 + x} \qquad \forall i \in N, s_i = \lfloor 1 - x \rfloor^+, \tag{1}$$

$$\frac{p_i}{s_i + x} < \frac{p_j}{s_j - 1 + x} \leq \frac{p_i}{s_i - 1 + x} \qquad \forall i \in N, s_i > \lfloor 1 - x \rfloor^+. \tag{2}$$

Proof i)→ii) We argue that $F_x(\boldsymbol{p}, H) = \boldsymbol{s}$ is such an $\boldsymbol{s}$ that satisfies the requirements of ii). Indeed, $\sum s_i = H$ and the iterative largest quotient procedure guarantees that $s_i \geq \lfloor 1 - x \rfloor^+$ for all i (here we rely on the implicit assumption that $H > n \lfloor 1 - x \rfloor$). Since j received the last seat, $s_j > \lfloor 1 - x \rfloor^+$ also follows. Thus, we only need to prove inequalities 1 and 2. Indirectly, if

$$\frac{p_j}{s_j - 1 + x} > \frac{p_i}{s_i - 1 + x},$$

then j has received its last seat before i has thereby violating $j = i(H)$. On the other hand, if $j = i(H)$, its claim $\frac{p_j}{s_j + x - 1}$ must be greater than the current claim of any other player $\frac{p_i}{s_i + x}$, proving the first inequality.

ii)→i) Suppose $\boldsymbol{s}$ is such that it satisfies the requirements of ii). Let $N^0 = \{i | i \in N, s_i = \lfloor 1 - x \rfloor^+\}$ and $N^+ = \{i | i \in N, s_i > \lfloor 1 - x \rfloor^+\}$. By the definition of s, $N = N^0 \cup N^+$.

The iterative largest quotient procedure, in step 1, allocates all players $\lfloor 1 - x \rfloor^+$. First, let $i \in N^0$. Inequality 1 shows that before allocating more than s_i seats to such players, player j will receive its s_jth seat. Now let $i \in N^+$. Since $\forall i \in N^+$ and $t \in \mathbb{N}$, $\lfloor 1 - x \rfloor^+ \leq t \leq s_i - 1$,

$$\frac{p_j}{s_j - 1 + x} \leq \frac{p_i}{s_i - 1 + x} \leq \frac{p_i}{t + x},$$

each player $i \in N^+$ must have received s_i seats before j got its s_jth seat. Thus, when j receives its s_jth seat in the iterative largest quotient procedure, there are no more seats to distribute.

The key point is that the iterative largest quotient procedure induces an ordering of the claims (see Example 1). The house size determines which state obtains the last seat, which in turn implies inequalities 1 and 2 and *vice versa*. The theorems that follow hinge very much on this lemma; hence, it is worth exploring a few examples.

Example 2 Just like in Example 1 we have three states, A, B and C with populations $\boldsymbol{p} = (600, 270, 140)$. Let us investigate a linear divisor method F_x with a negative parameter $x = -1$. The following table displays the claims for given seat allocations.

seat (s_i)	0	1	2	3	4	5	6
divisor ($f(s_i)$)	-1	0	1	2	3	4	5
claim of state A	∞	∞	600	300	200	150	120
claim of state B	∞	∞	270	135	90	67.5	54
claim of state C	∞	∞	140	70	46.7	35	28

Suppose that $H = 10$. Note that $\lfloor 1 - x \rfloor^+ = 2$, thus during the iterative largest quotient procedure, the first two seats correspond to infinite claims. After distributing two seats for each state, the claims become finite numbers. The remaining seats are distributed as follows. Two go to state A, then one to state B, and then the last one again to State A; thus, the final allotment is (5,3,2). The last claim of state A was $200 = \frac{600}{4-1}$ which is indeed more than the current claim of state B ($\frac{270}{3-1}$) or that of state C ($\frac{140}{2-1}$), but less than the previous claims of state B ($\frac{270}{2-1}$) and state C ($\frac{140}{1-1} = \infty$).

Example 3 Suppose the state populations are again the same as in Example 1, but now we choose the Imperiali method, which has a parameter $x = 2$.

s	0	1	2	3	4	5	6
divisor	2	3	4	5	6	7	8
claim of player A	300	200	150	120	100	85.7	75
claim of player B	135	90	67.5	54	45	38.6	33.8
claim of player C	70	46.7	35	28	23.3	20	17.5

A house size of seven results in the allotment (5,2,0) – state C ends up with no seats. The last seat is allotted to state B with a claim of $90 = \frac{270}{1+2}$. Note that state C belongs to N^0 as $s_C = \lfloor 1 - 2 \rfloor^+ = 0$. It is easy to verify that both inequalities 1 and 2 hold.

The Imperiali method, also known as the Belgian or Nohlen method, is often seen as somewhat exotic due to its failure to satisfy the exact quota (Serafini, 2020). Specifically, the Imperiali method sometimes fails to allocate the respective shares of each state, even when these shares are integers[2]. When placed next to the "standard" Adams and D'Hondt/Jefferson methods, it clearly favours the larger claimants: Belgian senator Imperiali came up with the idea to minimise the influence of minor secularist parties and is very effective as a means of electoral engineering to maintain the power of a dominant party (Golosov, 2014). This is not surprising given our majorization results in Section 5 and it is clear that higher x values, such as the Schepis method with $x = 4$ mentioned by (Kopfermann, 1991, p. 124) would yield apportionments benefiting the larger claimants even more.

Despite these criticisms, the Imperiali method does not necessarily perform poorly on empirical data and is not such an outrageous suggestion from the theoretical point either. Sziklai and Héberger (2020) found that if a reference ranking is created from the states'

[2] Consider two states with populations 350 and 140 and a House size of 7. Since the average constituency size $\bar{a} = 70$, there exists an exact proportional solution, namely, that the first state gets 5 seats while the second gets 2. However, it is easy to see that the Imperiali method awards 6 seats to the first state as the final claim of the first state $\left(\frac{350}{5+2} = 50\right)$ is bigger than that of the second state $\left(\frac{140}{1+2} = 46.66\right)$

respective shares, and the distances between this reference ranking and those induced by different apportionment methods are measured, then the Imperiali and (unsurprisingly) the Leximin methods turn out to be the closest to the reference ranking. On the other hand, the following Theorem 2 establishes the D'Hondt method as the neutral point with the – widely accepted – Adams and the – exotic – Imperiali being equally off, albeit in different directions. The Imperiali method is still used in local elections in Belgium.

We now show that all linear divisor methods induce a competitive equilibrium where players receive a uniform credit or debt.

Theorem 2 *Let F_x be a divisor method with claim function $f_x(s) = s + x$ with $x \in \mathbb{R}$. There exists a competitive equilibrium where players face modified budget constraints, and then the equilibrium price is determined by the claim of $i(H)$ in the iterative largest quotient procedure corresponding to $f_x(s)$.*

Proof Let $\pi = \frac{p_j}{s_j - 1 + x}$, where $j = i(H)$ is the last player who got a seat allocated in the procedure for F_x. As a first step, we modify the budgets in the following way. Each player receives an adjustment of $\pi(1-x)$, so that $p_i' = (p_i + \pi(1-x))^+$.

As a second step, we show that for this market, π is the equilibrium price. From Lemma 1 it follows that $\forall i \in N$, such that $s_i > \lfloor 1 - x \rfloor^+$,

$$\frac{p_i}{s_i + x} < \pi \leq \frac{p_i}{s_i - 1 + x}, \tag{3}$$

which, after some rearrangements, leads to

$$\pi(s_i - 1 + x) \leq p_i < \pi(s_i + x) \quad \forall i \in N, s_i > \lfloor 1 - x \rfloor^+. \tag{4}$$

For $s_i = \lfloor 1 - x \rfloor^+$ only the left-hand side of Inequality 3 applies due to Inequality 1.

As the third step, we show that the number of seats purchased under the competitive equilibrium matches the apportionment.

How many seats can they afford if the price is π? Each state has a budget $(p_i + \pi(1-x))^+$. For $i = j$ the problem simplifies to

$$\frac{(p_j + \pi(1-x))^+}{\pi} = \left(\frac{p_j + \pi(1-x)}{\pi}\right)^+ = \left(\frac{p_j}{\frac{p_j}{s_j - 1 + x}} + 1 - x\right)^+ = \left(s_j\right)^+ = s_j.$$

If $i \neq j$ then adding $\pi(1-x)$ to Equation 4 and dividing by π yields

$$s_i = \frac{\pi(s_i - 1 + x) + \pi(1-x)}{\pi} \leq \frac{p_i + \pi(1-x)}{\pi} \leq \frac{(p_i + \pi(1-x))^+}{\pi} \tag{5}$$

for all $s_i > \lfloor 1 - x \rfloor^+$ and

$$\begin{aligned}\frac{(p_i + \pi(1-x))^+}{\pi} &= \left(\frac{p_i + \pi(1-x)}{\pi}\right)^+ \\ &< \left(\frac{\pi(s_i + x) + \pi(1-x)}{\pi}\right)^+ = (s_i + 1)^+ = s_i + 1.\end{aligned}$$

for all $s_i \geq \lfloor 1 - x \rfloor^+$. Thus, each state can afford at least s_i seats but not more.

Theorem 2 generalises the competitive equilibrium property to such well-known methods as the Adams, Webster-Saint Laguë or the Imperiali (Niemeyer & Niemeyer, 2008). For the latter, the claim function $f(s) = s + 2$ means that each player has to pay the full price of a seat just to get involved in the apportionment process. The Adams method, on the other hand, gives each player enough credit to buy a full seat – effectively for free.

As the parameter of the claim function increases, players must pay higher and higher entry fees. Some players may become bankrupt before entering the game—but never all of them.

For a given apportionment, there are many possible competitive equilibria. We focus on what we call the *standard price*, $\pi = \frac{p_j}{s_j - 1 + x}$. The standard price is a market-clearing price, but so is any $\kappa \in [\pi - \epsilon, \pi]$ for $\epsilon > 0$ small enough. The next lemma asserts that the standard price decreases as the parameter of the divisor method increases.

Lemma 3 *Let X and Y be two divisor methods with claim function $f_x(s) = s + x$ and $f_y(s) = s + y$, respectively, such that $y > x$. Furthermore, for any apportionment problem $(\boldsymbol{p}, H)$, let π_x, π_y denote the standard prices for X and Y. It follows that $\pi_x > \pi_y$.*

Proof If for a given apportionment problem $(\boldsymbol{p}, H)$, X and Y induce the same distribution $\boldsymbol{s}$, then they agree on which state receives the last seat and due to $y > x$ it follows that

$$\pi_x = \frac{p_j}{s_j - 1 + x} > \frac{p_j}{s_j - 1 + y} = \pi_y.$$

Now assume that X prescribes the allotment $\boldsymbol{s}$ and Y the allotment $\boldsymbol{t}$. Furthermore, let j be the last player who obtained a seat in the iterative largest quotient procedure under X, while let k be the corresponding player under Y. If $\boldsymbol{s} \neq \boldsymbol{t}$, then there must exist a state ℓ which received more seats under Y than under X. Then

$$\pi_y = \frac{p_k}{t_k - 1 + y} \leq \frac{p_\ell}{t_\ell - 1 + y} < \frac{p_\ell}{t_\ell - 1 + x} \leq \frac{p_\ell}{s_\ell + x} \leq \frac{p_j}{s_j - 1 + x} = \pi_x$$

The first inequality arises from Lemma 1, the second from $y > x$, the third from $t_\ell \geq s_\ell + 1$ and the last, once more, from Lemma 1.

As the parameter tends to infinity, only the largest players will have a positive balance — it can happen that a player must pay a price higher than the cost of H seats just to be involved!

On the other hand, the seats obtained from free credit do count in the rest of the process. For example, the Adams method is not equivalent to a procedure where each state receives one free seat, and the rest is distributed according to the D'Hondt method.

Example 4 Consider a case when $\boldsymbol{p} = (600, 330, 120)$ and $H = 5$.

seat (s_i)	0	1	2	3	4
claim of state A	∞	600	300	200	125
claim of state B	∞	330	165	110	82.5
claim of state C	∞	120	60	40	30

After distributing one seat to each state, the Adams method awards a seat first to state A (with claim 600) then to state B (with claim 300). If the Jefferson/D'Hondt method is applied after distributing one seat to each state, then both remaining seats go to state A (with claims 300 and 200).

While the D'Hondt method remains the most widely used apportionment method, it is not without challengers. For the seat allocation of the House of Representatives of the United States, a non-linear divisor method, the method of Equal Proportions, is used.Huntington (1928) argued that successively reducing the relative differences between district sizes by transferring a seat from one state to another constitutes a convergent mechanism and leads to the Equal Proportions solution. Balinski and Young (1982) summarize this and related concepts (see pp. 100–105). Equality at the voter level is better expressed by optimisation methods such as the Burt-Harris (Burt & Harris, 1963) and Leximin (Biró et al., 2015). The latter is the only method that meets the fairness requirements outlined by the Venice Commission. It is natural to ask if the Theorem could also be generalised to such methods. Unfortunately, the answer is negative.

Theorem 4 *Let $(\boldsymbol{p}, \pi, c)$ be a CE, where $\boldsymbol{p}$ is the budget vector, π is the equilibrium price, and $c \in \mathbb{R}$ is a credit/debt. The total number of items distributed is $H = \sum_{i \in N} \left\lfloor \frac{p_i + c}{\pi} \right\rfloor^+$, where $^+$ denotes the positive part operator ($y^+ = \max(y, 0)$). Then, there exists a linear divisor method F for which*

$$F(\boldsymbol{p}, H) = \left(\left\lfloor \frac{p_1 + c}{\pi} \right\rfloor^+, \left\lfloor \frac{p_2 + c}{\pi} \right\rfloor^+, \ldots, \left\lfloor \frac{p_n + c}{\pi} \right\rfloor^+ \right).$$

Proof Let us fix $x = 1 - \frac{c}{\pi}$. We argue that the divisor method F_x with claim function $f_x(s) = s + x$ gives the desired result. Recall the definition of N^+ from Lemma 1. In our context, $N^+ = \{i | i \in N, s_i > \left\lfloor \frac{c}{\pi} \right\rfloor^+\}$. First, we prove that

$$\left\lfloor \frac{p_1 + c}{\pi} \right\rfloor - \frac{c}{\pi} > 0 \quad \forall i \in N^+ \tag{6}$$

Note that

$$s_i = \left\lfloor \frac{p_1 + c}{\pi} \right\rfloor^+ = \left\lfloor \frac{p_1 + c}{\pi} \right\rfloor > \left\lfloor \frac{c}{\pi} \right\rfloor^+ \geq 0,$$

thus $\left\lfloor \frac{p_1 + c}{\pi} \right\rfloor > 0$. If c is negative, that already implies Eq. 6. If c is non-negative, then $s_i > \left\lfloor \frac{c}{\pi} \right\rfloor^+ = \left\lfloor \frac{c}{\pi} \right\rfloor$, which in turn implies that $s_i \geq \left\lfloor \frac{c}{\pi} \right\rfloor + 1 > \frac{c}{\pi}$.

The first step ensured that the claims of any $i \in N^+$ in the last but one step of the iterative largest quotient procedure are positive, discrete numbers. Let j be such that

$$j = \arg\min_{i \in N^+} \frac{p_i}{\left\lfloor \frac{p_i + c}{\pi} \right\rfloor - \frac{c}{\pi}}.$$

From the definition, it follows that

$$\frac{p_j}{\left\lfloor \frac{p_j + c}{\pi} \right\rfloor - \frac{c}{\pi}} \leq \frac{p_i}{\left\lfloor \frac{p_i + c}{\pi} \right\rfloor - \frac{c}{\pi}} \quad \forall i \in N^+. \tag{7}$$

Furthermore,

$$\frac{p_i}{\left\lfloor \frac{p_i + c}{\pi} \right\rfloor + 1 - \frac{c}{\pi}} < \frac{p_i}{\frac{p_i + c}{\pi} - \frac{c}{\pi}} = \pi = \frac{p_j}{\frac{p_j + c}{\pi} - \frac{c}{\pi}} \leq \frac{p_j}{\left\lfloor \frac{p_j + c}{\pi} \right\rfloor - \frac{c}{\pi}} \quad \forall i \in N. \tag{8}$$

From Eq. 7 and 8

$$\frac{p_i}{\left\lfloor \frac{p_i + c}{\pi} \right\rfloor + 1 - \frac{c}{\pi}} < \frac{p_j}{\left\lfloor \frac{p_j + c}{\pi} \right\rfloor - \frac{c}{\pi}} \leq \frac{p_i}{\left\lfloor \frac{p_i + c}{\pi} \right\rfloor - \frac{c}{\pi}} \quad \forall i \in N^+.$$

while the lower bound is also true for all agents in N. Thus, by Lemma 1 there exists a divisor method with parameter x that replicates the distribution $\left(\left\lfloor \frac{p_1+c}{\pi} \right\rfloor^+, \left\lfloor \frac{p_2+c}{\pi} \right\rfloor^+, \ldots, \left\lfloor \frac{p_n+c}{\pi} \right\rfloor^+\right)$ and gives the last item to j.

Theorems 2 and 4 characterise competitive equilibrium for the apportionment problem.

Theorem 5 *An apportionment is a competitive equilibrium if and only if it can be obtained from a linear divisor method.*

Example 5 Let us revisit Examples 2 and 3. In the first case, the last state to receive a seat was A with a claim of 200. That will be our price, π. The divisor method we employed has a parameter $x = -1$, one less than the Adams method. From the formula $x = 1 - \frac{c}{\pi}$, we can calculate the budget adjustments, which is $c = 400$, twice as much as the price, basically giving away two free seats. The adjusted budgets are (1000, 670, 540). State A can afford five seats, while state B and C have three and two seats, respectively.

In the second case, the populations are the same, but the House size is reduced to seven, and the parameter changes to $x = 2$. The last seat was allotted to state B for a price of $\pi = 90$. The budgets are adjusted by an entry fee of -90, resulting in the endowment of (510, 180, 50). State A can afford five seats, state B two seats and state C none.

5 Majorization

Theorem 5 intimately links competitive equilibria ("one price") with linear divisor methods. These methods only differ in the credit or debit each player gets. When using the iterative largest quotient procedure to generate the apportionment, this translates into a number of free seats for each player, while the rest are allocated based on the players' populations. The more seats are free, and the fewer seats are allocated on the grounds of claims, the more equitable the allocation becomes. In the context of apportionment, this idea is formalised by the notion of majorization.

Our result provides insight into a popular majorization result for linear divisor methods: Given an apportionment a, majorization compares the vectors $a^M = (a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots)$ of cumulative number of seats allocated to the regions, where the regions are sorted in increasing order. The apportionment b majorizes a if $b^M \geq a^M$. Marshall et al. (2002) present conditions for one apportionment method majorizing another and finds that the majorization order of some well-known apportionment methods is (in descending order) Adams, Dean, Hill, Webster-Saint Laguë, and Jefferson-D'Hondt. Similar results have already been stated by Balinski and Young (1982) and Balinski (1993).

In our case, of two linear divisor methods with parameters x and y, the first majorizes the second in case $x < y$. This result follows from the observation that a smaller parameter implies that the discount is greater or the entry fee is smaller, which affects smaller states positively.

The following technical lemma is needed to prove the result of majorization.

Lemma 6 *Let* $h_i(x) = \frac{a_i}{b_i+x}$, $a_i, b_i > 0$ *for* $i = 1, 2$.

i. If $h_1(x) \leq h_2(x)$ *for some* $x > -\min\{b_1, b_2\}$ *and* $a_1 < a_2$ *then* $h_1(y) \leq h_2(y)$ *for all* $y > x$.
ii. If $h_1(y) \leq h_2(y)$ *for some* $y > 0$ *and* $a_1 > a_2$ *then* $h_1(x) \leq h_2(x)$ *for all* $-\min\{b_1, b_2\} < x < y$.

Proof The two cases are symmetric, thus we only prove the first. By rearranging $h_1(x) < h_2(x)$ we get

$$\frac{a_1}{a_2} < \frac{b_1 + x}{b_2 + x} \tag{9}$$

As $x \to \infty$, the right-hand side of Eq. 9 tends to 1, while the left-hand side is strictly less than 1. It follows that for very large y values $h_1(y) \le h_2(y)$. Since in the positive quadrant, h_1 and h_2 intersect at most once, it follows that $h_1(y) < h_2(y)$ for any $y > x$.

Next, we prove that an increase in the parameter benefits large players.

Theorem 7 *Let F_x and F_y be two divisor methods with claim function $f_x(s) = s + x$ and $f_y(s) = s + y$ respectively, such that $y > x$. Let $(\boldsymbol{p}, H)$ be an apportionment problem, such that $p_1 < p_2 < \cdots < p_n$ and $F_x(\boldsymbol{p}, H) = \boldsymbol{s}$, $F_y(\boldsymbol{p}, H) = \boldsymbol{t}$. It follows that $\boldsymbol{s}^M \ge \boldsymbol{t}^M$.*

Proof We proceed by contradiction. Let ℓ be the first index for which $\sum_{i=1}^{\ell} s_i < \sum_{i=1}^{\ell} t_i$. The assumption implies that $s_l < t_l$. Then there must exist a $k > \ell$ such that $t_k < s_k$. That is, when ℓ obtained its t_ℓth seat under F_y, k still had at most $s_k - 1$ seats. Thus

$$\frac{p_k}{s_k - 1 + y} \le \frac{p_\ell}{t_\ell - 1 + y}.$$

We use Lemma 6 backwards. For very large z

$$\frac{p_k}{p_\ell} \ge \frac{s_k + z - 1}{t_\ell + z - 1}$$

as the left-hand side is strictly greater than 1, while the right-hand side is close to 1. Somewhere between y and z the two functions intersect. Since they can have at most one intersection in the positive quadrant, it follows that for $0 < x < y$,

$$\frac{p_k}{s_k + x - 1} \le \frac{p_\ell}{t_\ell + x - 1}$$

which contradicts that under F_x state k received s_k seats while ℓ received s_ℓ.

Theorem 2 offers a different perspective on the majorization results. Traditionally, the D'Hondt/Jefferson method has been viewed as the opposite of the Adams method: while the latter is known to favour small states, the former benefits large ones. However, Competitive Equilibrium suggests that the D'Hondt/Jefferson method is the neutral choice (CE with no credit or debt imposed), with the Adams method standing in opposite to the Imperiali method. This result is seemingly at odds with the literature, in which the Webster method is considered the halfway compromise between methods that favor larger states and those that favor smaller ones.

Balinski and Young (1982) proved that Webster is the only divisor method that is pairwise unbiased on populations (see Prop. 5.2 on page 119 and subsequent propositions). There are different interpretations on what unbiased can mean, here we mention only one: an apportionment (a_1, a_2) between two states with populations (p_1, p_2) such that $p_1 > p_2$ is said to favor the larger state over the smaller state if $\frac{a_1}{p_1} > \frac{a_2}{p_2}$. Webster is the only divisor method that favors the larger state just as often as smaller states on an appropriate range of possible House sizes. Other interpretations model bias from probabilistic point of view. Nevertheless, Webster comes out as the unique unbiased solution in each case. However, from the point of view of competitive equilibrium, Webster offers a credit that worth half a seat to each state, which is clearly helping smaller states. In the next section, we reinforce the traditional view by showing that the D'Hondt/Jefferson and Adams methods are optimisation procedures with objective functions that are mirror images of each other.

6 One price or optimal price

Even though Theorem 5 establishes an if-and-only-if relation between competitive equilibria and linear divisor methods, this does not necessarily mean that other methods, such as the Equal Proportions or the Leximin, ever produce an allotment that cannot be written as a competitive equilibrium.

The Leximin method can be seen as an approach aimed at setting prices with minimal deviation from the average price. Since every state has its own price, each budget is spent, and hence, there is no inefficiency. In contrast, divisor methods have a universal price, but apart from the state that receives the last seat, each state ends up with some unused resources. Seats representing a state with a large leftover are divided among a larger body. Hence, the voters' influences get diluted.

Having a fixed price is fair from the point of view of the states. However, it necessarily creates inefficiency, which in turn leads to deviations in voters' influence. Setting different prices for each state is unfair from the states' perspectives, but it helps alleviate the unevenness of voting power.

The discrepancy between the same and the fair price is very specific to the setup where the budget is given in terms of a fake or virtual currency with no alternative usage: votes received in an election cannot be used to support anything else. This phenomenon is not entirely unique. There are various artificial markets where the intensity of preferences can be expressed by sharing an artificial budget among alternatives. Budish (2011) and references therein describe course allocation mechanisms at various prominent universities where students are given the same budget that can be used to bid for courses. Different mechanisms are used to find market prices, but as Budish (2011) highlights, in those markets, market-clearing prices may fail to exist and introduce approximate equilibria for approximately equal budgets (or incomes).

In course allocation problems, equilibrium prices must be found in several simultaneous markets using the same currency. Our setup is, of course, far simpler in the sense that there is only one resource, although players may buy more than one of this single resource. This guarantees the existence of a market-clearing price but makes it even more difficult to spend fake money.

For the vectors v and $w \in \mathbb{R}^N$ we say that v is *lexicographically smaller* than w and write $v \prec w$ if $v_1 < w_1$ or $v_j < w_j$ for some $j \in N$ and $v_i = w_i$ for all $i < j$. We say that a vector $v \in S$ is *lexicographically minimal in* S if $v \preceq w$ for all $w \in S$. For a vector v, let $v^{\downarrow}$ denote the sorted vector that contains the elements of v in a non-increasing order. Likewise, we use $v^{\uparrow}$ for nondecreasing sorting.

The *Leximin* method (Biró et al., 2015) lexicographically minimizes the sorted vector $d^{\downarrow}$ of maximum departures $d_i = \left|\frac{\frac{p_i}{a_i} - \overline{a}}{\overline{a}}\right|$, that is, the vector of absolute differences between the average populations of constituencies and the average constituency size $\overline{a}$.

Clearly, Leximin is not equivalent to any divisor method due to its incompatibility with various properties such as population monotonicity (Biró et al., 2015). However, this does not rule out the possibility that each Leximin allocation can be produced using a different divisor method.

Fortunately, this is something we can test. We can formulate a system of linear constraints whose intersection encompasses the set of feasible solutions.

Remark 1 Let $\boldsymbol{s}$ be a solution for the apportionment problem $(H, \boldsymbol{p})$. Then $\boldsymbol{s}$ can be written as a competitive equilibrium only if the intersection of the following set of inequalities is non-empty.

$$\begin{aligned} p_i + z &\geq s_i y \quad \forall i \in N \\ p_i + z &\leq (s_i + 1) y \quad \forall i \in N. \end{aligned} \tag{10}$$

Under the equilibrium price, y and uniform credit/debit z, each player i can afford exactly s_i seats. That is, the player's budget is at least $s_i y$, but at most $(s_i + 1)y$. Note that these are necessary but not sufficient conditions for the existence of a CE. The upper bounds are actually strict, and z is not independent of y. Nevertheless, there are apportionments that do not even satisfy these criteria. Experimenting with possible inputs quickly yielded the following counterexamples:

$$\textbf{Leximin:}\ H = 38,\ \boldsymbol{p} = [159, 448, 935, 1227, 1984];$$
$$\textbf{Equal Proportions:}\ H = 36,\ \boldsymbol{p} = [177, 681, 683, 812, 1930].$$

Thus, the Leximin and the Equal Proportions method sometimes produce allotments that cannot be constructed as competitive equilibria with a unique price. Naturally, when the market-clearing price does not have to be universal, then every allotment can be regarded as a competitive equilibrium, e.g., the price vector $\boldsymbol{\pi} = (p_i/s_i)_{i \in N}$ allows each state i to buy exactly s_i seats.

A variant of the Leximin method, the Leximinimax method was introduced by Kóczy et al. (2026): it lexicographically minimises the sorted difference vector $\delta^{\downarrow}$, where $\delta_i = \frac{\frac{p_i}{a_i} - \bar{a}}{\bar{a}}$ from the average constituency size, or equivalently lexicographically minimizes the sorted vector of average constituency sizes $\left(\left(\frac{p_i}{a_i}\right)_{i \in N}\right)^{\downarrow}$.

It is natural to introduce a corresponding method, *Leximaximin*, that lexicographically maximises the sorted difference vector $\delta^{\uparrow}$, where the sorting is *increasing* this time. Equivalently, the Leximaximin method maximizes the sorted vector $\left(\left(\frac{p_i}{a_i}\right)_{i \in N}\right)^{\uparrow}$.

While all of these are optimisation methods, the Leximaximin and the Leximinimax methods focus on one type of deviation from the average size only. The Leximinimax method minimises upward deviations from the average size. It makes sure that a voter's influence is never too far off the average. Kóczy et al. (2026) use it to ensure that the allocated workers are not overloaded – while, at the same time, it is acceptable if some workers have little to do. The Leximaximin method does the opposite: it tries to avoid overrepresentation while accepting the fact that some voters may get much less than average. When a bank allocates ATMs, there is little concern about busy locations, while an underused machine is simply not economical to place or maintain.

Balinski and Young (1982) already claimed, albeit without providing a proof, that both the Adams and Jefferson/D'Hondt methods can be obtained with a minimax-type algorithm (see Prop. 3.10 on page 105). (Edelman, 2006) and (Shechter, 2024) also formulate these two as optimization methods. In the next two theorems, we prove something slightly stronger.

Theorem 8 *The Adams and Leximinimax methods are equivalent.*

Proof Suppose the Leximinimax method yields the apportionment a, and the states are reindexed in such way that $\frac{p_1}{a_1} > \frac{p_2}{a_2} > \cdots > \frac{p_n}{a_n}$ holds true. Let us denote the apportionment prescribed by the Adams method by b. We proceed by contradiction and assume that $a \neq b$. Since the apportionments differ, there must be at least two states i and j for which $a_i \neq b_i$ and $a_j \neq b_j$. Without loss of generality, we assume that i happens earlier in the ordering than j, that is $i < j$.

Suppose first that $b_i < a_i$, and $b_j > a_j$. It follows that

$$\frac{p_i}{b_i} > \frac{p_i}{a_i} > \frac{p_j}{a_j} > \frac{p_j}{b_j}$$

holds. Construct the Adams apportionment step by step. When State j has a_j seats, then State i has at most b_i seats. The corresponding claims are $\frac{p_i}{b_i} > \frac{p_j}{a_j}$, so according to Adams, State j should not get another seat until State i has at least one more. This contradicts our assumption that $b_j > a_j$.

Now consider the opposite, namely that $b_i > a_i$, and $b_j < a_j$. Notice, that

$$\frac{p_j}{b_j} \geq \frac{p_j}{a_j - 1} > \frac{p_i}{a_i} > \frac{p_i}{b_i}.$$

The second inequality must hold; otherwise, a could be improved by giving one seat from j to i, which contradicts that a is lexicographically optimal. Again, construct Adams' apportionment. When State i has a_i seats, State j has at most b_j seats. Since $\frac{p_j}{b_j} > \frac{p_i}{a_i}$, State i would not get another seat until j gets at least one more. This contradicts our assumption that $b_i > a_i$.

Theorem 9 *The Jefferson-D'Hondt and Leximaximin methods are equivalent.*

Proof Suppose the Leximaximin method yields the apportionment a, and the states are reindexed in such way that $\frac{p_1}{a_1} < \frac{p_2}{a_2} < \cdots < \frac{p_n}{a_n}$ holds true. Let us denote the apportionment prescribed by the Jefferson-D'Hondt method by b. We proceed by contradiction and assume that $a \neq b$. Since the apportionments differ, there must be at least two states i and j for which $a_i \neq b_i$ and $a_j \neq b_j$. Without loss of generality, we assume that $i < j$.

Suppose first that $b_i < a_i$, and $b_j > a_j$. It follows that

$$\frac{p_j}{b_j} \leq \frac{p_j}{a_j + 1} < \frac{p_i}{a_i} = \frac{p_i}{a_i - 1 + 1} \leq \frac{p_i}{b_i + 1}$$

holds. The second inequality must hold; otherwise, a could be improved by giving one seat from i to j, which contradicts that a is lexicographically optimal. Construct the Jefferson-D'Hondt apportionment step by step. When State j has a_j seats, then State i has at least b_i seats. The corresponding claims are $\frac{p_i}{b_i+1} > \frac{p_j}{a_j+1}$, so according to Adams, State j should not get another seat until State i has at least one more. This contradicts our assumption that $b_j > a_j$.

Now consider the opposite, namely that $b_i > a_i$, and $b_j < a_j$. Notice, that

$$\frac{p_i}{b_i} \geq \frac{p_i}{a_i + 1} < \frac{p_i}{a_i} < \frac{p_j}{a_j} = \frac{p_j}{a_j - 1 + 1} \leq \frac{p_j}{b_j + 1}.$$

Again, construct the Jefferson-D'Hondt apportionment. When State i has a_i seats, State j has at most b_j seats. Since $\frac{p_j}{b_j+1} > \frac{p_i}{a_i+1}$, State i would not get another seat until j gets at least one more. This contradicts our assumption that $b_i > a_i$.

Next, we present efficient algorithms for the Leximinimax and Leximaximin algorithms. The idea is based on the heuristic developed by Biró et al. (2015) for the Leximin method.

A vector $a \in \mathbb{N}^n$ is a pre-allotment if the $\sum_{i \in N} a_i = H$ condition is relaxed. Let a^{i+} denote the pre-allotment where $a^{i+}_i = a_i + 1$ and $a^{i+}_j = a_j$ if $j \neq i, j \in N$. Analogously, let a^{i-} denote the pre-allotment where $a^{i-}_i = a_i - 1$ and $a^{i-}_j = a_j$ if $j \neq i, j \in N$. Each of the following two algorithms consists of a starting and a repeated step stage.

Algorithm A

Start For each state i, we set $a_i[0] = \left\lceil \frac{p_i}{\bar{a}} \right\rceil$, that is, the respective share rounded up.
Step While $\sum_{i \in N} a_i[k] > H$, let $a[k+1] = a^{i-}[k]$, where i minimizes $\frac{p_i}{a^{i-}_i[k]}$.

Algorithm D

Start For each state i, we set $d_i[0] = \left\lfloor \frac{p_i}{\bar{a}} \right\rfloor$, that is, the respective share rounded down.
Step While $\sum_{i \in N} d_i[k] < H$, let $d[k+1] = d^{i+}[k]$, where i maximizes $\frac{p_i}{d^{i+}_i[k]}$.

Lemma 10 *Both algorithms terminate in finite steps and*

- *Algorithm A produces the Adams apportionment and*
- *Algorithm D produces the D'Hondt-Jefferson apportionment.*

Proof Since $\lfloor x \rfloor \leq x \leq \lceil x \rceil$, we have that $\sum_{i \in N} a_i[0] \leq H$ and $\sum_{i \in N} d_i[0] \geq H$. Unless we have a very special case where the starting stage produces a pre-allotment that is also an apportionment, the step stage is run at least once. Observe that both sides are integers and that the Step changes (increases and decreases for the A and D algorithms, respectively) the left-hand side by at most 1. This proves that the algorithms terminate in finite steps.

Now focus on Algorithm A and show that it produces the Leximinimax and, therefore, by our earlier result, the Adams apportionment.

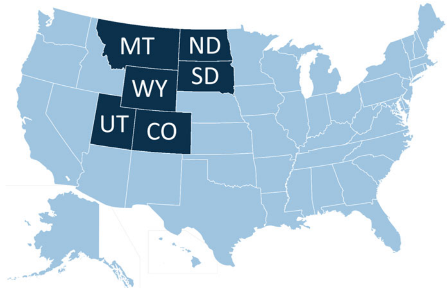
Our proof is by contradiction. Assume that the two allocations are different. Our algorithm produces a, such that $a \neq a'$, which is the Leximinimax allocation. Let the states be reindexed in such way that $\frac{p_1}{a'_1} > \frac{p_2}{a'_2} > \cdots > \frac{p_n}{a'_n}$ and assume that $j = \min_{h \in N} \left\{ h | a_h \neq a'_h \right\}$ is the first state where a and a' differ. We can rule out $a'_j < a_j$ by the definition of the Leximinimax allocation. Therefore $a'_j > a_j$. Note that the Adams method satisfies the upper quota and, therefore, $a'_j \leq a_j[0]$. This, however, implies that there was some t such that $a_j[t] = a'_j$ and $a_j[t+1] < a'_j$.

Now let $k = \min_{h \in N} \left\{ h | a_h < a'_h \right\}$ be the first state that gets less under the Leximinimax allocation a'. Since $k > j$, by the chosen indexing $\frac{p_j}{a'_j} > \frac{p_k}{a'_k}$. Notice, however, that Algorithm A chose to reduce $a_j[t] = a'_j$ rather than $a_k[t] > a'_k$ and so $\frac{p_j}{a'_j - 1} = \frac{p_j}{a_j[t]-1} < \frac{p_k}{a_k[t]-1} < \frac{p_k}{a'_k}$. $\frac{p_j}{a'_j} < \frac{p_j}{a'_j - 1}$ and therefore contradiction. The two allocations cannot be different.

The proof for Algorithm D is analogous.

We have, therefore, shown that at least *some* optimisation methods naturally produce competitive equilibria.

Remarkably, Leximin apportionment can be obtained by a combination of Algorithms A and D. As described in Biró et al. (2015), the procedure works as follows. First, the states' shares are rounded so that the departure from the average is minimised in each state. If the resulting allocation assigns more seats than the size of the House, Algorithm A is applied; if fewer, Algorithm D is used to reach the correct total. Thus, the key difference between Leximin and the Adams or Jefferson/D'Hondt methods lies in the starting allocation from which Algorithms A or D are initiated.



State		population	s_i
Colorado	(CO)	5 782	8
Utah	(UT)	3 275	4
Montana	(MT)	1 085	2
South Dakota	(SD)	888	1
North Dakota	(ND)	780	1
Wyoming	(WY)	578	1
Total		12 388	17

Fig. 1 The VIIIth Standard Federal Region of the United States with the populations (in 000's) according to the 2020 census, and the congressional seats (s_i) allocated in the subsequent apportionment.

7 Discussion

In this section, we highlight several remarkable connections and discuss the consequences of our results. We begin by examining the relation between the apportionment problem and the bankruptcy problem. Next, we present an example that demonstrates the differences between the Leximin, Leximinimax, and Leximaximin methods. Finally, we explore how fixed-base allocations can be reconciled with proportionality in real-world applications.

7.1 Parallels with the bankruptcy problem

A surprising analogy can be drawn between the apportionment problem and the bankruptcy problem. The latter concerns the division of a scarce resource among agents whose total claims exceed the available assets. The key difference between apportionment and bankruptcy is that, in the latter, the resource is divisible (though still homogeneous). Moreover, in a bankruptcy situation no agent receives more than its claim, while in apportionment both over- and underrepresentation are possible. Interestingly, in the bankruptcy setting, there also exists a rule that favours small agents, the Constrained Equal Awards (CEA) rule, and a rule that favours large agents, the Constrained Equal Losses (CEL) rule. Moreover, the Talmud rule, which lexicographically minimises the dissatisfaction of each agent group (or coalition), can be obtained as a combination of the CEA and CEL rules (Aumann & Maschler, 1985). In our analogy, the Adams method corresponds to CEA, the Jefferson/D'Hondt method to CEL and the Talmud rule to the Leximin method. Even though their computation is similar, as CEA and CEL can be formulated as lexicographic optimisation methods. There exists a fascinating hydraulic representation—with vessels representing claims and water representing the estate—that illustrates how the different rules are computed and relate to one another (Fleiner & Sziklai, 2012).

7.2 Example

As an illustration of these algorithms, we look at the problem of apportioning 17 seats among the members of the VIIIth Standard Federal Region: Colorado, Utah, Montana, South Dakota, North Dakota, and Wyoming. This region will suitably illustrate the working of the different apportionment algorithms. Figure 1 summarises the population figures and the

state	q	d_l	d_u	pre-allotment	LM
CO	7.9	13	1	8	8
UT	4.5	12	10	5	4
MT	1.5	49	26	2	2
SD	1.2	22	39	1	1
ND	1.1	7	47	1	1
WY	0.8	∞	21	1	1
Total	17			18	17

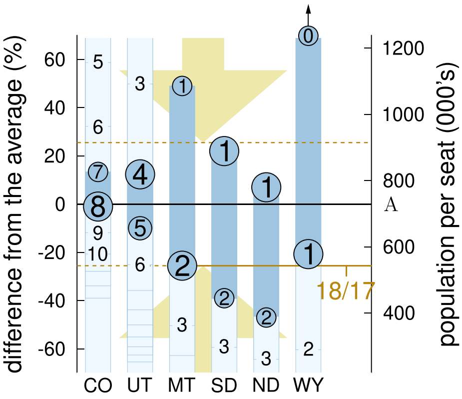


Fig. 2 In the Leximin Algorithm, the pre-allotment takes the lower or upper quota for which the difference (d_l or d_u) is the smallest. The initial pre-allotment (8, 5, 2, 1, 1, 1) allocates one too many seats, so it must be taken back. We remove it while keeping the sorted difference vector lexicographically minimal.
In the visualisation, the vertical axis shows the relative difference from the average, which naturally translates into absolute population/seat ratios as shown on the right. The relevant range for each state is highlighted. The final allotment is in larger figures.

current congressional apportionment for these states. Note that the average constituency size is $\bar{a} = 728\,707$.

We begin by finding the Leximin apportionment: in the first stage, we determine the respective share $\frac{p_i}{\bar{a}}$ for each state (exact quota), and we determine the lower and upper differences

$$\frac{\frac{p_i}{\left\lfloor \frac{p_i}{\bar{a}} \right\rfloor} - \bar{a}}{\bar{a}} = \frac{p_i}{\bar{a}\left\lfloor \frac{p_i}{\bar{a}} \right\rfloor} - 1 \text{ and } \frac{\bar{a} - \frac{p_i}{\left\lceil \frac{p_i}{\bar{a}} \right\rceil}}{\bar{a}} = 1 - \frac{p_i}{\bar{a}\left\lceil \frac{p_i}{\bar{a}} \right\rceil},$$

and choose the neighbour of the exact quota that gives the lower difference. The result is the pre-allotment (8,5,2,1,1,1) where the allocated number of seats does not add up to the required total. Here we must take one of the allocated seats back, choosing the one that results in the lowest difference (Figure 2). While in general, one would have to look at all possibilities, removing the last seat of one of the smaller states would result in an unbounded error, and for the three larger states, rounding the exact quota up was optimal, so our table also shows the differences from rounding down. Removing one seat from Utah gives the smallest difference. The algorithm terminates here, giving the final allocation (8,4,2,1,1,1).

For the Leximinimax allocation, we apply Algorithm A. First, we allocate each state the upper quota $a_i[0] = q_u\,\lceil p_i/\bar{a} \rceil$, that is, its exact quota rounded up. We get a pre-allotment that, in this case, has three more seats allocated than we have. Using the greedy algorithm, we take away three seats, keeping the p_i/s_i fractions minimal, that is, without increasing the district sizes too much. This is achieved by lowering the allotment to North Dakota, Utah, and Colorado by one seat each. The result is an apportionment (Fig. 3, Table 1)

For the Leximaximin case, we use Algorithm D. First, we allocate each state the lower quota $a_i[0] = q_l = \lfloor p_i/\bar{a} \rfloor$, that is, its exact quota rounded down. We get a pre-allotment that, in this case, has three fewer seats allocated than what we have. Using the greedy algorithm, we add three seats, keeping the p_i/s_i fractions as high as possible, that is, without decreasing the district sizes too much. This is achieved by increasing the allotment to Colorado by two

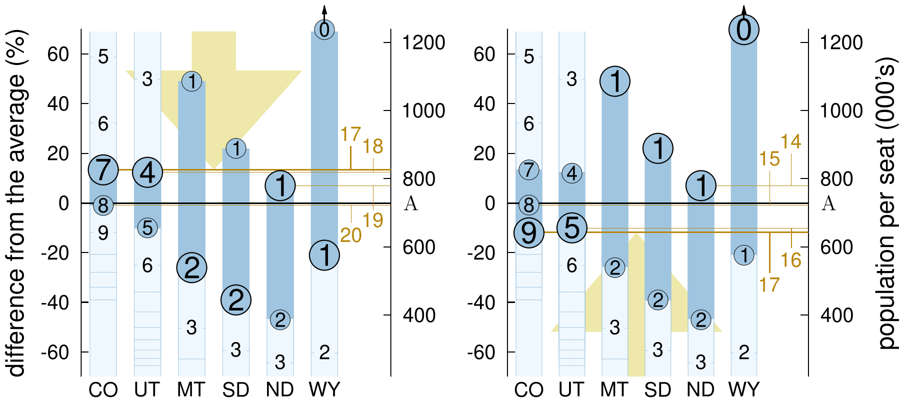


Fig. 3 A visualisation of Algorithms A and D for the example. The vertical axis shows the relative difference from the average if a certain number of seats are allocated to each of the six states. This naturally translates into population/seat ratios, as shown on the right-hand side. The relevant range for each state is highlighted. For A, the initial pre-allotment (8, 5, 2, 2, 2, 1) allocates too many seats, so some must be taken back. We remove these while keeping the maximal population/seat ratio minimal. For D, the initial pre-allotment (7, 4, 1, 1, 1, 0) allocates too few seats, so more must be distributed. We assign these while keeping the minimal population/seat ratio maximal. The final allotments are in larger figures

Table 1 Algorithm A: Finding the initial allotment and the optimal greedy steps. For the latter, the average claims/seats ratios are calculated in columns four to six for the initial allotment and for possible reduction of seats for each state. The number of seats should not go below 1 for a bounded ratio.

state	exact quota	upper quota	claim/seat (000's) 0	-1	-2	Adams allotment
CO	7.93	8	723	826	964	7
UT	4.49	5	655	819	1092	4
MT	1.49	2	543	1085		2
SD	1.22	2	444	888		2
ND	1.07	2	390	780		1
WY	0.79	1	578			1
Total		20				17

and to Utah by one seat. Remarkably, Wyoming does not need any seats allocated. The result is an apportionment (Table 2, Fig. 3).

7.3 Fixed base allocations and proportionality

Apportionment practices with elements of disproportionality are not uncommon. Proportionality makes larger states dominant, while on certain matters, size is irrelevant. The United States Senate has two senators from each state, completely ignoring size differences; in a sense, all seats are given away for free. In Spain, a mixed system is used (Márquez & Ramírez, 1998). The Congress of Deputies is composed of 350 members. Each of the fifty provinces gets two free seats, the cities of Ceuta and Melilla get one each, while the remaining 248 seats are allocated proportionally to the populations.[3] Disregarding the two North African

[3] http://www.electionresources.org/es/index_en.html

Table 2 Algorithm D: Finding the initial allotment and the optimal greedy steps. For the latter, the average claims/seats ratios are calculated in columns four to six for the initial allotment and for the possible addition of seats for each state

state	exact quota	lower quota	claim/seat (000's) 0	+1	+2	D'Hondt allotment
CO	7.93	7	826	723	642	9
UT	4.49	4	819	655	546	5
MT	1.49	1	1085	543	362	1
SD	1.22	1	888	444	296	1
ND	1.07	1	780	390	260	1
WY	0.79	0		578	289	0
Total		14				17

cities that receive only one free seat, it would be very natural to obtain this apportionment method by giving a complete discount for the first two seats. This would correspond to the case when x is set to -1.

Those allocating the seats of the European Parliament face a similar problem. Here, the will of the majority should prevail... but at the same time, we do not want MEPs from a few large countries dominating decision-making and sidelining smaller member states. The idea of *degressive proportionality* was put forward by Lamassoure and Severin (2007): "the ratio of between the population and the number of seats for each Member State must vary in relation to their respective populations in such a way that each Member from a more populous Member State represents more citizens than each Member from a less populous Member State and conversely, but also that no less populous Member State has more seats than a more populous Member State". The Cambridge Compromise (Grimmett, 2012) was put forward as a simple, practical implementation of this idea. Under the Cambridge Compromise, a base+prop method was proposed where first a fixed number of seats is allocated to each state, then the rest is allocated proportionally to their populations, subject to rounding and capping at the maximum[4]. The proposal also specifies that the base should be 5, and for the remaining seats, they find a market-clearing price taking the cap into account, with the additional twist that rounding is *upwards*. Grimmett (2012) argues that the method is equivalent to the Jefferson-D'Hondt method with a cap.

Unfortunately, the Cambridge Compromise has never been accepted by the European Parliament, and the discussion about the fair method goes on (Cegiełka et al., 2019). Implementing the constraints on the apportionment – each country gets at least 6 but at most 96 seats – is easy. There remains much freedom in what happens between these, and naturally, a multiplicity of options may create factions. The possibilities extend beyond choosing one or the other apportionment method. For example, the divisor function need not be linear; $f(s) = 2^s$ will guarantee degressive proportionality. Also, the apportionment methods are based on the players' budgets. One may choose to modify these budgets to achieve the desired distortions. In the case of European representation, taking the square root of populations could be such a modification.

In the context of resource allocation, apportionment can arise, for instance, when balancing the workload between workers across divisions. When we focus on equalising the amount of tasks assigned to employees, we are, in a way, implementing the One Man, One Vote principle. However, it is also natural to consider fairness from the perspective of the divisions. Resource- and population-monotonicity-related questions may arise during periodic revisions, which can steer decision-makers toward solutions that consider fairness at the division level. The

[4] Currently, no member state can have more than 96 seats in the EP.

apportionment literature has explored at great length the trade-offs between individual- and state-level solutions (Kóczy et al., 2017). Divisor methods are paradox-free, while remainder methods adhere to the Hare quota; however, both fail to fully enforce the One Man, One Vote principle.

8 Conclusion

Solutions that emerge as competitive equilibria are often deemed fair, not only due to their favourable properties but also because of their inherent impartiality and stability. The characterisation of apportionment methods that lead to competitive equilibria advances our comprehension of the relationship between voters' influence and seat allocations, while also offering insights into the problem of selecting solutions within a purely resource-allocation context.

Our research presents several intriguing questions for further exploration. The Adams method, known to be favourable to small states (or players with lesser endowments), compensates by giving a full discount for purchasing the first seat (or item). This universal credit is similar to apportionments where each state has a fixed base allocation, and the rest is divided in proportion to its size. This is a natural idea which aims to balance the fairness of an allocation between states and voters. The x part of the claim function $f(s) = s + x$ stands for the universal credit or debt the players face. Divisor methods with negative parameters will give more free seats to the states. We are not aware of other papers that consider divisor methods with negative parameters, although such methods can be directly linked to real-life applications, such as the apportionment of the Congress of Deputies in Spain.

Another straightforward question is how leximin can be characterised with respect to majorization. Simulation results place leximin between the Adams and the Jefferson methods. The fact that leximin can be obtained by the combination of the leximinimax and leximaximin methods also points toward the possibility that this might be true.

Finally, optimisation methods have gathered less attention in the literature. The apportionment problem cares about both under- and overrepresentation, but in some resource allocation tasks, only one side is relevant. Formulating apportionment methods as optimization procedures helps us understand how the lessons learned in apportionment can be extended to resource allocation in general.

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References

Aumann, R., & Maschler, M. (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory, 36*, 195–213.

Balinski, M. L. (1993). The problem with apportionment. *Journal of the Operations Research Society of Japan, 36*(3), 134–148.

Balinski, M. L., & Demange, G. (1989). Algorithms for proportional matrices in reals and integers. *Mathematical Programming, 45*(1–3), 193–210.

Balinski, M. L., & Ramirez, V. (2012). Parametric vs. divisor methods of apportionment. *Annals of Operations Research*.

Balinski, M. L., & Young, H. P. (1978). The Jefferson Method of Apportionment. *SIAM Review, 20*(2), 278–284.

Balinski, M. L., & Young, H. P. (1980). The Webster method of apportionment. *Proceedings of the National Academy of Sciences of the United States of America, 77*(1), 1–4.

Balinski, M. L., & Young, H. P. (1982). *Fair Representation: Meeting the Ideal of One Man, One Vote*. New Haven: Yale University Press.

Balinski, M. L., & Young, H. P. (1994). Apportionment. In S. M. Pollock, M. H. Rothkopf, and A. Barnett (Eds.), *Handbooks in Operations Research and Management Science, Volume 6*, Chapter 15, pp. 529–560. Amsterdam: North-Holland.

Biró, P., Kóczy, L. Á., & Sziklai, B. R. (2015). Fair apportionment in the view of the Venice Commission's recommendation. *Mathematical Social Sciences, 77*, 32–41.

Brill, M., Gölz, P., Peters, D., Schmidt-Kraepelin, U., & Wilker, K. (2024). Approval-based apportionment. *Mathematical Programming, 203*(1–2), 77–105.

Budish, E. (2011). The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy, 119*, 1061–1103.

Burt, O. R., & Harris, C. C. (1963). Apportionment of the US House of Representatives: A minimum range, integer solution, allocation problem. *Operations Research, 11*(4), 648–652.

Cegiełka, K., Łyko, J., & Rudek, R. (2019). Beyond the Cambridge Compromise algorithm towards degressively proportional allocations. *Operational Research, 19*(2), 317–332.

Cembrano, J., Correa, J., & Verdugo, V. (2021). Multidimensional Apportionment through Discrepancy Theory. In *Proceedings of the 22nd ACM Conference on Economics and Computation*, New York, NY, USA, pp. 287–288. ACM.

Cembrano, J., Correa, J., & Verdugo, V. (2022). Multidimensional political apportionment. *Proceedings of the National Academy of Sciences 119*(15).

Chase, J. S., Anderson, D. C., Thakar, P. N., Vahdat, A. M., & Doyle, R. P. (2001). Managing energy and server resources in hosting centers. *ACM SIGOPS Operating Systems Review, 35*, 103–116.

Debreu, G. (1982). Existence of competitive equilibrium. In *Handbook of Mathematical Economics*, Volume 2, Chapter 15, pp. 697–743.

Eckman, S. J. (2021, November 22). Apportionment and redistricting process for the U.S. House of Representatives. Technical Report R45951, Congressional Research Service.

Edelman, P. H. (2006). Minimum total deviation apportionments. In B. Simeone & F. Pukelsheim (Eds.), *Mathematics and Democracy, Berlin, Heidelberg* (pp. 55–64). Berlin Heidelberg: Springer.

Fleiner, T., & Sziklai, B. (2012). The Nucleolus Of The Bankruptcy Problem By Hydraulic Rationing. *International Game Theory Review 14*(01), 1250007–1–1.

Golosov, G. V. (2014). 11). Authoritarian electoral engineering and its limits: A curious case of the imperiali highest averages method in russia. *Europe-Asia Studies, 66*, 1611–1628.

Gölz, P., D. Peters, & Procaccia, A. D. (2025). In this apportionment lottery, the house always wins. *Operations Research*. ahead of print.

Grimmett, G. R. (2012). European apportionment via the Cambridge Compromise. *Mathematical Social Sciences, 63*(2), 68–73.

Huntington, E. V. (1928). The apportionment of representatives in congress. *Transactions of the American Mathematical Society, 30*(1), 85–110.

Ibaraki, T., & Katoh, N. (1988, 3). *Resource Allocation Problems. Algorithmic Approaches*. The MIT Press.

Johannesson, M., & Weinstein, M. C. (1993). 12). On the decision rules of cost-effectiveness analysis. *Journal of Health Economics, 12*, 459–467.

Jones, M. A., D. McCune, & Wilson J. M. (2023). *Delegate Apportionment in the US Presidential Primaries*. Studies in Choice and Welfare. Cham: Springer International Publishing.

Kóczy, L. Á., Biró, P., & Sziklai, B. (2017). US vs. European apportionment practices: The conflict between monotonicity and proportionality. In U. Endriss (Ed.), *Trends in Computational Social Choice*, Chapter 16. AI Access.

Kóczy, L. Á., Koltai, T., & Sziklai, B. R. (2026). Apportionment methods in resource allocation. *Omega, 138*, Article 103403.
Kopfermann, K. (1991). *Mathematische Aspekte der Wahlverfahren*. Mandatsverteilung bei Abstimmungen: Mandatsverteilungen bei Abstimmungen. Wissenschaftsverlag.
Lamassoure, A. & Severin, A. (2007). European Parliament resolution on "Proposal to amend the Treaty provisions concerning the composition of the European Parliament, Adopted on 11 October 2007 (INI/2007/2169).
Lin, C. M., & Gen, M. (2008). 5). Multi-criteria human resource allocation for solving multistage combinatorial optimization problems using multiobjective hybrid genetic algorithm. *Expert Systems with Applications, 34*, 2480–2490.
Luce, R. (1930). *Legislative Principles*. Boston-New York: Houghton Mifflin Company.
Luss, H. (2012). *Equitable resource allocation: models, algorithms, and applications*. Hoboken, New Jersey: Wiley.
Márquez, M. L., & Ramírez, V. (1998). The spanish electoral system:proportionality and governability. *Annals of Operations Research, 84*, 45–59.
Marshall, A. W., Olkin, I., & Pukelsheim, F. (2002). A majorization comparison of apportionment methods in proportional representation. *Social Choice and Welfare, 19*(4), 885–900.
Mojica-Nava, E., Macana, C. A., & Quijano, N. (2014). Dynamic population games for optimal dispatch on hierarchical microgrid control. *IEEE Transactions on Systems, Man, and Cybernetics: Systems, 44*, 306–317.
Niemeyer, H. F., & Niemeyer, A. C. (2008). Apportionment methods. *Mathematical Social Sciences, 56*(2), 240–253.
Pretolani, D. (2014). Apportionments with minimum gini index of disproportionality: a quadratic knapsack approach. *Annals of Operations Research, 215*(1), 257–267.
Pukelsheim, F. (2014). *Proportional Representation*. Heidelberg: Springer.
Segal-Halevi, E. (2020). Competitive equilibrium for almost all incomes: existence and fairness. *Autonomous Agents and Multi-Agent Systems* (Vol. 34, pp. 1–50). US: Springer.
Serafini, P. (2020). *Legislative Territorial Representation*, pp. 69–82. Springer International Publishing.
Shechter, S. M. (2024). Congressional apportionment: A multiobjective optimization approach. *Management Science, 71*(2), 1464–1487.
Supreme Court of the United States (1962). Baker v. Carr. U.S. Supreme Court decision. 369 U.S. 186.
Supreme Court of the United States (1964). Reynolds v. Sims. U.S. Supreme Court decision. 377 U.S. 533.
Supreme Court of the United States (1983). Karcher v. daggett. U.S. Supreme Court decision. 462 U.S. 725.
Sziklai, B. R., & Héberger, K. (2020). Apportionment and districting by sum of ranking differences. *PLOS ONE 15*(3), 1–20.
Commission, V. (2002). Code of Good Practice in Electoral Matters. *CDL-AD, 23*(190), 1–33.