



# A Dividend-Based Approach to the Solidarity Value: Structural Decomposition and Axiomatic Characterizations

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## Abstract

This paper introduces a novel recursive approach to define Harsanyi-type dividends for the class of solidarity values in cooperative games with transferable utilities. This approach facilitates the reduction of computational complexities, simplifies the characterizations of the solidarity value, and strengthens the value's theoretical foundation while enhancing conceptual clarity and computational tractability via dividend-based decomposition. We also introduce a set of axioms that offer alternative characterizations of the solidarity value.

**Keywords** Shapley value · solidarity value · Harsanyi dividend

## 1 Introduction

Cooperative games with transferable utilities, or TU games, in short, have widespread applications, particularly in economics, political science, and social sciences. They focus on collaborations, where players form coalitions to achieve shared objectives, and the outcomes depend on the level of cooperation and mutual support among participants. Such games model real-world scenarios, such as trade negotiations or team efforts, emphasizing the importance of pooling resources and coordinating strategies. They also address the equitable distribution of collective gains through solutions like the Shapley value [16] or the core [8], ensuring fairness and stability in coalitions. By analyzing collective decision-making, TU games provide insights into optimizing mutual benefits across various fields.

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In TU game theory, fair allocation among players is essential to maintain coalition stability and incentivize cooperative behavior. Traditional solutions, such as the Shapley value [16], focus on marginal contributions, rewarding players in proportion to their contributions to coalition values. However, in scenarios where solidarity and collective welfare are prioritized, marginalistic approaches may not fully capture the cooperative essence of coalition formation. To address this, Sprumont [17] introduced the solidarity value recursively to ensure that even players who contribute minimally to coalitions receive a portion of the surplus. Later, Nowak and Radzik [15] provided an alternative definition of the solidarity value by considering the average marginal contributions of coalition members.

The characterization of the solidarity value replaces the null player axiom of the Shapley value with the average null player axiom, which states that a player receives a payoff of zero if their average marginal contribution across all coalitions they belong to is zero. This makes the solidarity value distinct from the Shapley value, as it permits null players to receive non-zero payoffs in specific scenarios, emphasizing cooperative fairness.

In later years, studies have extended the applications of the solidarity value, examining its potential in diverse cooperative settings and exploring approaches to strengthen its theoretical foundation. Xu et al. [19] provided a potential form of the solidarity value and characterized it using the quasi-balanced contribution property. Gutierrez et al. [9] provided an axiomatic characterization of the egalitarian solidarity value that emphasizes the equal distribution between players while considering their contributions. Casajus and Huettner [5] introduced a one-parameter family of solidarity values, which generalizes existing solution concepts by determining the type of player whose removal does not affect the payoffs of the remaining players, with the solidarity value positioned centrally in this family. Recently, Zou et al. [20] explored the integration of Shapley and solidarity value, a two-step method that highlights the importance of balancing solidarity with efficiency in TU games.

Harsanyi [10] introduced the concept of dividends to calculate the Shapley value by fairly distributing coalition gains, forming the foundation of its axiomatic characterization. It provides a systematic way to decompose the value function into marginal contributions that account for the interplay among all subsets of players and offers alternative formulations based on the decomposition of a TU game into simpler components. This approach simplifies the process of attributing the value of the coalition fairly among players by focusing on their average marginal contributions.

In this paper, we apply a dividend approach similar to that used by Harsanyi [10] in characterizing the Shapley value to study the solidarity value due to Nowak and Radzik [15]. The primary aim of this paper is to represent the solidarity value in terms of Harsanyi-type dividends and also characterize it using those dividends. We have not seen any study on this approach for the solidarity value to date. The first characterization is done using the fairness property along with the efficiency and the average null player property. The second characterization is done using the differential marginality, which comes as a result of the first characterization and Proposition 59 of [13]. Later, we introduce two new axioms, namely average  $\alpha$ -player property and shifted average marginality, and we provide two more characterizations based on these axioms.

The structure of the paper is as follows: Sect. 2 covers the necessary preliminaries. Section 3 introduces the dividend form of the solidarity value for TU games. Section 4 presents the characterization of the solidarity value. Finally, Sect. 5 concludes the paper.

## 2 Preliminaries

This section provides the fundamental definitions and concepts necessary for the study of TU games and their solutions. A TU game is represented by a pair  $(N, v)$ , where  $N$  is a finite set of players and  $v : 2^N \mapsto \mathbb{R}$  is a characteristic function that assigns a value  $v(S)$  to each subset  $S \subseteq N$ . Also,  $(N \setminus \{i\}, v)$  represents the game where  $v$  is restricted to the player set  $N \setminus \{i\}$ . The value  $v(S)$ , referred to as the worth of the coalition  $S$ , represents the total utility achievable by coalition  $S$  through cooperation, and it is conventionally assumed that  $v(\emptyset) = 0$ . The set of all TU games with the player set  $N$  is denoted by  $G^N$ , which is a linear space. Denote by  $G$  the set of all TU games with a variable player set. For each  $\emptyset \neq S \subseteq N$ , define the class of unanimity games  $u_S : 2^N \mapsto \mathbb{R}$  as

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise.} \end{cases}$$

The class  $\{u_S \mid \emptyset \neq S \subseteq N\}$  is a basis for  $G^N$ . A value function is defined as a mapping  $\Phi$  that associates each game  $(N, v)$  with a payoff vector  $\Phi(N, v) = (\Phi_i(N, v))_{i \in N} \in \mathbb{R}^{|N|}$ , called a value where  $\Phi_i(N, v)$  specifies the payoff assigned to the player  $i$ . The marginal contribution of a player  $i \in S$  to a coalition  $S \subseteq N$  is defined as  $\Delta_i(S, v) = v(S) - v(S \setminus \{i\})$ . Two players  $i, j \in N$  are said to be symmetric in  $(N, v)$  if  $v(S \cup \{i\}) = v(S \cup \{j\}) \forall S \subseteq N \setminus \{i, j\}$ . A player  $i \in N$  is said to be a null player if  $v(S \cup \{i\}) = v(S)$  for all  $S \subseteq N \setminus \{i\}$ .

The Shapley value [16] is a widely studied value function in TU game theory that allocates payoffs to players based on their contributions across all coalitions. It is given by the formula.

$$\Phi_i^{Sh}(N, v) = \sum_{S \subseteq N: i \in S} \frac{(|N| - |S|)!(|S| - 1)!}{|N|!} \Delta_i(S, v), \tag{1}$$

where  $|N|$  denotes the total number of players and  $|S|$  represents the size of coalition  $S$ .

An alternative formula of the Shapley value was given by Harsanyi [10] in terms of the Harsanyi dividends, where the Harsanyi dividends are defined inductively: the dividend of the empty set is zero, and the dividend of any other possible coalition of the player set is the remaining worth of the coalition after excluding the dividends of all sub-coalitions. Formally, we have the following. Let  $(N, v)$  be a TU game. For all  $S \subseteq N$ , the Harsanyi dividends  $\lambda_v(S)$ , are defined recursively by

$$\lambda_v(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ v(S) - \sum_{T \subsetneq S} \lambda_v(T) & \text{otherwise.} \end{cases} \tag{2}$$

Every TU game  $(N, v)$  can be expressed as a unique linear combination of the unanimity games, with coefficients (unanimity coefficients) being the Harsanyi dividends given by

$$v(S) = \sum_{T \subseteq N} u_S(T) \lambda_v(T). \quad (3)$$

Let us call those dividends the “Harsanyi dividends of the unanimity games” so that later we define similar dividends on an alternative basis for  $G^N$  and distinguish them based on such basis. The Shapley value  $\Phi^{Sh}$  is obtained using Harsanyi dividends of the unanimity games as

$$\Phi_i^{Sh}(N, v) = \sum_{S \subseteq N: i \in S} \frac{\lambda_v(S)}{|S|}. \quad (4)$$

It follows that the Harsanyi dividend of the unanimity games provides a clear decomposition of the value function into contributions from each coalition, making it easier to interpret the Shapley value as a fair distribution of these contributions. The Shapley value is characterized using different axioms, including the following:

**Axiom 1 Efficiency:** The total value allocated to all players is equal to the value of the grand coalition, that is,  $\sum_{i \in N} \Phi_i(N, v) = v(N)$ .

**Axiom 2 Symmetry:** If two players  $i, j \in N$  are symmetric in  $(N, v)$ , then  $\Phi_i(N, v) = \Phi_j(N, v)$ .

**Axiom 3 Additivity:** For any two games  $(N, v)$  and  $(N, w)$ ,  $\Phi$  satisfies  $\Phi(N, v+w) = \Phi(N, v) + \Phi(N, w)$ .

**Axiom 4 Null player property:** If a player  $i \in N$  is a Null Player, then  $\Phi_i(N, v) = 0$ .

**Axiom 5 Fairness:** If two players  $i, j \in N$  are symmetric in  $(N, v)$ , it holds  $\Phi_i(N, u+v) - \Phi_i(N, u) = \Phi_j(N, u+v) - \Phi_j(N, u)$ .

**Axiom 6 Differential marginality:** If  $\Delta_i(K, u) - \Delta_j(K, u) = \Delta_i(K, v) - \Delta_j(K, v)$ , for all  $K \subseteq N \setminus \{i, j\}$ , then  $\Phi_i(N, u) - \Phi_j(N, u) = \Phi_i(N, v) - \Phi_j(N, v)$ .

Shapley [16] showed that efficiency, symmetry, additivity, and null player property uniquely characterize the Shapley value. Alternative characterizations, among others, were proposed by van den Brink [18] using the fairness property in place of additivity and symmetry. Huettner [13] showed that fairness and differential marginality are equivalent, and thus, the Shapley value is characterized by differential marginality along with efficiency and null player property. Later, Casajus [6] extended these characterizations to the classes of convex games and super-additive games.

Recall from the introduction that the solidarity value, introduced by Sprumont [17] and later studied in detail by Nowak and Radzik [15], offers an alternative solution concept that emphasizes cooperative principles over marginal contributions. It modifies the Shapley value by considering the average marginal contributions of coalitions instead of individual contributions. For a coalition  $S \subseteq N$ , the average marginal contribution is defined as

$$\Delta^{av}(S, v) = \frac{1}{|S|} \sum_{i \in S} \Delta_i(S, v). \quad (5)$$

The solidarity value for player  $i \in N$  is then defined as

$$\Phi_i^{Sol}(N, v) = \sum_{S \subseteq N: i \in S} \frac{(|N| - |S|)! (|S| - 1)!}{|N|!} \Delta^{av}(S, v). \tag{6}$$

Denote by  $b_T : G^N \mapsto \mathbb{R}$ , the game defined by

$$b_T(S) = \begin{cases} \binom{|S|}{|T|}^{-1} & \text{if } T \subseteq S \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The class  $\{b_T | T \in 2^N \setminus \emptyset\}$  forms a basis for the space of TU games  $G^N$ . The solidarity value being linear for every  $v \in G^N$  can be uniquely expressed as a linear combination of the solidarity values of the basis games. The solidarity value for player  $i \in T$  for the basis game  $(N, b_T)$  is given by,<sup>1</sup>

$$\Phi_i^{Sol}(N, b_T) = \frac{1}{|T|} \binom{|N|}{|T|}^{-1}. \tag{8}$$

Alternatively, the solidarity value can be defined recursively. For any  $S \subseteq N$  and  $i \in S$ , it is given by

$$\Phi_i^{Sol}(S, v) = \frac{1}{|S|} \Delta^{av}(S, v) + \frac{1}{|S|} \sum_{j \in S \setminus \{i\}} \Phi_j^{Sol}(S \setminus \{j\}, v),$$

with the initial condition  $\Phi_i^{Sol}(\{i\}, v) = v(\{i\})$ .

In their characterization of the solidarity value, Nowak and Radzik [15] replaced the null player property with the average null player property, which states that a player receives zero payoff only if her average marginal contribution across all coalitions (equivalently, her net contribution to the game) is zero. This adjustment ensures that even players with minimal contributions are taken into account in the allocation process. Formally, we have

**Definition 1 Average null player:** A player  $i \in N$  is called an average null player if, for every coalition  $S \subseteq N$  such that  $i \in S$ ,  $v(S) = \frac{1}{|S|} \sum_{k \in S} v(S \setminus \{k\})$ .

**Axiom 7 Average null player property:** If  $i \in N$  is an Average Null Player in  $(N, v)$ , then  $\Phi_i(N, v) = 0$ .

The solidarity value is often studied in conjunction with other properties, such as the quasi-balanced contribution property introduced by Xu et al. [19], which balances the contributions of any player when other players leave the game. This is a modification of the balanced contribution property due to Myerson [14] used in the characterization of the Shapley value for games with conference structures.

<sup>1</sup> See Nowak and Radzik [15].

**Axiom 8 Quasi-balanced contribution property:** For  $(N, v) \in G^N$  and players  $i, j \in N$  with  $i \neq j$ ,  $\Phi_i(N, v) - \Phi_i(N \setminus \{j\}, v) + \frac{1}{n}v(N \setminus \{j\}) = \Phi_j(N, v) - \Phi_j(N \setminus \{i\}, v) + \frac{1}{n}v(N \setminus \{i\})$ .

The following characterization theorems of the solidarity value are respectively due to Nowak and Radzik [15] and Xu et al. [19].

**Theorem 1** [15] *A value  $\Phi$  on  $G^N$  satisfies efficiency, additivity, symmetry, and average null player property if and only if  $\Phi = \Phi^{Sol}$ .*

**Theorem 2** [19] *A value  $\Phi$  on  $G^N$  satisfies efficiency and quasi-balanced contribution property if and only if  $\Phi = \Phi^{Sol}$ .*

### 3 The solidarity Value in Terms of the Basis Coefficients

#### 3.1 The Harsanyi-Type Dividends

This section introduces Harsanyi-type dividends of the basis  $\{b_T | T \subseteq N : T \neq \emptyset\}$  given by Eq. 7, which will be later used in representing the solidarity value. Recall that, any TU-game  $(N, v)$ , the value function  $v$  can be uniquely expressed as a linear combination of the elements of the basis  $\{b_T | T \subseteq N : T \neq \emptyset\}$  as  $v = \sum_{T \subseteq S} \mu_v(T)b_T$ , where the coefficients  $\mu_v(T)$  for each  $T \subseteq N$  play a similar role as that of the unanimity coefficients in Eq. 3. It follows that

$$v(S) = \sum_{T \subseteq S} \mu_v(T)b_T(S) = \sum_{T \subseteq S} \left(\frac{|S|}{|T|}\right)^{-1} \mu_v(T) = \mu_v(S) + \sum_{T \subset S} \left(\frac{|S|}{|T|}\right)^{-1} \mu_v(T) \text{ for all } S \subseteq N. \tag{9}$$

Thus, keeping our notations aligned with those in Eq. 2, we denote the coefficients  $\mu_v(S)$  for each  $S \subseteq N$  as the Harsanyi-type dividends of the basis  $\{b_T | \emptyset \neq T \subseteq N\}$  and define by

$$\mu_v(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ v(S) - \sum_{T \subset S} \left(\frac{|S|}{|T|}\right)^{-1} \mu_v(T) & \text{otherwise.} \end{cases}$$

Using this recursive formula, all the values of  $\mu_v(S)$  can be calculated for all  $S \subseteq N$ . It is obvious that  $\mu_v(S) = v(S)$  for  $|S| = 1$ .

Next, we obtain some properties of the Harsanyi-type dividends of the basis  $\{b_T | \emptyset \neq T \subseteq N\}$ .

**Proposition 1** *For any two TU-games  $(N, u)$  and  $(N, v)$ ,  $\mu_{u+v}(T) = \mu_u(T) + \mu_v(T)$ .*

**Proof** The proof is straightforward; therefore, it is omitted. □

**Proposition 2** *For any coalition  $T \subseteq N$ ,  $\Delta^{av}(T) = \mu_v(T)$ , where  $\Delta^{av}$  is given by Eq. 5.*

**Proof** By the definition of average marginal contributions, for all  $S \subseteq T$ , we have

$$\begin{aligned} \Delta^{av}(T) &= v(T) - \frac{1}{|T|} \sum_{i \in T} v(T \setminus \{i\}) \\ &= v(T) - \frac{1}{|T|} \sum_{i \in T} \sum_{S \subseteq T \setminus \{i\}} \binom{|T \setminus \{i\}|}{|S|}^{-1} \mu_v(S) \\ &= v(T) - \frac{1}{|T|} \sum_{S \subseteq T} (|T| - |S|) \binom{|T| - 1}{|S|}^{-1} \mu_v(S) \\ &= v(T) - \sum_{S \subseteq T} \binom{|T|}{|S|}^{-1} \mu_v(S) \\ &= \mu_v(T) \end{aligned}$$

End of proof. □

As a result of Proposition 2 and the definition of average null player, we have the following corollary.

**Corollary 1** For any coalition  $T$  containing an average null player,  $\mu_v(T) = 0$ .

Recall from Eq. 6 that calculating the solidarity value involves computing the average marginal contributions by summing over all possible coalitions, which becomes computationally intensive as the number of players increases. Thus, we obtain the solidarity value using the Harsanyi-type dividend of  $\{b_T \mid \emptyset \neq T \subseteq N\}$  to simplify the complex computation of each player’s share in a TU game. The formula breaks down the computing into smaller and more manageable parts. It expresses the value of any coalition in terms of its subsets, thereby reducing the complexity involved in determining each player’s average marginal contribution. Thus, we have the following proposition.

**Proposition 3** The solidarity value  $\Phi_i^{Sol}(N, v)$  can be expressed in terms of the Harsanyi-type dividends of the basis  $\{b_T \mid \emptyset \neq T \subseteq N\}$  as

$$\Phi_i^{Sol}(N, v) = \sum_{T \in 2^N: i \in T} \frac{\mu_v(T)}{|T|} \binom{|N|}{|T|}^{-1}.$$

**Proof** Using Eqs. 6 and 8 and the definition of the dividends, we have

$$\begin{aligned} \Phi_i^{Sol}(N, v) &= \sum_{S \subseteq N} \frac{(|N| - |S| - 1)! (|S|)!}{|N|!} \left[ \frac{1}{|S|} \sum_{k \in S} (v(S) - v(S \setminus \{k\})) \right] \\ &= \sum_{S \subseteq N \setminus \{i\}} \frac{(|N| - |S| - 1)! (|S|)!}{|N|!} \left[ \frac{1}{|S|} \sum_{k \in S} \left( \sum_{T \in 2^N} \mu_v(T) b_T(S) - \sum_{T \in 2^N} \mu_v(T) b_T(S \setminus \{k\}) \right) \right] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{T \in 2^N, i \in T} \mu_v(T) \left\{ \sum_{S \subseteq N \setminus \{i\}} \frac{(|N| - |S| - 1)!(S)!}{|N|!} \left[ \frac{1}{|S|} \sum_{k \in S} (b_T(S) - b_T(S \setminus \{k\})) \right] \right\} \\
 &= \sum_{T \in 2^N, i \in T} \mu_v(T) \Phi_i^{Sol}(N, b_T) \\
 &= \sum_{T \in 2^N, i \in T} \frac{\mu_v(T)}{|T|} \binom{|N|}{|T|}^{-1} \tag{10}
 \end{aligned}$$

This completes the proof. □

The Harsanyi-type dividend,  $\mu_v(T)$  of the basis  $\{b_T\}$  that is used to represent the solidarity value given by Eq. 10, has significant implications. It represents the net average marginal contribution of coalition  $T$  after accounting for the contributions of its proper sub-coalitions. Unlike the Shapley value, where Harsanyi dividends  $\lambda_v(T)$  are distributed equally among members based on the unanimity basis, the solidarity value uses  $\mu_v(T)$  along with the combinatorial term  $\binom{|N|}{|T|}^{-1}$  to reflect a more cooperative framework. Thus, a player’s payoff depends not only on her contribution but also on the collective contributions of her coalitions. The Harsanyi-type dividends for the solidarity value emphasize mutual support and shared responsibility, focusing on “what their group adds,” whereas the dividend for the Shapley value highlights a marginalistic approach by concentrating on “what they add.”

It is evident from the formula given by Eq. 10 that it can be used to calculate the solidarity value more efficiently, particularly in games with many players. Note that by “efficiency,” we mean that it is easier to interpret the value and due to its recursive nature, efficient algorithms can be designed for its computation.<sup>2</sup> In the following sub-section, we obtain an alternative expression of the A-potential function due to Xu et al. [19] in terms of the Harsanyi-type dividends of the basis  $\{b_T \mid \emptyset \neq T \subseteq N\}$ .

### 3.2 The A-Potential Function for the Solidarity Value

The potential function is introduced by Hart and Mas-Colell [12] to characterize the Shapley value. Xu et al. [19] later modified this function for the solidarity value and termed it the A-potential function with some adjustments to the prior function. The A-potential function is a function  $P^* : G^N \mapsto \mathbb{R}$  that satisfies the following:

$$\sum_{i \in N} [P^*(N, v) - P^*(N \setminus \{i\}, v)] = v(N) - \frac{1}{n} \sum_{i \in N} v(N \setminus \{i\}) \tag{11}$$

which can be equivalently written as

$$P^*(N, v) = \frac{1}{n} \left[ \sum_{i \in N} P^*(N \setminus \{i\}, v) + v(N) - \frac{1}{n} \sum_{i \in N} v(N \setminus \{i\}) \right]. \tag{12}$$

<sup>2</sup> In this regard, we follow similar arguments by Harsanyi [10, 11], Dehez [7], and Albizuri et al. [1] for the Harsanyi dividends in computing the Shapley value. Also, see [2].

Xu et al. [19] provided a relationship of the A-potential function with the solidarity value. In the following, we provide an alternative expression of  $P^*(N, v)$  using the Harsanyi-type dividends of the basis  $\{b_T \mid \emptyset \neq T \subseteq N\}$ .

**Theorem 3** *The function  $P^* : G^N \mapsto \mathbb{R}$  defined by  $P^*(N, v) = \sum_{S \subseteq N} \binom{|N|}{|S|}^{-1} \frac{\mu_v(S)}{|S|}$  represents the A-potential function for the solidarity value.*

**Proof** We proceed with the proof using the method of induction. The result is obvious for  $|N| = 1$ . Suppose it holds for each  $P^*(N \setminus \{i\}, v)$ , whenever  $i \in N$ . Then, from the recursive definition of the A-potential function, we have

$$\begin{aligned}
 P^*(N, v) &= \frac{1}{n} \left[ \sum_{i \in N} P^*(N \setminus \{i\}, v) + v(N) - \frac{1}{n} \sum_{i \in N} v(N \setminus \{i\}) \right] \\
 &= \frac{1}{n} \left[ \sum_{i \in N} \sum_{S \subseteq N \setminus \{i\}} \binom{|N \setminus \{i\}|}{|S|}^{-1} \frac{\mu_v(S)}{|S|} + v(N) - \frac{1}{n} \sum_{i \in N} v(N \setminus \{i\}) \right] \\
 &= \frac{1}{n} \left[ \sum_{i \in N} \sum_{S \subseteq N \setminus \{i\}} \binom{|N \setminus \{i\}|}{|S|}^{-1} \frac{\mu_v(S)}{|S|} \right] + \frac{1}{n} \left[ v(N) - \frac{1}{n} \sum_{i \in N} v(N \setminus \{i\}) \right] \\
 &= \sum_{S \subset N} \binom{|N|}{|S|}^{-1} \frac{\mu_v(S)}{|S|} + \frac{1}{n} \lambda_N(v) \\
 &= \sum_{S \subseteq N} \binom{|N|}{|S|}^{-1} \frac{\mu_v(S)}{|S|}.
 \end{aligned}$$

This concludes the proof. □

The following section provides four characterizations of the solidarity value in terms of the Harsanyi-type dividends of the basis  $\{b_T \mid \emptyset \neq T \subseteq N\}$ . First and second characterizations are done using the fairness and differential marginality axioms, which are in line with their counterparts for the Shapley value given by van den Brink [18] and Casajus [6]. The other two characterizations are based on our newly defined axioms, namely average  $\alpha$ -player property and shifted average marginality.

### 4 Characterizations

We begin with the following Lemma.

**Lemma 1** *If  $\Phi$  satisfies fairness and average null player property, and then  $\Phi$  also satisfies symmetry.*

**Proof** Suppose  $\Phi$  satisfies fairness and average null player property. Let  $(N, v_0)$  be the null game, then  $v_0(S) = 0$  for all  $S \subseteq N$ . By the average null player property,  $\Phi_i(N, v_0) = 0$  for all  $i \in N$ . By the fairness property, for any two players  $i, j \in$

$N$ , symmetric in  $(N, v)$  and for the null game  $(N, v_0)$ , we have  $\Phi_i(N, v + v_0) - \Phi_i(N, v_0) = \Phi_j(N, v + v_0) - \Phi_j(N, v_0)$ . Substituting  $\Phi_i(N, v_0) = \Phi_j(N, v_0) = 0$ , and  $v + v_0 = v$  in the above equation, we have  $\Phi_i(N, v) = \Phi_j(N, v)$ . Thus,  $\Phi$  satisfies symmetry.  $\square$

The first characterization theorem goes as follows.

**Theorem 4** *The solidarity value is the unique value satisfying efficiency, average null player property, and fairness.*

**Proof** It is well-known that the solidarity value satisfies efficiency and average null player property. Also, as the solidarity value satisfies additivity and symmetry; thus, the solidarity value also satisfies fairness.<sup>3</sup>

Now, we prove the converse part of the theorem by the method of induction. Let  $\Phi : G^N \mapsto \mathbb{R}^{|N|}$  satisfy efficiency, average null player property, and fairness. For the Harsanyi dividends  $\mu_v(S)$ ,  $S \subseteq N$ , define  $D(v) = \{S \subseteq N : \mu_v(S) \neq 0\}$  and  $d(v) = |D(v)|$ .

1. If  $d(v) = 0$ , then  $(N, v)$  is the null game, so  $v(S) = 0$  for all  $S \subseteq N$ . By the average null player property  $\Phi_i(N, v) = 0$  for all  $i \in N$ . Therefore,  $\Phi(N, v) = 0$ , which is consistent with the solidarity value for the null game.
2. If  $d(v) = 1$ , then there exists a coalition  $T \subseteq N$  such that  $v = \mu_v(T)b_T$ . Therefore, we can write  $v(N) = b_T(N)\mu_v(T)$ . By the average null player property  $\Phi_i(N, v) = 0$  for all  $i \in N \setminus T$ . By efficiency,  $\sum_{i \in T} \Phi_i(N, v) = v(N) = b_T(N)\mu_v(T) = \binom{|N|}{|T|}^{-1} \mu_v(T)$ . Since all players in  $T$  are symmetric in  $(N, b_T)$ , thus for all  $i \in T$ , by Lemma 1,  $\Phi_i(N, v) = \frac{\mu_v(T)}{|T|} \binom{|N|}{|T|}^{-1} = \Phi_i^{Sol}(N, v)$ .
3. Assume  $\Phi(N, v) = \Phi^{Sol}(N, v)$  for all  $(N, v) \in G^N$  with  $d(v) \leq k$ . Now, we show that it also holds for all games with  $d(v) = k + 1$ . Let  $v \in G^N$  with  $d(v) = k + 1$ . Then, we can write  $v = \sum_{S \in D(v)} \mu_v(S)b_S$ . Pick a coalition  $T \in D(v)$  and define  $v'$  as  $v' = v - \mu_v(T)b_T$ . Here,  $d(v') = k$ . Let  $\Phi_i(N, v') = \delta_i$ . Then, by the induction hypothesis, each  $\delta_i$  can be determined uniquely and is equal to  $\Phi_i^{Sol}(N, v')$ . Let us consider the effect of adding the marginal game  $(N, \mu_v(T)b_T)$  on the players' payoffs, denoted by  $\Delta\Phi_i = \Phi_i(N, v) - \Phi_i(N, v')$ . We then evaluate this  $\Delta\Phi_i$  depending on the players' status in the added game:
  - The players  $k \in N \setminus T$  are average null players in the game  $(N, \mu_v(T)b_T)$ . Because their average marginal contribution to this specific component is zero across all coalitions, the addition of this game does not affect their payoff. Therefore,  $\Delta\Phi_i = 0$ , which means

$$\Phi_i(N, v) = \Phi_i(N, v') = \delta_i.$$

- The players  $i, j \in T$  are symmetric in  $(N, \mu_v(T)b_T)$ . By the fairness property, their change in payoff must be identical. Thus,  $\Delta\Phi_i = \Delta\Phi_j = c$  for some

<sup>3</sup> See Proposition 2.4 of [18].

constant  $c$ . To find the constant  $c$ , we use the efficiency axiom. The total change in payoffs must equal the change in the worth of the grand coalition. Therefore,

$$\sum_{i \in N} \Delta \Phi_i = \sum_{i \in N} \{\Phi_i(N, v) - \Phi_i(N, v')\} = v(N) - v'(N) = \mu_v(T) b_T(N).$$

Now, splitting the summation over  $N$  into our two disjoint sets  $T$  and  $N \setminus T$ , we have

$$\sum_{i \in T} \Delta \Phi_i + \sum_{i \in N \setminus T} \Delta \Phi_i = \mu_v(T) b_T(N).$$

Substituting the known values for  $\Delta \Phi_i, i \in N$ , we get  $|T|c = \mu_v(T) b_T(N)$  which gives  $c = \frac{\mu_v(T)}{|T|} \left(\frac{|N|}{|T|}\right)^{-1}$ . Since  $\Phi_i(N, v) = \delta_i + c$ , we have

$$\Phi_i(N, v) = \delta_i + \frac{\mu_v(T)}{|T|} \left(\frac{|N|}{|T|}\right)^{-1}.$$

Therefore,

$$\Phi_i(N, v) = \begin{cases} \delta_i + \frac{\mu_v(T)}{|T|} \left(\frac{|N|}{|T|}\right)^{-1} & \text{for } i \in T \\ \delta_i & \text{for } i \notin T \end{cases}$$

Since  $\delta_i$ 's can be uniquely obtained from the induction hypothesis,  $\Phi(N, v)$  is also uniquely determined. □

Huettner [13] showed that fairness and differential marginality are equivalent. Thus, as a Corollary to Theorem 4, we have the following result.

**Corollary 2** *The solidarity value is the unique solution satisfying efficiency, average null player property, and differential marginality.*

**Remark** The logical independence of each of the three sets of axioms in Theorem 4 and Corollary 2 is shown below:

- (i) The solution  $\Phi_i(N, v) = 0 \quad \forall i \in N$  and  $(N, v) \in G^N$  satisfies average null player property and fairness (hence differential marginality) but not efficiency.
- (ii) The Egalitarian rule  $\Phi^{ED} : G^N \rightarrow \mathbb{R}^N$  given by  $\Phi_i^{ED}(N, v) = \frac{v(N)}{|N|} \quad \forall i \in N$  and  $(N, v) \in G^N$  satisfies efficiency and fairness (hence differential marginality) but not average null player property.
- (iii) The normalized Banzhaf value  $\beta : G^N \rightarrow \mathbb{R}^N$  given by  $\beta_i(N, v) = \frac{\beta_i(v)}{\sum_{j \in N} \beta_j(v)} v(N)$  where  $\beta_i(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N: i \in S} [v(S) - v(S \setminus \{i\})] \quad \forall i \in N$  and  $(N, v) \in G^N$  satisfies efficiency, average null player property but not fairness (hence differential marginality).

We now introduce two new axioms to offer alternative characterizations of the solidarity value.

**Axiom 9 Average  $\alpha$ -player property:** If for some constant  $\alpha$ ,  $\Delta^{av}(T, v) = \alpha$  for all  $T \subseteq N$  with  $i \in T$ , then  $\Phi_i(N, v) = \alpha$ .

This property posits that if the average marginal contributions for a player over all possible coalitions that the player could join are identical, then the players' payoffs will also be identical to their average marginal contributions. This property generalizes the average null player property, which is obtained just by putting  $\alpha = 0$ .

**Axiom 10 Shifted average marginality:** For two games  $(N, v)$  and  $(N, w)$ , if  $\Delta^{av}(T, v) - \Delta^{av}(T, w) = \alpha$  for some constant  $\alpha$  and for all  $T \subseteq N$  with  $i \in T$ , then  $\Phi_i(N, v) - \Phi_i(N, w) = \alpha$ .

This property states that if the game changes such that a player's average marginal contribution increases (or decreases) uniformly by a constant amount in every coalition she is part of, then her payoff should increase (or decrease) by the same amount.

**Lemma 2** *If  $\Phi$  satisfies shifted average marginality and average null player property, then it also satisfies average  $\alpha$ -player property.*

**Proof** Suppose  $(N, w)$  is the null game with  $\Delta^{av}(T, w) = 0$ . Let  $\Delta^{av}(T, v) - \Delta^{av}(T, w) = \alpha$  for all  $T \subseteq N$  with  $i \in T$ . Then, by shifted average marginality,  $\Phi_i(N, v) - \Phi_i(N, w) = \alpha$ . Since  $(N, w)$  is the null game, by average null player property,  $\Phi_i(N, w) = 0$ . Hence, for  $\Delta^{av}(T, v) = \alpha$  for all  $T \subseteq N$  with  $i \in T$ , we have  $\Phi_i(N, v) = \alpha$ . Thus,  $\Phi$  satisfies average  $\alpha$ -player property.  $\square$

**Lemma 3** *If  $\Phi$  satisfies additivity and average  $\alpha$ -player property, then  $\Phi$  also satisfies Shifted average marginality.*

**Proof** Suppose  $\Phi$  satisfies additivity and average  $\alpha$ -player property. Let  $(N, v)$  and  $(N, w)$  be two games such that  $\Delta^{av}(T, v) - \Delta^{av}(T, w) = \alpha$  for some constant  $\alpha$  and for all  $T \subseteq N$  with  $i \in T$ . In view of Proposition 2,  $\mu_v(T) - \mu_w(T) = \alpha$ . Define a new game  $(N, u)$  such that  $u = v - w$ . Then, by Proposition 1, we have  $\mu_u(T) = \mu_{v-w}(T) = \mu_v(T) - \mu_w(T)$  for all  $T \subseteq N$ . For the player  $i$ , let  $\mu_u(T) = \alpha$  for all  $T \ni i$ . By average  $\alpha$ -player property, we have  $\Phi_i(N, u) = \alpha$ . By the additivity axiom, we have  $\Phi(N, v) = \Phi(N, u + w) = \Phi(N, u) + \Phi(N, w)$ . Therefore,  $\Phi_i(N, v) - \Phi_i(N, w) = \Phi_i(N, u)$ . Since  $\Phi_i(N, u) = \alpha$ , it follows that  $\Phi_i(N, v) - \Phi_i(N, w) = \alpha$ . Thus,  $\Phi$  satisfies shifted average marginality.  $\square$

**Proposition 4**  *$\Phi^{Sol}$  satisfies average  $\alpha$ -player property and shifted average marginality.*

**Proof** Suppose  $\mu_v(T) = \alpha$  for all  $T \ni i$ . We prove  $\Phi_i^{Sol}(N, v) = \alpha$ . From Eq. 10, we get

$$\Phi_i^{Sol}(N, v) = \sum_{T \ni i} \binom{|N|}{|T|}^{-1} \frac{\mu_v(T)}{|T|} = \sum_{T \ni i} \binom{|N|}{|T|}^{-1} \frac{\alpha}{|T|} = \alpha \sum_{T \ni i} \binom{|N|}{|T|}^{-1} \frac{1}{|T|} = \alpha \cdot 1 = \alpha$$

Hence,  $\Phi^{Sol}$  satisfies average  $\alpha$ -player property. Also, by Lemma 3,  $\Phi^{Sol}$  satisfies shifted average marginality.  $\square$

**Lemma 4** *If  $\Phi$  satisfies shifted average marginality, then  $\Phi$  also satisfies Fairness.*

**Proof** Suppose  $\Phi$  satisfies shifted average marginality. Let  $i$  and  $j$  be two symmetric players in  $(N, v)$ . Then, there exist coalitions  $T \ni i$  and  $S \ni j$  such that  $\mu_v(T) = \mu_v(S)$  where  $T \setminus \{i\} = S \setminus \{j\}$ . By Proposition 1, we have  $\mu_{u+v}(T) - \mu_u(T) = \mu_{u+v}(S) - \mu_u(S)$ . If we assume this value to be  $\alpha$  for all  $T \ni i$  and  $S \ni j$ , then by Shifted Average Marginality, we have  $\Phi_i(N, u + v) - \Phi_i(N, u) = \Phi_j(N, u + v) - \Phi_j(N, u)$ . Thus,  $\Phi$  satisfy fairness.  $\square$

Following our earlier discussion on Lemma 4 and Proposition 4, we introduce a further characterization of the solidarity value related to these foundational results.

**Theorem 5** *A value  $\Phi$  on  $G^N$  satisfies efficiency, average null player property, and shifted average marginality if and only if  $\Phi = \Phi^{Sol}$ .*

As a result of Lemma 4, the proof of this new characterization mirrors the proof of Theorem 4, so we skip the formal proof of Theorem 5. Below, we present an additional characterization of the solidarity value based on the average  $\alpha$ -player property, along with other axioms. The proof of this characterization also proceeds following that of Theorem 4.

**Theorem 6** *A value  $\Phi$  on  $G^N$  satisfies Efficiency, Additivity, Average  $\alpha$ -player property, and symmetry if and only if  $\Phi = \Phi^{Sol}$ .*

**Proof** It is well known that  $\Phi^{Sol}$  satisfies efficiency, additivity, and symmetry. Also, by Proposition 4, it satisfies average  $\alpha$ -player property.

For the converse part, consider  $D(v) = \{S \subseteq N : \mu_v(S) \neq 0\}$  and  $d(v) = |D(v)|$ . We employ the method of induction on  $d(v)$  to show the uniqueness of the solidarity value.

1. If  $d(v) = 0$ , then  $\mu_v(S) = 0$  for all  $S \subseteq N$ . This implies that  $v(S) = 0$  for all  $S \subseteq N$ , which is the null game. For any player  $i \in N$ , since  $\mu_v(T) = 0$  for all coalitions  $T$  containing  $i$ , by the average  $\alpha$ -player property,  $\Phi_i(N, v) = 0$  which coincides with the solidarity value for the null game.
2. If  $d(v) = 1$ , there exists a unique non-empty coalition  $T \subseteq N$  such that  $\mu_v(T) \neq 0$ , and  $\mu_v(S) = 0$  for all other  $S \neq T$ . Then,  $v$  can be written as  $v = \mu_v(T)b_T$ . In this scenario,  $v(N) = \mu_v(T)b_T(N)$ .  
 If  $i \notin T$ , then  $i \in S$  for some  $S \subseteq N \setminus T$ . Since  $\mu_v(S) = 0$  for all such coalitions, by the average  $\alpha$ -player property,  $\Phi_i(N, \mu_v(T)b_T) = 0$ . This outcome is consistent with the solidarity value, as  $\Phi_i^{Sol}(N, \mu_v(T)b_T) = 0$  for players not in the unique non-zero dividend coalition  $T$ .  
 If  $i \in T$ , then by the efficiency axiom, the sum of payoffs equals the grand coalition's worth. Therefore,

$$\sum_{j \in N} \Phi_j(N, \mu_v(T)b_T) = v(N) = \mu_v(T)b_T(N) = \mu_v(T) \left( \frac{|N|}{|T|} \right)^{-1}.$$

Since we established that  $\Phi_j(N, \mu_v(T)b_T) = 0$  for  $j \notin T$ , this equation simplifies to

$$\sum_{j \in T} \Phi_j(N, \mu_v(T)b_T) = \mu_v(T) \left( \frac{|N|}{|T|} \right)^{-1}.$$

Here, all players are symmetric in the game  $(N, b_T)$ , and consequently in  $(N, v)$ . Thus, using the symmetry axiom, we have

$$\Phi_j(N, \mu_v(T)b_T) = \frac{\mu_v(T)}{|T|} \left( \frac{|N|}{|T|} \right)^{-1} = \Phi_j^{Sol}(N, \mu_v(T)b_T).$$

Thus, the statement  $\Phi(N, v) = \Phi^{Sol}(N, v)$  holds for games with  $d(v) = 1$ .

3. Assume that  $\Phi(N, v) = \Phi^{Sol}(N, v)$  holds for all games  $(N, v) \in G^N$  with  $d(v) \leq k$ . Now, we show it for games  $(N, v) \in G^N$  with  $d(v) = k + 1$ . We can express  $v$  as a sum of basis games weighted by their Harsanyi-type dividends:  $v = \sum_{S \in D(v)} \mu_v(S)b_S$ . Choose an arbitrary coalition  $T \in D(v)$  such that  $\mu_v(T) \neq 0$ . Define a new game  $v' = v - \mu_v(T)b_T$ , which means  $d(v') = k$ . By the inductive hypothesis, since  $d(v') = k$ , we have  $\Phi(N, v') = \Phi^{Sol}(N, v')$ . Since we can write  $v$  as  $v = v' + \mu_v(T)b_T$ , using the Additivity axiom, we can decompose the value of  $\Phi(N, v)$  as

$$\Phi(N, v) = \Phi(N, v') + \Phi(N, \mu_v(T)b_T) = \Phi^{Sol}(N, v') + \Phi^{Sol}(N, \mu_v(T)b_T). \tag{13}$$

Since the solidarity value  $\Phi^{Sol}$  is itself Additive, we know that:

$$\Phi^{Sol}(N, v) = \Phi^{Sol}(N, v' + \mu_v(T)b_T) = \Phi^{Sol}(N, v') + \Phi^{Sol}(N, \mu_v(T)b_T). \tag{14}$$

Comparing the Eqs. 13 and 14, we conclude that  $\Phi(N, v) = \Phi^{Sol}(N, v)$ .

Therefore, by the principle of mathematical induction, the theorem holds for all games  $(N, v) \in G^N$ . □

**Remark** The logical independence of each of the three sets of axioms in Theorems 5–6 is shown below:

- (i) Define  $\Phi : G^N \mapsto \mathbb{R}^{|N|}$  given by  $\Phi_i(N, v) = \sum_{S \subseteq N: i \in S} \frac{\Delta^{av}(S, v)}{2^{|M|-1}} \quad \forall i \in N$  and  $(N, v) \in G^N$  where  $\Delta^{av}(S, v)$  is given in Eq. 5 satisfies average null player property, shifted average marginality, additivity, average  $\alpha$ -player property, and symmetry but does not satisfy efficiency.
- (ii) Define  $\Phi_i(N, v) = \begin{cases} \Phi_j^{Sol}(N, v) + 1 & \text{if } j = 1 \\ \Phi_j^{Sol}(N, v) - \frac{1}{|N|-1} & \text{if } j \neq 1 \end{cases} \quad \forall i \in N \text{ and } (N, v) \in G^N$  satisfies efficiency and shifted average marginality, but does not satisfy average null player property.

- (iii) Define a set  $E(v) = \{i \in N \mid i \text{ is not an Average Null Player in } (N, v)\}$ . The value  $\Phi : G^N \mapsto \mathbb{R}^{|N|}$  defined by  $\Phi_i(N, v) = \begin{cases} \frac{v(N)}{|E(v)|} & \text{if } i \in E(v) \\ 0 & \text{otherwise.} \end{cases} \forall i \in N$  and  $(N, v) \in G^N$  satisfies efficiency and average null player property, but does not satisfy shifted average marginality.
- (iv) The Shapley value,  $\Phi^{Sh} : G^N \mapsto \mathbb{R}^{|N|}$  defined in Eq. 1 and given by  $\Phi_i^{Sh}(N, v) = \sum_{S \subseteq N: i \in S} \frac{(|N|-|S|)! (|S|-1)!}{|N|!} \Delta_i(S, v) \forall i \in N$  and  $(N, v) \in G^N$  satisfies efficiency, additivity, and symmetry but fails to satisfy average  $\alpha$ -player property.
- (v) The weighted solidarity value,  $\Phi^{w-Sol} : G^N \mapsto \mathbb{R}^{|N|}$ , defined in [4] for the basis game  $b_T$  is given by  $\Phi^{w-Sol}(N, b_T) = \sum_{j \in N} \frac{w_j}{w_j} b_T$  satisfies efficiency, average  $\alpha$ -player property, and additivity, but not symmetry.
- (vi) Consider a game  $(N, v)$  such that  $K(v) = \{i \in N \mid \Delta^{av}(T, v) = \alpha_i \forall T \ni i\} \neq \emptyset$  where  $\alpha_i$ 's are constants. Define a value function  $\Phi : G^N \mapsto \mathbb{R}^{|N|}$  as

$$\Phi_i(N, v) = \begin{cases} \alpha_i & \text{if } i \in K(v) \\ \frac{v(N) - \sum_{j \in K(v)} \alpha_j}{|N \setminus K(v)|} & \text{if } i \in N \setminus K(v). \end{cases}$$

This value satisfies efficiency, average  $\alpha$ -player property, and symmetry, but the value does not satisfy additivity.

## 5 Conclusion

We introduced the Harsanyi-type dividends of a class of basis vectors  $\{b_T \mid \emptyset \neq T \subseteq N\}$  and obtained an alternative expression for the solidarity value in TU games. This is in line with the Harsanyi dividends used to represent the Shapley value. We also obtain the A-potential function, as presented by Xu et al. [19], in terms of these Harsanyi-type dividends. It is evident from the expressions that their computational complexities reduce to a great extent in this manner. Moreover, we have used these dividends to characterize the solidarity value. This opens up new avenues for obtaining similar dividends for other existing values in TU game theory and generalizes the dividend to cover all such values hitherto found in the literature. The computational complexity becomes more significant when we switch to network games. In a follow-up paper, we plan to introduce Harsanyi-type dividends to network games and derive simple expressions in comparison to existing frameworks.

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## Declarations

**Conflict of Interest** The authors declare no competing interests.

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