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Predictor-corrector interior-point algorithm for sufficient linear complementarity problems based on AET functions with bounded proportional growth rate

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HIGHLIGHTS

- A new class of algebraically equivalent transformation functions is defined.
- A new predictor-corrector interior-point algorithm for $P_*(\kappa)$-LCPs is proposed.
- We prove polynomial complexity in the size of the problem and in the parameter κ.
- Numerical results show the efficiency of the presented algorithm.

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ABSTRACT

In this paper we define the first class of algebraically equivalent transformation (AET) functions for predictor-corrector (PC) interior-point algorithms (IPAs) for sufficient linear complementarity problems (LCPs) and we provide the unified complexity analysis of the introduced PC IPA. This new class is called the class of AET functions with bounded proportional growth rate for PC IPAs. The main difference between our PC IPA and those from the literature is that the AET function is an input data of our algorithm. The new class of AET functions is defined by several inequalities that are directly used in the analysis of the new PC IPA. We show that the PC IPA using any function of the new class of AET functions has polynomial iteration complexity in the size of the problem, the starting point's duality gap, the accuracy parameter and the handicap of the problem's matrix. We also provide promising numerical results on a newly generated test set of problems that contain matrices with large handicap, as well.

1. Introduction

Linear complementarity problems (LCPs) have a wide variety of applications in different fields, see [1,2]. It is well-known that linear programming (LP) and linearly constrained (convex) quadratic programming (QP) problems are special cases of LCPs. Bimatrix games, the Arrow-Debreu competitive market equilibrium problem with linear and Leontief utility functions can also be formulated as LCPs, see [3,4]. The books by Cottle et al. [1] and Kojima et al. [5] provide insights into the theory of LCPs. Note that LCPs are members of the class of NP-complete problems, see [6]. However, it is known that if the problem's matrix is skew-symmetric [7–9] or positive semidefinite [10], IPAs can find an approximate solution to LCPs in polynomial time. Starting from a properly selected ϵ-optimal solution we can obtain an exact solution using the rounding procedure in strongly polynomial time, see [7,11].

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Furthermore, other matrix classes have also been introduced. The class of sufficient matrices has been defined by Cottle, Pang, and Venkateswaran [12]. Kojima et al. [5] proposed the class of $P_*(\kappa)$-matrices. Considering the union of the sets $P_*(\kappa)$ for all nonnegative κ we get the class P_*, see [5]. Väliaho [13] showed that the class of P_*-matrices is equivalent to the class of sufficient matrices. It is important that in general, interior-point algorithms (IPAs) for solving $P_*(\kappa)$-LCPs have polynomial iteration complexity in the size of the problem, the starting point's duality gap, the accuracy parameter and the special parameter $\kappa \geq 0$. The rounding procedure of Illés et al. [11] works for $P_*(\kappa)$-LCPs, as well. Although, De Klerk and E.-Nagy [14] proved that the handicap of the problem's matrix could be exponential in the bit length of the data. In spite of the fact that the complexity analyses of IPAs for $P_*(\kappa)$-LCPs depend on the special parameter κ, there are computational results for LCPs with matrices having exponential value κ, where the iteration numbers are much better than predicted by the complexity results, see [15–17]. This shows that it is worth trying to analyse different versions of IPAs for $P_*(\kappa)$-LCPs.

There are several types of IPAs. Predictor-corrector (PC) IPAs turned out to be efficient in practice. In a main iteration they perform one predictor and several corrector steps. The goal of the predictor step is to approach the optimal solution in a greedy way. However, after a predictor step a certain amount of deviation from the central path is obtained. The aim of the corrector step is to return to the designated small neighbourhood of the central path. The first PC IPA was independently proposed by Mehrotra [18] and Sonnevend [19]. Note that Mehrotra's PC IPA is the practical one. The PC IPAs that use only one corrector step in a main iteration are called Mizuno-Todd-Ye (MTY) algorithms [20]. Potra and Sheng [21] introduced an MTY type IPA for solving $P_*(\kappa)$-LCPs from arbitrary positive starting points. Miao [22] defined an MTY type IPA for solving $P_*(\kappa)$-LCPs and showed superlinear convergence for problems admitting strict complementarity solutions. Illés and Nagy [23], Kheirfam [24] and Darvay et al. [16] also provided several MTY type PC IPAs. Potra and Liu [25] defined PC IPAs for $P_*(\kappa)$-LCPs acting in a wide neighbourhood of the central path.

The method by which we define the search directions in the case of IPAs also plays a key role in this theory. For example, Peng et al. [26] proposed self-regular functions, and Bai et al. [27] introduced the class of eligible kernel functions. Furthermore, Lešaja and Roos [28] provided a unified analysis using eligible kernel functions for IPAs solving $P_*(\kappa)$-LCPs. Darvay [29] introduced the algebraically equivalent transformation (AET) technique for defining search directions in the case of IPAs for LP. After that, several IPAs were introduced using specific AET functions. Rigó [30] presented different IPAs using the AET technique for LP, sufficient LCPs and symmetric cone optimization. However, it is an interesting question, whether we can define a whole class of functions in the AET approach, for which IPAs with polynomial complexity can be defined. In the case of primal-dual IPAs for $P_*(\kappa)$-LCPs this question was answered in [31], where the authors defined a new class of AET functions in the case of short-step IPAs. In [32], the authors presented a new class of AET functions for long-step IPAs for LPs. Haddou et al. [33] proposed a family of concave functions to determine search directions in the case of IPAs for solving monotone LCPs. It should be mentioned that they used another type of transformation of the central path system. However, in the case of PC IPAs there is no class of AET functions for which IPAs can be defined. The aim of this paper is to introduce a new class of AET functions and to provide a unified analysis of PC IPAs for $P_*(\kappa)$-LCPs for the new class of AETs. Several PC IPAs presented in [16,24] are based on AET functions that are members of our new class of AETs, thus a unified complexity analysis is provided for those cases, as well. It should be mentioned that there are only a few test set problems for $P_*(\kappa)$-LCPs with $\kappa > 0$. Illés and Morapitiye [34] initiated the development of test set problems for sufficient LCPs. In [16] numerical results are presented in the case of three families of LCPs including a special matrix having an exponential handicap. Our aim was to expand the currently known test set of sufficient LCPs towards problems with larger sizes and with matrices having large handicaps. For this reason, Török and Illés generated new test problems that are presented in [35]. In this paper we present our numerical results and experiments with the PC IPAs based on the functions belonging to our new class of AET functions and based on a kernel function on the test problems from [35].

The paper is organized as follows. Section 2 presents results related to the theory of $P_*(\kappa)$-LCPs and the classical AET approach. We define the new class of AET functions for the PC IPA used in this paper. In Sections 3 and 4 we propose the PC IPA and provide the complexity analysis of the proposed IPA. We also describe two examples of functions belonging to our new class of AET functions for which we derive the complexity bounds. In Section 5 we present numerical results on a new test set of problems generated by Török and Illés, see [35]. Section 6 is devoted to concluding remarks and further research topics.

2. A new class of algebraically equivalent transformations for predictor-corrector interior-point algorithms

In the first part of this section we present the $P_*(\kappa)$-LCPs.

2.1. $P_*(\kappa)$-linear complementarity problems

In the LCP we are looking for the vectors $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$ for which the following constraints are satisfied:

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \;\; \mathbf{x}, \mathbf{s} \geq \mathbf{0}, \;\; \mathbf{x}\mathbf{s} = \mathbf{0}, \tag{LCP}$$

where $M \in \mathbb{R}^{n \times n}$, $\mathbf{q} \in \mathbb{R}^n$ and $\mathbf{xs}$ is the Hadamard product of vectors $\mathbf{x}$ and $\mathbf{s}$. We will use the following notations to determine the feasible region, the interior and the solution set of LCPs:

$$\begin{aligned} \mathcal{F} &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^n_\oplus \times \mathbb{R}^n_\oplus \, : \, -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \\ \mathcal{F}^+ &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \, : \, -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \\ \mathcal{F}^* &:= \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F} \, : \, \mathbf{xs} = \mathbf{0}\}. \end{aligned}$$

Here we denote by $\mathbb{R}^n_\oplus$ the n-dimensional nonnegative orthant and by $\mathbb{R}^n_+$ the positive orthant.

As we have already mentioned, LCPs belong to the class of NP-complete problems. However, Kojima et al. [10] proved that if the problem's matrix possesses the so-called $P_*(\kappa)$-property, IPAs for LCPs can find an approximate solution in polynomial time in the size of the problem, the starting point's duality gap, the accuracy parameter and the special parameter κ.

Definition 2.1 (Kojima et al. [5]). Let $\kappa \geq 0$ be a nonnegative real number. A matrix $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$-matrix if

$$(1+4\kappa)\sum_{i\in I_+(\mathbf{x})} x_i(Mx)_i + \sum_{i\in I_-(\mathbf{x})} x_i(Mx)_i \geq 0, \quad \forall \mathbf{x}\in\mathbb{R}^n, \tag{1}$$

where

$$I_+(\mathbf{x}) = \{1\leq i\leq n : x_i(Mx)_i > 0\} \text{ and } I_-(\mathbf{x}) = \{1\leq i\leq n : x_i(Mx)_i < 0\}.$$

We call an LCP $P_*(\kappa)$-LCP if the problem's matrix is a $P_*(\kappa)$-matrix. Throughout the paper we deal with $P_*(\kappa)$-LCPs and we suppose that $\mathcal{F}^+ \neq \emptyset$. The handicap of a matrix M [13] is defined by $\hat{\kappa}(M) := \inf\{\kappa | M \in P_*(\kappa)\}$.

Definition 2.2 (Kojima et al. [5]). A matrix $M \in \mathbb{R}^{n\times n}$ is a P_*-matrix if it is a $P_*(\kappa)$-matrix for some $\kappa \geq 0$.

Consider $P_*(\kappa)$, which denotes the set of $P_*(\kappa)$-matrices. We will use P_* to denote the set of all P_*-matrices, i.e.,

$$P_* = \bigcup_{\kappa\geq 0} P_*(\kappa).$$

The central path problem is the following:

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q}, \qquad \mathbf{x},\mathbf{s}>\mathbf{0}, \qquad \mathbf{x}\mathbf{s}=\mu\mathbf{e}, \tag{CPP}$$

where $\mathbf{e}$ denotes the n-dimensional all-ones vector and $\mu > 0$. Kojima et al. [5] proved that if M is a $P_*(\kappa)$-matrix, then (CPP) has a unique solution for every $\mu > 0$. The unique solution of the central path system for a certain μ is called the μ-center. The set of μ-centers forms a trajectory towards the solution when μ runs through positive real numbers.

In the following subsection we present the AET technique for determining search directions in the case of IPAs.

2.2. A new class of algebraically equivalent transformations in case of predictor-corrector interior-point algorithms

Firstly, we present the AET technique in the case of $P_*(\kappa)$-LCPs, which was first introduced by Darvay [29] for LPs. Let $\varphi : (\bar{\xi},\infty) \to \mathbb{R}$, with $0 \leq \bar{\xi} < 1$, be a continuously differentiable and invertible function, such that $\varphi'(t) > 0,\ \forall t > \bar{\xi}$. The following notation is used: $\varphi(\mathbf{x}) = [\varphi(x_1), \varphi(x_2)\ \ldots, \varphi(x_n)]^T$. We can write system (CPP) in the following equivalent form:

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q} \qquad \mathbf{x},\mathbf{s}>\mathbf{0}, \qquad \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)=\varphi(\mathbf{e}). \tag{CPP_φ}$$

Applying Newton's method to (CPP_φ), we get

$$\begin{aligned} -M\Delta\mathbf{x}+\Delta\mathbf{s}&=\mathbf{0},\\ S\Delta\mathbf{x}+X\Delta\mathbf{s}&=\mathbf{a}_\varphi, \end{aligned} \tag{3}$$

where

$$\mathbf{a}_\varphi = \mu\frac{\varphi(\mathbf{e})-\varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)}{\varphi'\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)}, \tag{4}$$

and $X = \operatorname{diag}(\mathbf{x})$, $S = \operatorname{diag}(\mathbf{s})$, $(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+$. System (3) has unique solution, see Lemma 4.1 in [5]. The scaling technique plays a key role in the theory of IPAs. Let

$$\mathbf{v}=\sqrt{\frac{\mathbf{x}\,\mathbf{s}}{\mu}}, \quad \mathbf{d}=\sqrt{\frac{\mathbf{x}}{\mathbf{s}}}, \quad \mathbf{d}_x=\frac{\mathbf{d}^{-1}\,\Delta\mathbf{x}}{\sqrt{\mu}}=\frac{\mathbf{v}\,\Delta\mathbf{x}}{\mathbf{x}}, \quad \mathbf{d}_s=\frac{\mathbf{d}\,\Delta\mathbf{s}}{\sqrt{\mu}}=\frac{\mathbf{v}\,\Delta\mathbf{s}}{\mathbf{s}}. \tag{5}$$

After some substitutions and calculations we obtain the scaled Newton system:

$$\begin{aligned} -\bar{M}\mathbf{d}_x+\mathbf{d}_s&=\mathbf{0},\\ \mathbf{d}_x+\mathbf{d}_s&=\mathbf{p}_\varphi, \end{aligned} \tag{6}$$

where $\bar{M} = DMD$, $D = \operatorname{diag}(\mathbf{d})$ and

$$\mathbf{p}_\varphi = \frac{\varphi(\mathbf{e})-\varphi(\mathbf{v}^2)}{\mathbf{v}\varphi'(\mathbf{v}^2)}. \tag{7}$$

Illés et al. [31] presented a new class of AET functions in the case of short-step IPAs for solving $P_*(\kappa)$-LCPs. Later on, Darvay and Rigó [36] extended this approach to symmetric cone horizontal linear complementarity problems. Based on the idea proposed in [31,36] we introduce a new class of AET functions for determining search directions in the case of PC IPAs for solving $P_*(\kappa)$-LCPs. To the best of our knowledge, this is the first class of AET functions to determine search directions in the case of PC IPAs. We call this class *AET functions with bounded proportional growth rate* for PC IPAs.

Definition 2.3. Let $\varphi : (\xi^2, \infty) \to \mathbb{R}$ be a continuously differentiable, invertible function, such that $\varphi'(t) > 0,\ \forall t > \xi^2$, where $0 \le \xi \le \frac{1}{2}$. All functions φ satisfying the following conditions belong to the new class of AET functions. There exist positive real numbers $c_0 \ge \frac{1}{4}$, $c_1, c_2, c > 0$ such that

AET1. $c_0 \left|t(1-t)\varphi'(t^2)\right| \le \left|\varphi(1)-\varphi(t^2)\right| \le 2c_1(1+t)\left|t(1-t)\varphi'(t^2)\right|$
holds for all $t > \xi$,

AET2.

$$4t^2\varphi'(t^2)(\varphi(1)-\varphi(t^2)) - c_2(\varphi(1)-\varphi(t^2))^2 \le 4t^2(1-t^2)\left(\varphi'(t^2)\right)^2 \le 4t^2\varphi'(t^2)(\varphi(1)-\varphi(t^2)) + (\varphi(1)-\varphi(t^2))^2$$

holds for all $t > \xi$,

AET3.

$$\lim_{t\to\infty} \frac{\varphi(1)-\varphi(t^2)}{t^2\varphi'(t^2)} = -c.$$

This class differs from the one proposed in [31], which was developed for primal-dual IPAs. Since the corrector step can be interpreted as a full-Newton step, we use some results from [31] in the analysis of the corrector step, and certain similarities between the two classes of functions can therefore be observed. However, in order to define PC IPAs, it is necessary to decompose the right-hand side $\mathbf{a}_\varphi$ of the Newton system into two terms that satisfy specific conditions. Such a decomposition does not exist for all functions; for example, it is not possible for the function $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$. In this case, a PC IPA cannot be defined within our framework. To guarantee the existence of the required decomposition, we therefore introduced the third condition on the function φ, namely condition AET3 of Definition 2.3. Moreover, the left-hand side of the first condition, namely condition AET1 of Definition 2.3 ensures the feasibility of the predictor step. In this way, we obtained a subclass of the class of functions defined in [31].
Let us introduce the following function: $f : (\xi, \infty) \to \mathbb{R}$:

$$f(t) = \frac{\varphi(1)-\varphi(t^2)}{t\varphi'(t^2)}. \tag{8}$$

Then, we obtain the following formulation of the definition for the new class of AET functions.

Proposition 2.1. *Consider the function f given in (8). The conditions given in Definition 2.3 can be formulated in the following way. There exist positive constants $c_0 \ge \frac{1}{4}$, $c_1, c_2, c > 0$ such that*

AETa. $c_0|1-t| \le |f(t)| \le 2c_1|1-t^2|$, *holds for all $t > \xi$,*
AETb. $-c_2\frac{f(t)^2}{4} \le 1 - t^2 - tf(t) \le \frac{f(t)^2}{4}$, *holds for all $t > \xi$,*
AETc. $\lim_{t\to\infty} \frac{f(t)}{t} = -c$.

Proof. By substituting (8) into Definition 2.3, after some calculations we obtain the result. □

Lemma 2.1. *Assume that $\mathbf{x}$ and $\mathbf{s}$ are the fixed current iterates. If Condition AET3 of Definition 2.3 or AETc of Proposition 2.1 is satisfied, then*

$$\lim_{\mu\to 0} \mathbf{a}_\varphi = -c\mathbf{x}\mathbf{s}.$$

Proof. By using (7) and Condition AETc from Proposition 2.1 and $t_i = \sqrt{\frac{x_i s_i}{\mu}}$, $i = 1, \dots n$, we get

$$\lim_{t_i\to\infty} \frac{\varphi(1)-\varphi(t_i^2)}{t_i^2\varphi'(t_i^2)} = \lim_{\mu\to 0} \frac{\varphi(1)-\varphi\left(\frac{x_i s_i}{\mu}\right)}{\frac{x_i s_i}{\mu}\varphi'\left(\frac{x_i s_i}{\mu}\right)} = \lim_{\mu\to 0} \frac{(\mathbf{a}_\varphi)_i}{x_i s_i} = -c,$$

which leads to the result. □

In our new PC IPA, we will first take a corrector step and after that a predictor step. The reason for this is that after a corrector step we reach a proper neighbourhood of the central path. These methods are named corrector-predictor IPAs, see Potra [37].

Darvay et al. [16] provided a unified approach for defining search directions in the case of PC IPAs by using the AET technique. We will use this approach in our PC IPA, as well.
The corrector step is a full-Newton step, hence the corrector system coincides with system (3) and the scaled corrector system coincides with system (6), which has the following solution:

$$\mathbf{d}_x^c = (I+\bar{M})^{-1}\mathbf{p}_\varphi, \quad \mathbf{d}_s^c = \bar{M}(I+\bar{M})^{-1}\mathbf{p}_\varphi.$$

From this and (5) we can calculate the search directions $\Delta^c\mathbf{x}$ and $\Delta^c\mathbf{s}$. The point after a corrector step is

$$\mathbf{x}^c = \mathbf{x} + \Delta^c\mathbf{x}, \quad \mathbf{s}^c = \mathbf{s} + \Delta^c\mathbf{s}. \tag{9}$$

Table 1
Functions belonging to the new class AET functions.

φ	a_φ	c	p_φ
$\varphi_1(t) = t$	$\mu\mathbf{e} - \mathbf{xs}$	1	$\mathbf{v}^{-1} - \mathbf{v}$
$\varphi_2(t) = \sqrt{t}$	$2(\sqrt{\mu\mathbf{xs}} - \mathbf{xs})$	2	$2(\mathbf{e} - \mathbf{v})$
$\varphi_3(t) = t - \sqrt{t}$	$\frac{\sqrt{\mu}\mathbf{xs}}{2\sqrt{\mathbf{xs}}-\sqrt{\mu}\mathbf{e}} - \mathbf{xs}$	1	$\frac{2(\mathbf{v}-\mathbf{v}^2)}{2\mathbf{v}-\mathbf{e}}$
$\varphi_4(t) = t^2 - t + \sqrt{t}$	$\frac{4\mu^2\sqrt{\mathbf{xs}}+2\mu\mathbf{xs}\sqrt{\mathbf{xs}}-3\mu\mathbf{xs}\sqrt{\mu}}{2(4\mathbf{xs}\sqrt{\mathbf{xs}}-2\mu\sqrt{\mathbf{xs}}+\mu\sqrt{\mu})} - \frac{\mathbf{xs}}{2}$	$\frac{1}{2}$	$\frac{2(\mathbf{e}-\mathbf{v})(\mathbf{e}+\mathbf{v}^2+\mathbf{v}^3)}{4\mathbf{v}^3-2\mathbf{v}+\mathbf{e}}$

The predictor step aims to find the optimal solution in a greedy way. This means, we should set $\mu = 0$ in $\mathbf{a}_\varphi$ of system (3). By setting $\mu = 0$ in $\mathbf{a}_\varphi$ and from Lemma 2.1 we obtain the predictor system

$$\begin{aligned} -M\Delta^p\mathbf{x} + \Delta^p\mathbf{s} &= \mathbf{0}, \\ S\Delta^p\mathbf{x} + X\Delta^p\mathbf{s} &= -c\mathbf{x}^c\mathbf{s}^c. \end{aligned} \tag{10}$$

The scaled predictor system follows from (3) and (5):

$$\begin{aligned} -\bar{M}_+\mathbf{d}_x^p + \mathbf{d}_s^p &= \mathbf{0}, \\ \mathbf{d}_x^p + \mathbf{d}_s^p &= -c\mathbf{v}^+, \end{aligned} \tag{11}$$

where $\mathbf{v}^+ = \sqrt{\frac{\mathbf{x}^c\mathbf{s}^c}{\mu}}$, and μ corresponds to $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+$ appearing in (9) and (10). Moreover, $\mathbf{d}^+ = \sqrt{\frac{\mathbf{x}^c}{\mathbf{s}^c}}$, $D^+ = \operatorname{diag}(\mathbf{d}^+)$, $\bar{M}_+ = D^+MD^+$ and which has the solution:

$$\mathbf{d}_x^p = -c\mathbf{v}^+\left(I + \bar{M}_+\right)^{-1} \quad \text{and} \quad \mathbf{d}_s^p = -c\mathbf{v}^+\,\bar{M}_+(I + \bar{M}_+)^{-1}.$$

Using (5) the predictor search directions $\Delta^p\mathbf{x}$ and $\Delta^p\mathbf{s}$ can be calculated. We define the point after a predictor step as

$$\mathbf{x}^p = \mathbf{x}^c + \theta\Delta^p\mathbf{x}, \quad \mathbf{s}^p = \mathbf{s}^c + \theta\Delta^p\mathbf{s}, \quad \mu^p = (1 - c\theta)\,\mu,$$

where $\theta \in \left(0, \frac{1}{c_4}\right)$, with $c_4 = \max\{1, c\}$ is the update parameter.
Table 1 contains functions belonging to the new class of AET functions from Definition 2.3 and the corresponding vectors $\mathbf{a}_\varphi$, $\mathbf{p}_\varphi$ and the values of c, respectively.

3. New predictor-corrector interior-point algorithm for solving $P_*(\kappa)$-linear complementarity problems

The proximity measure is defined as $\delta : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$:

$$\delta(\mathbf{x}, \mathbf{s}; \mu) := \delta(\mathbf{v}) = \frac{\left\|\mathbf{p}_\varphi\right\|}{2}, \tag{12}$$

where $\|\cdot\|$ is the standard Euclidean norm, and $\mathbf{p}_\varphi$ is given in (7).
The τ-neighbourhood of a fixed point of the central path is given by

$$\mathcal{N}_2(\tau, \mu) := \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \delta(\mathbf{x}, \mathbf{s}; \mu) \leq \tau\}, \tag{13}$$

where $\delta(\mathbf{x}, \mathbf{s}; \mu)$ is from (12), τ is a threshold parameter and $\mu > 0$ is fixed.
The PC IPA for solving $P_*(\kappa)$-LCPs is given in Algorithm 3.1.
The next subsection provides the complexity analysis of the PC IPA.

3.1. *Complexity analysis of the predictor-corrector interior-point algorithm*

In this subsection we present the complexity analysis of the proposed PC IPA. First, we address the corrector step and after that the analysis of the predictor step will be provided.

3.1.1. *Corrector step*

The corrector step is a full-Newton step, so the analysis of this part is very similar to that of the short-step IPA presented in [31]. We summarize the lemmas and corollaries used in this case. It should be mentioned that the parameter c_3 given in [31] corresponds in this case to c_2. Furthermore, we introduce $c_3 = \max\{1, c_2\}$. This corresponds to c_2 in [31].

The first lemma proves the strict feasibility of the corrector step.

Lemma 3.1 ([31]). *Let $(x, s) \in \mathcal{F}^+$ be given, such that $\delta(x, s; \mu) < \frac{1}{\sqrt{1+4\kappa}}$ and $\mathbf{v} > \xi\mathbf{e}$. For any function satisfying AET2 of Definition 2.3, we have that $(\mathbf{x}^c, \mathbf{s}^c) \in \mathcal{F}^+$.*

The quadratic convergence of the corrector step is given in the following lemma.

Algorithm 3.1 PC IPA for $P_*(\kappa)$-LCPs based on a new class of AET functions.

Input:
Function φ satisfying the properties from Definition 2.3;
Accuracy parameter $\epsilon > 0$ and proximity parameter $\tau > 0$;
Update parameter $\theta \in \left(0, \frac{1}{c_4}\right)$, with $c_4 = \max\{1, c\}$;
$P_*(\kappa)$-matrix M with known upper bound κ of the handicap $\hat{\kappa}(M)$;
Starting point $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^+$, $\mu^0 = \frac{(\mathbf{x}^0)^T \mathbf{s}^0}{n}$, such that $\delta(\mathbf{x}^0, \mathbf{s}^0; \mu^0) \le \tau$ and $\mathbf{v}^0 > \xi\mathbf{e}$.
begin
  $k := 0$;
  while $\left(\mathbf{x}^k\right)^T \mathbf{s}^k > \epsilon$ ***do begin***

    (corrector step)
    compute $(\Delta^c x^k, \Delta^c s^k)$ from system (6) by using (5) with φ;
    let $(\mathbf{x}^c)^k := \mathbf{x}^k + \Delta^c \mathbf{x}^k$ and $(\mathbf{s}^c)^k := \mathbf{s}^k + \Delta^c \mathbf{s}^k$;

    (predictor step)
    compute $(\Delta^p x^k, \Delta^p s^k)$ from system (11) by using (5);
    let $(\mathbf{x}^p)^k := (\mathbf{x}^c)^k + \theta\Delta^p \mathbf{x}^k$ and $(\mathbf{s}^p)^k := (\mathbf{s}^c)^k + \theta\Delta^p \mathbf{s}^k$;

    (update of the parameters and the iterates)
    $(\mu^p)^k := (1 - c\theta)\,\mu^k$;
    $\mathbf{x}^{k+1} := (\mathbf{x}^p)^k, \quad \mathbf{s}^{k+1} := (\mathbf{s}^p)^k, \quad \mu^{k+1} := (\mu^p)^k, \quad k := k + 1;$

  end
end.

Lemma 3.2 ([31]). *Let $(\boldsymbol{x}, \boldsymbol{s}) \in \mathcal{F}^+$, $\mathbf{v} > \xi\mathbf{e}$ and $\mathbf{v}^+ = \sqrt{\frac{\boldsymbol{x}^c \boldsymbol{s}^c}{\mu}}$ be given and suppose that $\delta(\mathbf{x}, \mathbf{s}; \mu) < \sqrt{\frac{1-\xi^2}{1+4\kappa}}$. For any function φ satisfying AET1 and AET2 of Definition 2.3, we can say that after a full-Newton step we have $\mathbf{v}^+ > \xi\mathbf{e}$ and*

$$\delta(\boldsymbol{x}^c, \boldsymbol{s}^c; \mu) \le c_1(c_3 + 2 + 4\kappa)\delta(\boldsymbol{x}, \boldsymbol{s}; \mu)^2,$$

where $c_1, c_3 = \max\{1, c_2\} \in \mathbb{R}_+$.

The next lemma investigates the effect of the full-Newton step on the duality gap.

Lemma 3.3 ([31]). *Let $\delta := \delta(\boldsymbol{x}, \boldsymbol{s}; \mu)$ and suppose that the vectors $\boldsymbol{x}^c$ and $\boldsymbol{s}^c$ are obtained using a full-Newton step, thus $\boldsymbol{x}^c = \boldsymbol{x} + \Delta\boldsymbol{x}$ and $\boldsymbol{s}^c = \boldsymbol{s} + \Delta\boldsymbol{s}$. For any function φ satisfying AET2 of Definition 2.3 with $c_2 \in \mathbb{R}_+$, we have that*

$$(\boldsymbol{x}^c)^T \boldsymbol{s}^c \le \mu\left(n + (c_2 + 1)\delta^2\right).$$

In the following part we present several technical lemmas that will be used later in the analysis.

3.1.2. Technical lemmas

We present the first technical lemma which is similar to Lemma 4.5 in [38].

Lemma 3.4. *Let f be a function satisfying AETa from Proposition 2.1. Then,*

$$c_0\|\mathbf{e} - \mathbf{v}\| \le \|f(\mathbf{v})\| \le 2c_1\|\mathbf{e} - \mathbf{v}^2\|.$$

Proof. The proof is similar to the proof of Lemma 4.5 in [38]. The lower bound is the same as in [38], except we have c_0 instead of $\bar{k}$. To prove the upper bound part, we need to use AETa from Proposition 2.1 and the same technique. □

The following lemma is a generalization of Lemma 5.3 in [16], where the PC IPA is based on the function $\varphi(t) = t - \sqrt{t}$. In this case $c_0 = c = 1$.

Lemma 3.5. *Consider a function φ from the new class of AET functions given in Definition 2.3. Then,*

$$\|\mathbf{d}_x^p \mathbf{d}_s^p\| \le \frac{c^2\, n(1 + 2\kappa)\left(1 + \frac{2}{c_0}\,\delta^+\right)^2}{2},$$

where $\delta^+ = \delta(\mathbf{x}^c, \mathbf{s}^c; \mu)$.

Proof.

Similarly to [16] (page 2641), we have

$$(1 + 4\kappa)\sum_{i \in I_+} d_{x_i}^p d_{s_i}^p + \sum_{i \in I_-} d_{x_i}^p d_{s_i}^p \ge 0. \tag{14}$$

The second equation of system (11) yields

$$\sum_{i\in I_+} d^p_{x_i} d^p_{s_i} \le \frac{1}{4}\sum_{i\in I_+}\left(d^p_{x_i}+d^p_{s_i}\right)^2 \le \frac{1}{4}\sum_{i=1}^{n}\left(d^p_{x_i}+d^p_{s_i}\right)^2 = \frac{1}{4}\|\mathbf{d}^p_x+\mathbf{d}^p_s\|^2 = \frac{1}{4}\left\|c\mathbf{v}^+\right\|^2. \tag{15}$$

By combining (14) with (15), we obtain

$$\begin{aligned} c^2\|\mathbf{v}^+\|^2 &= \|\mathbf{d}^p_x+\mathbf{d}^p_s\|^2 = \|\mathbf{d}^p_x\|^2+\|\mathbf{d}^p_s\|^2 + 2\left(\sum_{i\in I_+} d^p_{x_i}d^p_{s_i} + \sum_{i\in I_-} d^p_{x_i}d^p_{s_i}\right)\\ &\ge \|\mathbf{d}^p_x\|^2+\|\mathbf{d}^p_s\|^2 - 8\kappa\sum_{i\in I_+} d^p_{x_i}d^p_{s_i} \ge \|\mathbf{d}^p_x\|^2+\|\mathbf{d}^p_s\|^2 - 2c^2\kappa\,\|\mathbf{v}^+\|^2. \end{aligned}$$

Hence, $\|\mathbf{d}^p_x\|^2+\|\mathbf{d}^p_s\|^2 \le c^2(1+2\kappa)\|\mathbf{v}^+\|^2$. Similar to the computations given in [16] (page 2642), we obtain

$$\|\mathbf{v}^+\| \le \sqrt{n}(\sigma^+ + 1), \tag{16}$$

where $\sigma^+ = \|\mathbf{e}-\mathbf{v}^+\|$. We will compute a lower bound on δ^+. Lemma 3.4 implies

$$\delta^+ = \frac{1}{2}\left\|\mathbf{p}_{\mathbf{v}^+}\right\| = \frac{1}{2}\|f(\mathbf{v}^+)\| \ge \frac{c_0}{2}\left\|\mathbf{e}-\mathbf{v}^+\right\| = \frac{c_0}{2}\sigma^+, \tag{17}$$

where f is given in (8). From (16) and (17) we derive

$$\|\mathbf{v}^+\| \le \sqrt{n}\left(1+\frac{2}{c_0}\,\delta^+\right). \tag{18}$$

Taking everything into consideration, we have

$$\|\mathbf{d}^p_x\mathbf{d}^p_s\| \le \frac{1}{2}\left(\|\mathbf{d}^p_x\|^2+\|\mathbf{d}^p_s\|^2\right) \le \frac{c^2}{2}(1+2\kappa)\left\|\mathbf{v}^+\right\|^2 \le \frac{c^2\, n(1+2\kappa)\left(1+\frac{2}{c_0}\,\delta^+\right)^2}{2}.$$

which gives the result. □

Note that the technique of this proof is similar to that of Lemma 5.3 in [16]. However, in this case we deal with AET functions belonging to the class defined in Definition 2.3.

In the following part we deal with the predictor step.

3.1.3. Predictor step

In the following lemma we prove the strict feasibility of the predictor step.

Lemma 3.6. *Let $(\mathbf{x}^c,\mathbf{s}^c)\in\mathcal{F}^+$ and $\mu>0$ such that $\delta^+ := \delta(\mathbf{x}^c,\mathbf{s}^c;\mu) < \frac{c_0}{2}$. Furthermore, let $\theta\in\left(0,\frac{1}{c_4}\right)$, where $c_4=\max\{1,c\}$. Let $\mathbf{x}^p=\mathbf{x}^c+\theta\Delta^p\mathbf{x}$, $\quad \mathbf{s}^p=\mathbf{s}^c+\theta\Delta^p\mathbf{s}$ be the iterates after a predictor step. Then, for all functions φ satisfying the conditions given in Definition 2.3, we have $(\mathbf{x}^p,\mathbf{s}^p)\in\mathcal{F}^+$, if $u(\delta^+,\theta,n)>0$, where*

$$u(\delta^+,\theta,n) := \left(1-\frac{2}{c_0}\delta^+\right)^2 - \frac{\theta^2c^2\,n(1+2\kappa)\left(1+\frac{2}{c_0}\,\delta^+\right)^2}{2(1-c\theta)}. \tag{19}$$

Proof. By using the definition of $\mathbf{v}^+$ after system (11) we have

$$\mathbf{x}^p(\alpha) = \frac{\mathbf{x}^c}{\mathbf{v}^+}(\mathbf{v}^+ + \alpha\,\theta\,\mathbf{d}^p_x), \quad \mathbf{x}^p(\alpha) = \frac{\mathbf{s}^c}{\mathbf{v}^+}(\mathbf{v}^+ + \alpha\,\theta\,\mathbf{d}^p_s).$$

By combining the second equation of system (11) and the computations given in (5.17), page 2643 in [16], we have

$$\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha) = \mu\left(\left(\mathbf{v}^+\right)^2 + \alpha\theta\mathbf{v}^+\left(-c\,\mathbf{v}^+\right) + \alpha^2\,\theta^2\,\mathbf{d}^p_x\mathbf{d}^p_s\right) = \mu\left((1-c\alpha\theta)\left(\mathbf{v}^+\right)^2 + \alpha^2\,\theta^2\,\mathbf{d}^p_x\mathbf{d}^p_s\right). \tag{20}$$

Similarly to the idea from [16] and from (20), we get

$$\begin{aligned} \min\left(\frac{\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha)}{\mu(1-c\alpha\theta)}\right) &= \min\left(\left(\mathbf{v}^+\right)^2 + \frac{\alpha^2\theta^2}{1-c\alpha\theta}\mathbf{d}^p_x\mathbf{d}^p_s\right)\\ &\ge \min\left(\left(\mathbf{v}^+\right)^2 - \frac{\alpha^2\theta^2}{1-c\alpha\theta}\|\mathbf{d}^p_x\mathbf{d}^p_s\|_\infty\mathbf{e}\right) \ge \min\left(\left(\mathbf{v}^+\right)^2 - \frac{\theta^2}{1-c\theta}\|\mathbf{d}^p_x\mathbf{d}^p_s\|_\infty\mathbf{e}\right). \end{aligned} \tag{21}$$

The last inequality holds due to the fact that

$$h(\alpha) = \frac{\alpha^2\theta^2}{1-c\alpha\theta}$$

is strictly monotone increasing for $0\le\alpha\le1$ and for each fixed $\theta\in\left(0,\frac{1}{c_4}\right)$, with $c_4=\max\{1,c\}$, $c>0$. From

$$|1-v_i^+| \le \|\mathbf{e}-\mathbf{v}^+\|, \quad \forall i=1,\dots,n$$

using the definition of σ_+ after (16) we have

$$1 - \sigma^+ \le v_i^+ \le 1 + \sigma^+, \quad \forall i = 1, \dots, n.$$

This and (17) imply

$$\min \left(\mathbf{v}^+\right)^2 \ge (1-\sigma^+)^2 \ge \left(1 - \frac{2}{c_0}\,\delta^+\right)^2. \tag{22}$$

By applying Lemmas 3.5 and (22), and by using that $\|\cdot\|_\infty \le \|\cdot\|$, we obtain

$$\min\left(\frac{\mathbf{x}^p(\alpha)\,\mathbf{s}^p(\alpha)}{\mu\,(1 - c\,\alpha\,\theta)}\right) \ge \left(1 - \frac{2}{c_0}\delta^+\right)^2 - \frac{\theta^2 c^2\, n(1+2\kappa)\left(1 + \frac{2}{c_0}\,\delta^+\right)^2}{2(1-c\theta)} =: u(\delta^+, \theta, n). \tag{23}$$

If $u(\delta^+, \theta, n) > 0$, then $\mathbf{x}^p(\alpha)\,\mathbf{s}^p(\alpha) > 0$ for $0 \le \alpha \le 1$. Therefore, $\mathbf{x}^p(\alpha)$ and $\mathbf{s}^p(\alpha)$ do not change sign on $0 \le \alpha \le 1$. Since $\mathbf{x}^p(0) = \mathbf{x}^c > \mathbf{0}$ and $\mathbf{s}^p(0) = \mathbf{s}^c > \mathbf{0}$, we obtain $\mathbf{x}^p(1) = \mathbf{x}^p > \mathbf{0}$ and $\mathbf{s}^p(1) = \mathbf{s}^p > \mathbf{0}$.

□

Consider

$$\mathbf{v}^p = \sqrt{\frac{\mathbf{x}^p\mathbf{s}^p}{\mu^p}}, \tag{24}$$

where $\mu^p = (1 - c\theta)\,\mu$. By substituting $\alpha = 1$ in (23), we get

$$\min\,(\mathbf{v}^p)^2 \ge u(\delta^+, \theta, n) > 0. \tag{25}$$

In the following lemma the effect of a predictor step and the update of μ on the proximity measure is analysed.

Lemma 3.7. *Consider a function φ from the new class of AET functions given in Definition 2.3. Furthermore, let $\delta^+ := \delta(\mathbf{x}^c, \mathbf{s}^c; \mu) < \frac{c_0}{2}$, $u(\delta^+, \theta, n) > \xi^2$, $\mathbf{v} > \xi\mathbf{e}$, with $0 \le \xi \le \frac{1}{2}$, $\mu^p = (1 - c\theta)\,\mu$, where $\theta \in \left(0, \frac{1}{c_4}\right)$, with $c_4 = \max\{1, c\}$, and let $\mathbf{x}^p$ and $\mathbf{s}^p$ denote the iterates after a predictor step. Then, $\mathbf{v}^p > \xi\mathbf{e}$ and*

$$\delta^p := \delta(\mathbf{x}^p, \mathbf{s}^p; \mu^p) \le c_1((2 + c_3) + 4\kappa)\delta^2 + \frac{c_1\, c^2\, n\,\theta^2}{2(1-c\theta)}\,(1+2\kappa)\left(1 + \frac{2}{c_0}\delta^+\right)^2,$$

where $\delta := \delta(\mathbf{x}, \mathbf{s}; \mu)$ and $c_3 = \max\{1, c_2\}$.

Proof. Since $u(\delta^+, \theta, n) > \xi^2$, from Lemma 3.6 we get $\mathbf{x}^p, \mathbf{s}^p > \mathbf{0}$. By using (25), we have

$$\min\,(\mathbf{v}^p) \ge \sqrt{u(\delta^+, \theta, n)} > \xi,$$

which proves the first part of the lemma. Let us compute δ^p by using (7)

$$\delta^p := \frac{\|f(\mathbf{v}^p)\|}{2} \le c_1 \left\|\mathbf{e} - (\mathbf{v}^p)^2\right\|, \tag{26}$$

where the inequality in (26) follows from Lemma 3.4.

After some calculations, (20) implies

$$(\mathbf{v}^p)^2 = \left(\mathbf{v}^+\right)^2 + \frac{\theta^2}{1 - c\theta}\mathbf{d}_x^p\mathbf{d}_s^p, \tag{27}$$

hence we get

$$\delta^p \le c_1 \left\|\mathbf{e} - \left(\mathbf{v}^+\right)^2 - \frac{\theta^2}{1-c\theta}\mathbf{d}_x^p\mathbf{d}_s^p\right\| \le c_1\left(\left\|\mathbf{e} - \left(\mathbf{v}^+\right)^2\right\| + \frac{\theta^2}{1-c\theta}\left\|\mathbf{d}_x^p\mathbf{d}_s^p\right\|\right). \tag{28}$$

By using (28) and inequality (5.25) from [16] we obtain

$$\left\|\mathbf{e} - \left(\mathbf{v}^+\right)^2\right\| \le \|\mathbf{e} - \mathbf{v}^2 - \mathbf{v}\mathbf{p}_\varphi\| + \left\|\frac{\mathbf{p}_\varphi^2}{4}\right\| + \left\|\frac{\mathbf{q}_\varphi^2}{4}\right\|, \tag{29}$$

where $\mathbf{q}_\varphi = \mathbf{d}_x^c - \mathbf{d}_s^c$ and $\mathbf{p}_\varphi = f(\mathbf{v})$. After some calculations we obtain

$$\left\|\mathbf{e} - \mathbf{v}^2 - \mathbf{v}\mathbf{p}_\varphi\right\| = \left\|\mathbf{e} - \mathbf{v}^2 - \mathbf{v}f(\mathbf{v})\right\| \le c_3\left\|\frac{\mathbf{p}_\varphi^2}{4}\right\|, \tag{30}$$

where $c_3 = \max\{1, c_2\}$ and the last inequality follows from condition AETb of Proposition 2.1. We also used a modification of Lemma 3.4. By applying Lemma 4.4 from [38], (29) and (30) we have

$$\left\|\mathbf{e} - \left(\mathbf{v}^+\right)^2\right\| \le (1+c_3)\left\|\frac{\mathbf{p}_\varphi^2}{4}\right\| + \left\|\frac{\mathbf{q}_\varphi^2}{4}\right\| \le (1+c_3)\left(\frac{\|\mathbf{p}_\varphi\|}{2}\right)^2 + \left(\frac{\|\mathbf{q}_\varphi\|}{2}\right)^2 \le ((2+c_3)+4\kappa)\delta^2. \tag{31}$$

Lemmas 3.5, (28) and (31) yields the desired result

$$\delta^p \leq c_1((2+c_3)+4\kappa)\delta^2 + \frac{c_1\, c^2\, n\, \theta^2}{2(1-c\theta)}\,(1+2\kappa)\left(1+\frac{2}{c_0}\delta^+\right)^2. \qquad \square$$

In the following lemma we give an upper bound for the duality gap after a main iteration.

Lemma 3.8. *Let φ be a member of the new class of functions proposed in Definition 2.3 and let $\theta \in \left(0, \frac{1}{c_4}\right)$, where $c_4 = \max\{1, c\}$. If $\mathbf{x}^p$ and $\mathbf{s}^p$ are the iterates obtained after the predictor step of Algorithm 3.1, then*

$$(\mathbf{x}^p)^T\mathbf{s}^p \leq \left(1 - c\theta + \frac{c^2\theta^2}{2}\right)(\mathbf{x}^c)^T\mathbf{s}^c < \frac{\left(n+(c_2+1)\delta^2\right)\mu^p}{1-c\theta}.$$

Proof. By substituting $\alpha = 1$ into (20) and using (24), we obtain

$$(\mathbf{x}^p)^T\mathbf{s}^p = \mu^p\mathbf{e}^T(\mathbf{v}^p)^2 = \mu\mathbf{e}^T\left((1-c\theta)\left(\mathbf{v}^+\right)^2 + \theta^2\mathbf{d}_x^p\mathbf{d}_s^p\right) = (1-c\theta)(\mathbf{x}^c)^T\mathbf{s}^c + \mu\theta^2\left(\mathbf{d}_x^p\right)^T\mathbf{d}_s^p. \tag{32}$$

By applying the technique given in the proof of Lemma 5.7 from [16], we get

$$(\mathbf{x}^p)^T\mathbf{s}^p \leq \left(1 - c\theta + \frac{c^2\theta^2}{2}\right)(\mathbf{x}^c)^T\mathbf{s}^c. \tag{33}$$

If $\theta \in \left(0, \frac{1}{c_4}\right)$, where $c_4 = \max\{1, c\}$, then

$$1 - c\theta + \frac{c^2\theta^2}{2} < 1. \tag{34}$$

From (33) and (34), $\mu^p = (1-c\theta)\,\mu$ and Lemma 3.3 we have

$$(\mathbf{x}^p)^T\mathbf{s}^p \leq \left(1 - c\theta + \frac{c^2\theta^2}{2}\right)(\mathbf{x}^c)^T\mathbf{s}^c < (\mathbf{x}^c)^T\mathbf{s}^c \leq \mu\left(n+(c_2+1)\delta^2\right) = \frac{\left(n+(c_2+1)\delta^2\right)\mu^p}{1-c\theta},$$

and we get the result. $\square$

In the following section we prove that the algorithm is well defined.

4. Complexity bound

In this section we prove that the algorithm is well defined and we obtain its polynomial iteration complexity in the size of the problem, starting point's duality gap, the accuracy parameter and in the special parameter $\kappa \geq 0$. For this, we need to assure that the function u defined in (19) is greater or equal to ξ^2. For this, we introduce the following notation:

$$Q(\tau) = \frac{\left(1 - \frac{1}{c_0}\tau\right)^2 - \xi^2}{\left(1 + \frac{1}{c_0}\tau\right)^2}. \tag{35}$$

When $\xi = 0$, then $Q(\tau)$ is a positive number. However, when $\xi > 0$, we have to ensure that $Q(\tau) > 0$ is satisfied, hence we need $\tau < (1-\xi)c_0$.
Furthermore, we introduce

$$Q_L = \frac{\theta^2 c^2\, n(1+2\kappa)}{2(1-c\theta)}. \tag{36}$$

The next lemma gives a condition related to the parameters τ and θ for which the PC IPA is well defined.

Lemma 4.1. *Let φ be a function from the new class of AET functions given in Definition 2.3. Let $0 \leq \xi \leq \frac{1}{2}$, $\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3+2+4\kappa)}$, where $r \geq 2$, and $0 < \theta \leq \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3+2+4\kappa)\sqrt{n}}$, where $q \geq 1$, $c > 0$, $c_5 = \max\left\{\frac{1}{4}, c_1\right\}$ and $c_3 = \max\{1, c_2\}$. If (i) $r \leq 2q$ and (ii) $Q_L < Q(\tau)$, then the PC IPA proposed in Algorithm 3.1 is well defined.*

Proof. We show that with these values for θ the condition $\theta \in \left(0, \frac{1}{c_4}\right)$ is satisfied, where $c_4 = \max\{1, c\}$. By using $c_5 \geq \frac{1}{4}$, $c_3 \geq 1$, $0 < \theta \leq \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3+2+4\kappa)\sqrt{n}}$, $n \geq 1$, $q \geq 1$ and $\kappa \geq 0$, we obtain

$$\theta \leq \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3+2+4\kappa)\sqrt{n}} < \frac{1}{c_4}. \tag{37}$$

In the following part we will use Lemma 3.6. We want to show that $u(\delta^+, \theta, n) > 0$. For this, first we give an upper bound for Q_L.

From $\theta \leq \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3+2+4\kappa)\sqrt{n}}$, $c_3 \geq 1$, $c_5 \geq \frac{1}{4}$, $n \geq 1$, $c_4 = \max\{1, c\}$ and $\kappa \geq 0$ we get

$$\frac{1}{2(1-c\theta)} \leq \frac{6q}{12q-1}. \tag{38}$$

We also consider

$$\theta \leq \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3+2+4\kappa)\sqrt{n}} \leq \frac{1}{16qc_5c(1+2\kappa)\sqrt{n}}. \tag{39}$$

By using (38) and (39), we get

$$Q_L = \frac{\theta^2c^2\, n(1+2\kappa)}{2(1-c\theta)} \leq c^2 n\,(1+2\kappa)\ \frac{6q}{12q-1}\ \frac{1}{16qc_5c(1+2\kappa)\sqrt{n}}\ \frac{\sqrt{1-\xi^2}}{16qc_5c(c_3+2+4\kappa)\sqrt{n}} \leq \frac{3}{4(12q-1)}\frac{r}{q}\tau, \tag{40}$$

where in the last inequality we used that $\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3+2+4\kappa)}$ and $c_5 \geq \frac{1}{4}$. Following the computations in (40) we can show that

$$c_5 Q_L < \frac{3}{16(12q-1)}\frac{r}{q}\tau < \frac{3}{4(12q-1)}\frac{r}{q}\tau.$$

After that we give different upper bounds for δ^+. Let $(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+$ such that $\mathbf{v} > \xi\mathbf{e}$ and $\delta(\mathbf{x},\mathbf{s};\mu) \leq \tau$. In order to use Lemma 3.2, we have to show that $\tau < \sqrt{\frac{1-\xi^2}{1+4\kappa}}$. This follows from $\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3+2+4\kappa)}$, $c_5 \geq \frac{1}{4}$, $c_3 \geq 1$ and $r \geq 2$. After a corrector step, by using Lemma 3.2, we have

$$\delta^+ := \delta(\mathbf{x}^c,\mathbf{s}^c;\mu) \leq c_1(c_3+2+4\kappa)\delta(\mathbf{x},\mathbf{s};\mu)^2,$$

where $c_3 = \max\{1, c_2\} \in \mathbb{R}_+$. By using $\delta(\mathbf{x},\mathbf{s};\mu) \leq \tau$, $r \geq 2$ and $c_1 \leq c_5$, we obtain

$$\delta^+ \leq c_5(c_3+2+4\kappa)\tau^2 = \frac{\sqrt{1-\xi^2}}{8r}\tau \leq \frac{1}{16}\tau. \tag{41}$$

From $c_5 \geq \frac{1}{4}$, $c_3 \geq 1$ and $\kappa \geq 0$ we obtain

$$\delta^+ \leq c_5(c_3+2+4\kappa)\tau^2 \leq \frac{1-\xi^2}{64r^2c_5(c_3+2+4\kappa)} \leq \frac{1}{48r^2}. \tag{42}$$

Moreover, by using $\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3+2+4\kappa)}$, $c_3 \geq 1$, $c_5 \geq \frac{1}{4}$, $c_0 \geq \frac{1}{4}$, $r \geq 2$ and (41), we have

$$\delta^+ \leq \frac{1}{16}\tau < \frac{1}{2}\tau \leq \frac{1}{4r(3+4\kappa)} \leq \frac{1}{24} < \frac{c_0}{4} < \frac{c_0}{2}. \tag{43}$$

Hence, $\tau < \frac{c_0}{2}$. Using $\xi \leq \frac{1}{2}$, we have

$$\tau < \frac{c_0}{2} \leq (1-\xi)c_0, \tag{44}$$

hence the $Q(\tau) > 0$ is also satisfied. Now we want to show that $u(\delta^+,\theta,n) > 0$. For this, consider the function

$$u(\delta^+,\theta,n) = \left(1 - \frac{2}{c_0}\delta^+\right)^2 - \frac{\theta^2c^2\, n(1+2\kappa)\left(1+\frac{2}{c_0}\,\delta^+\right)^2}{2(1-c\theta)},$$

which is monotone decreasing with respect to δ^+ in the interval $\left(0, \frac{c_0}{2}\frac{1+Q_L}{1-Q_L}\right)$. From (40) we have $Q_L < 1$. By using this and (43) we have

$$\delta^+ \leq \frac{1}{16}\tau < \frac{c_0}{2} < \frac{c_0}{2}\frac{1+Q_L}{1-Q_L}. \tag{45}$$

Using that $u(\delta^+,\theta,n)$ is monotone decreasing with respect to δ^+ in the above mentioned interval, from condition (ii) given in the lemma and (43) we obtain

$$u(\delta^+,\theta,n) > u\left(\frac{1}{2}\tau,\theta,n\right) = \left(1-\frac{1}{c_0}\tau\right)^2 - Q_L\left(1+\frac{1}{c_0}\tau\right)^2 > \xi^2 > 0. \tag{46}$$

Hence, conditions $u(\delta^c,\theta,n) > 0$ from Lemma 3.6 and $u(\delta^c,\theta,n) > \xi^2$ from Lemma 3.7 are satisfied.
From Lemma 3.7 and $c_1 \leq c_5$, we get

$$\delta^p \leq c_5((2+c_3)+4\kappa)\delta^2 + \frac{c_5\,c^2\,n\,\theta^2}{2(1-c\theta)}\,(1+2\kappa)\left(1+\frac{2}{c_0}\delta^+\right)^2, \tag{47}$$

where $\delta := \delta(\mathbf{x},\mathbf{s};\mu)$.

By using (40), (42) and $c_0 \geq \frac{1}{4}$ and the bound obtained on $c_5 Q_L$ we have

$$c_5 Q_L \left(1 + \frac{2}{c_0}\delta^+\right)^2 < \frac{3}{4(12q-1)} \frac{r}{q} \left(1 + \frac{1}{6r^2}\right)^2 \tau. \tag{48}$$

From $r \geq 2$, $q \geq 1$ and $r \leq 2q$ we obtain

$$\frac{3}{4(12q-1)} \frac{r}{q} \left(1 + \frac{1}{6r^2}\right)^2 \leq \frac{3}{44} \cdot 2 \cdot \frac{25^2}{24^2} < \frac{15}{16}. \tag{49}$$

From (41), (47), (48) and (49) we get

$$\delta^p < \left(\frac{1}{16} + \frac{15}{16}\right)\tau = \tau, \tag{50}$$

hence if we have a feasible solution for which $\delta(\mathbf{x}, \mathbf{s}; \mu) \leq \tau$, then after a corrector and predictor step we will get that $\delta^p(\mathbf{x}^p, \mathbf{s}^p, \mu^p) < \tau$. □

Example 4.1. Kheirfam [24] introduced a PC IPA for horizontal $P_*(\kappa)$-LCPs using the function $\varphi_2(t) = \sqrt{t}$ in the AET technique. Note that this function belongs to the class given in Definition 2.3. For $c_0 = c_1 = c_2 = 1$ and $c = 2$ the function satisfies the conditions given in Definition 2.3. Hence, $c_3 = 1$, $c_4 = 2$ and $c_5 = 1$. In [24], the parameters $\tau = \frac{1}{2(1+2\kappa)}$ and $\theta = \frac{1}{4(1+2\kappa)\sqrt{n}}$ were used as proximity and update parameters. In our case, using the parameters we have $\tau = \frac{1}{8r(3+4\kappa)}$, with $r \geq 2$ and $0 < \theta \leq \frac{1}{32q(3+4\kappa)\sqrt{n}}$, $q \geq 1$. In this case the complexity result is obtained not only for a specific value of the parameters τ and θ. However, it should be noted that these are not the only possible values for the parameters for which conditions from Definition 2.3 are satisfied for the function φ. This means that we can derive other possible values and bounds for the parameters τ and θ.

The following lemma gives a sufficient condition for satisfying condition (ii) from Lemma 4.1.

Lemma 4.2. *Let φ be a function belonging to the AET class defined in Definition 2.3. Consider $0 \leq \xi \leq \frac{1}{2}$, $\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3+2+4\kappa)}$, $r \geq 2$, and $0 < \theta \leq \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3+2+4\kappa)\sqrt{n}}$, where $q \geq \frac{1}{2}r$, $c_5 = \max\left\{\frac{1}{4}, c_1\right\}$, $c_3 = \max\{1, c_2\}$ and $c_4 = \max\{1, c\}$ with $c > 0$. Consider the function $Q(\tau)$ given in (35). If*

$$\frac{1}{16q^2} < Q\left(\frac{1}{6r}\right), \tag{51}$$

then condition (ii) of Lemma 4.1 is satisfied.

Proof. By using (38) we have

$$\frac{1}{2(1-c\theta)} < 1. \tag{52}$$

From the properties of the function φ given in Definition 2.3, $c_5 \geq \frac{1}{4}$, $\kappa \geq 0$ and (39), we obtain

$$Q_L = \frac{\theta^2 c^2\, n(1+2\kappa)}{2(1-c\theta)} < c^2 n(1+2\kappa)\, \frac{1}{256q^2c^2c_5^2(1+2\kappa)^2 n} \leq \frac{1}{16q^2}. \tag{53}$$

Furthermore, from $\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3+2+4\kappa)}$, $c_3 \geq 1$, $c_5 \geq \frac{1}{4}$ and $\kappa \geq 0$ we obtain

$$\tau \leq \frac{1}{6r}. \tag{54}$$

Using $Q'(\tau)$ it can be shown that the function $Q(\tau)$ is strictly decreasing on $\left(0, \frac{c_0(2-\xi^2)}{2}\right)$. However, by using $r \geq 2$, $0 \leq \xi \leq \frac{1}{2}$, and $c_0 \geq \frac{1}{4}$, we get that $\frac{1}{6r} < \frac{c_0}{2}(2-\xi^2)$. Hence, the function $Q(\tau)$ is strictly decreasing with respect to τ on $(0, \frac{1}{6r}]$, as well. Thus, by using (54), we get

$$Q(\tau) \geq Q\left(\frac{1}{6r}\right). \tag{55}$$

From (51), (52), (53) and (55) we obtain

$$Q_L = \frac{\theta^2 c^2\, n(1+2\kappa)}{2(1-c\theta)} < \frac{1}{16q^2} < Q\left(\frac{1}{6r}\right) \leq Q(\tau),$$

which yields the result. □

In the following lemma we show that condition (i) of Lemma 4.1 implies condition (ii) of Lemma 4.1.

Lemma 4.3. *Let φ be a function belonging to the AET class defined in Definition 2.3. Consider $\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3+2+4\kappa)}$, $0 \leq \xi \leq \frac{1}{2}$, $r \geq 2$, and $0 < \theta \leq \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3+2+4\kappa)\sqrt{n}}$, where $q \geq 1$, $c_5 = \max\left\{\frac{1}{4}, c_1\right\}$ and $c_3 = \max\{1, c_2\}$. Consider the function $Q(\tau)$ given in (35). If condition (i) from Lemma 4.1 is satisfied, then condition (ii) from Lemma 4.1 also holds.*

Proof. Let us introduce the following function

$$g(r) = \frac{5}{4r} \frac{\left(1 + \frac{1}{6rc_0}\right)}{\sqrt{\left(1 - \frac{1}{6rc_0}\right)^2 - \xi^2}},$$

which is decreasing with respect to r, where for a given AET function φ, c_0 and ξ are constants. Thus, for $r \geq 2$, using $c_0 \geq \frac{1}{4}$ and $\xi \leq \frac{1}{2}$, we have

$$g(r) \leq g(2) < 2. \tag{56}$$

From (56) and condition (i) of Lemma 4.1 we get

$$\frac{1}{\sqrt{Q\left(\frac{1}{6r}\right)}} < \frac{8}{5} r < 4q.$$

Hence, if condition (i) of Lemma 4.1 holds, then (51) is satisfied. Using Lemma 4.2, we obtain that condition (ii) from Lemma 4.1 also holds. □

We summarize the obtained results in the following corollary.

Corollary 4.4. *Let φ be a function belonging to the AET class defined in Definition 2.3. Consider $\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3+2+4\kappa)}$, $r \geq 2$, and $0 < \theta \leq \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3+2+4\kappa)\sqrt{n}}$, where $q \geq 1$, $c_5 = \max\left\{\frac{1}{4}, c_1\right\}$ and $c_3 = \max\{1, c_2\}$. If $q \geq \frac{1}{2}r$, then the PC IPA proposed in Algorithm 3.1 is well defined.*

In the following lemma we provide a bound on the number of iterations performed by the introduced PC IPA.

Lemma 4.5. *Let φ be a function belonging to the AET class defined in Definition 2.3. Consider $\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3+2+4\kappa)}$, $r \geq 2$, and $0 < \theta \leq \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3+2+4\kappa)\sqrt{n}}$, where $q \geq \frac{1}{2}r$, $c_5 = \max\left\{\frac{1}{4}, c_1\right\}$ and $c_3 = \max\{1, c_2\}$. Furthermore, let $\mu^0 = \frac{\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{n}$ and $\delta(\mathbf{x}^0, \mathbf{s}^0; \mu^0) \leq \tau$. Moreover, we denote by $\mathbf{x}^k$ and $\mathbf{s}^k$ the iterates obtained after k iterations of Algorithm 3.1. Then, $\left(\mathbf{x}^k\right)^T \mathbf{s}^k \leq \epsilon$ for*

$$k \geq 1 + \left\lceil \frac{1}{c\theta} \log \frac{(37 + c_2)\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{36\epsilon} \right\rceil.$$

Proof. From $\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3+2+4\kappa)}$, $r \geq 2$, $c_5 \geq \frac{1}{4}$, $c_3 \geq 1$, $n \geq 1$ and $\kappa \geq 0$, we get

$$\delta^2 < \frac{1}{4(3 + 4\kappa)^2} < \frac{n}{36}. \tag{57}$$

By using Lemma 3.8, we obtain

$$\left(\mathbf{x}^k\right)^T \mathbf{s}^k < \frac{\left(n + (c_2 + 1)\delta^2\right)\mu^k}{1 - c\theta} < \frac{(37 + c_2)n\mu^k}{36(1 - c\theta)} = \frac{37 + c_2}{36}(1 - c\theta)^{k-1}\left(\mathbf{x}^0\right)^T \mathbf{s}^0.$$

Note that the inequality $\left(\mathbf{x}^k\right)^T \mathbf{s}^k \leq \epsilon$ holds if $\frac{37+c_2}{36}(1 - c\theta)^{k-1}\left(\mathbf{x}^0\right)^T \mathbf{s}^0 < \epsilon$. By taking logarithms, we obtain

$$(k - 1)\log(1 - c\theta) + \log \frac{(37 + c_2)\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{36} < \log \epsilon.$$

From $\log(1 + \theta) \leq \theta$, $\theta \geq -1$, we get that the above inequality holds if

$$-c\theta(k - 1) + \log \frac{(37 + c_2)\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{36} \leq \log \epsilon,$$

and we obtain the result. □

We give another example, where $\varphi_4(t) = t^2 - t + \sqrt{t}$.

Example 4.2. Illés et al. [38] proposed a PC IPA for $P_*(\kappa)$-LCPs using the function $\varphi_4(t) = t^2 - t + \sqrt{t}$ in the AET technique. Note that this function belongs to the class given in Definition 2.3. For $c_0 = \frac{1}{2}$, $c_1 = 2$, $c_2 = 8$ and $c = \frac{1}{2}$, the function satisfies the conditions given in Definition 2.3. Hence, $c_3 = 8$, $c_4 = 1$ and $c_5 = 2$. In [38] the function $L(\tau) = \frac{(1-2\tau)^2}{(1+2\tau)^2}$ was used, which corresponds to $Q(\tau)$ given in (35) with $c_0 = \frac{1}{2}$ and $\xi = 0$. In [38] $\tau = \frac{1}{2r(10+4\kappa)}$, $r \geq 2$ and $0 < \theta \leq \frac{1}{q(10+4\kappa)\sqrt{n}}$, $q \geq 2$. In our case $\tau = \frac{1}{16r(10+4\kappa)}$, with $r \geq 2$ and $0 < \theta \leq \frac{1}{32q(10+4\kappa)\sqrt{n}}$, $q \geq 1$. Hence, with the corresponding values of parameters, we get different values and bounds for τ and θ. However, this does not affect the complexity result obtained in Theorem 4.6.

In the following theorem we show that the PC IPA has polynomial iteration complexity in the size of the problem, the starting point's duality gap, the accuracy parameter and the special parameter $\kappa \geq 0$.

Theorem 4.6. *Let φ be a function belonging to the AET class defined in Definition 2.3. Consider $\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3+2+4\kappa)}$, $r \geq 2$, and $0 < \theta \leq \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3+2+4\kappa)\sqrt{n}}$, where $q \geq \frac{1}{2}r$, $c_5 = \max\left\{\frac{1}{4}, c_1\right\}$ and $c_3 = \max\{1, c_2\}$. Then, the PC IPA proposed in Algorithm 3.1 is well defined and it performs at most*

$$\mathcal{O}\left((c_3 + 2 + 4\kappa)\sqrt{n}\log\frac{(37 + c_2)n\mu^0}{36\epsilon}\right)$$

iterations. The output is a pair $(\mathbf{x}, \mathbf{s})$ satisfying $\mathbf{x}^T\mathbf{s} \leq \epsilon$.

Proof. It follows from Corollary 4.4 and Lemma 4.5. □

Note that the constants do not play an important role in Theorem 4.6, because this is about worst-case complexity and it shows that it depends on the parameter κ and the size of the problem.

In the following proposition we provide bounds on the proximity and update parameters for which the PC IPA is well defined and for which the complexity bound from Theorem 4.6 is obtained.

Proposition 4.1. *Let φ be a function belonging to the AET class defined in Definition 2.3 and consider $0 < \tau \leq \frac{\sqrt{1-\xi^2}}{16c_5(c_3+2+4\kappa)}$ and $0 < \theta \leq \frac{1}{c_4\sqrt{n}}\tau$. Then, the PC IPA proposed in Algorithm 3.1 is well defined and it performs at most*

$$\mathcal{O}\left((c_3 + 2 + 4\kappa)\sqrt{n}\log\frac{(37 + c_2)n\mu^0}{36\epsilon}\right)$$

iterations. The output is a pair $(\mathbf{x}, \mathbf{s})$ satisfying $\mathbf{x}^T\mathbf{s} \leq \epsilon$.

Proof. If $\tau \leq \frac{\sqrt{1-\xi^2}}{16c_5(c_3+2+4\kappa)}$, then we can find $r \geq 2$ such that

$$\tau = \frac{\sqrt{1-\xi^2}}{8rc_5(c_3 + 2 + 4\kappa)}. \tag{58}$$

From $\theta \leq \frac{1}{c_4\sqrt{n}}\tau$ and $\tau \leq \frac{\sqrt{1-\xi^2}}{16c_5(c_3+2+4\kappa)}$ we have

$$\theta \leq \frac{1}{c_4\sqrt{n}}\tau \leq \frac{\sqrt{1-\xi^2}}{16c_5c_4(c_3 + 2 + 4\kappa)\sqrt{n}}.$$

Hence, we can find $q \geq 1$ such that

$$\theta = \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3 + 2 + 4\kappa)\sqrt{n}}. \tag{59}$$

By using (58), (59) and $\theta \leq \frac{1}{c_4\sqrt{n}}\tau$, we get

$$\theta = \frac{\sqrt{1-\xi^2}}{16qc_5c_4(c_3 + 2 + 4\kappa)\sqrt{n}} = \frac{r}{2q}\frac{\tau}{c_4\sqrt{n}} \leq \frac{1}{c_4\sqrt{n}}\tau,$$

therefore $q \geq \frac{1}{2}r$ holds. Hence, all conditions from Lemma 4.1 are satisfied. By using Corollary 4.4 and Lemma 4.5, we get the result. □

In the following section we present the implemented algorithm and the numerical results.

5. Numerical results

In the C+ + programming language we implemented Algorithm 5.1. All computations were made on a desktop computer with an Intel quad-core 3.3 GHz processor and 32 GB RAM. We set $\epsilon = 10^{-5}$. Note that Algorithm 5.1 differs from Algorithm 3.1. We used $\theta = 0.999$ instead of the theoretical one. In three cases the algorithm could not find an ϵ-optimal solution with this large value of θ. Therefore, we started to halve the value of θ and in these cases we stopped at $\frac{\theta}{16}$. Furthermore, it is important to mention that we aimed to stay in the feasible set and we did not take into consideration the neighbourhood of the central path. We used the proximity measure to warn the user if the new iterate is not in the corresponding and reasonably large neighbourhood of the central path. We used $\sigma = 0.95$ as a decrease parameter for the step length to avoid reaching the boundary of the feasible set. Finally, we calculated in

Algorithm 5.1 Implemented version of PC IPA for $P_*(\kappa)$-LCPs.

Let $\epsilon = 10^{-5}$ *be the accuracy parameter,* $\theta = 0.999$*,* $\sigma = 0.95$ *decrease parameter,* $0 < \alpha < 3$ *the step length,* $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^+$ *with* $\left(\mathbf{x}^0\right)^T \mathbf{s}^0 = n\mu^0$.
begin
  $k := 0;$
  while $\left(\mathbf{x}^k\right)^T \mathbf{s}^k > \epsilon$ ***do begin***
    (predictor step)
    compute $(\Delta^p \mathbf{x}^k, \Delta^p \mathbf{s}^k)$ *from system (11) using (5);*

    calculate $\kappa(\Delta^p \mathbf{x}^k)$*;*
    compute α *with*
    $\alpha_{x_h} := \left\{ -\frac{x_i}{\Delta x_i} : \Delta x_i < 0 \right\}$ *and* $\alpha_{s_h} := \left\{ -\frac{s_i}{\Delta s_i} : \Delta s_i < 0 \right\}$
    $\alpha_h := \sigma \cdot \min \{\alpha_{x_h}, \alpha_{s_h}\}$
    $\alpha := \min\{\alpha_h, 3\}$
    let $(\mathbf{x}^p)^k := \mathbf{x}^k + \alpha \Delta^p \mathbf{x}^k$ *and* $(\mathbf{s}^p)^k := \mathbf{s}^k + \alpha \Delta^p \mathbf{s}^k$;

    (corrector step)
    compute $(\Delta^c \mathbf{x}^k, \Delta^c \mathbf{s}^k)$ *from system (3) using (5);*
    compute α *with*
    $\alpha_{x_h} := \left\{ -\frac{x_i}{\Delta x_i} : \Delta x_i < 0 \right\}$ *and* $\alpha_{s_h} := \left\{ -\frac{s_i}{\Delta s_i} : \Delta s_i < 0 \right\}$
    $\alpha_h := \sigma \cdot \min \{\alpha_{x_h}, \alpha_{s_h}\}$
    $\alpha := \min\{\alpha_h, 3\}$
    let $\mathbf{x}^{k+1} = (\mathbf{x}^p)^k + \alpha \Delta^c \mathbf{x}^k, \quad \mathbf{s}^{k+1} := (\mathbf{s}^p)^k + \alpha \Delta^c \mathbf{s}^k$ *and* $\mu^{k+1} := (1-\theta)\frac{(\mathbf{x}^{k+1})^T \mathbf{s}^{k+1}}{n}$*;*

    $k := k+1$*;*

  end
end.

each predictor step the local κ defined as:

$$\kappa(\Delta x) = -\frac{1}{4} \frac{\Delta \mathbf{x}^T M \Delta \mathbf{x}}{\sum_{i \in \mathcal{I}_+(\Delta \mathbf{x})} \Delta \mathbf{x}_i (M \Delta \mathbf{x})_i}.$$

We can read more details regarding the implementation in [39].

However, it is important to mention that we solved the Newton system in a different way in this implementation, which caused fewer numerical errors. For this reason, we could set the maximum of α to 3.

Numerical results regarding PC IPA based on member functions of our new class of AET functions are shown in [16,38]. Darvay et al. [16] presented their numerical results by using their PC IPA based on the AET function φ_3 from Table 1 and the results were based on test cases that were generated by Illés and Morapitiye [34]. On the other hand, Illés et al. [38] used AET functions φ_1, φ_2 and φ_4 from Table 1 and presented their results on other test cases based on a special matrix having exponential κ. Other numerical results regarding PC IPAs using the AET technique can be found in [40], although that algorithm works in a wide neighbourhood of the central path. It is important to mention that the implemented version of the algorithm mentioned in [40] considered the neighbourhood during each iteration, hence the implementation of that algorithm was closer to the theoretical algorithm.

In this paper we present our numerical results using the matrices from the website [35]. First, consider a matrix A coming from the constraints of LP problems from the NETLIB test collection [41]. Hence, such a matrix captures the structure of those LP problems. Using these matrices we generated PSD matrices by computing $A^T A$ and AA^T. We called these "TR_PSD" matrices. Using these we generated bisymmetric matrices, called "TR_BS" matrices, as follows:

$$M = \begin{bmatrix} P & B \\ -B^T & Q \end{bmatrix}, \tag{60}$$

where P and Q are PSD matrices of size $n \times n$ and $m \times m$ and $B \in \mathbb{R}^{n \times m}$ is an arbitrary matrix. The bisymmetric matrices with the structure shown in (60) first appeared in a paper by Klafszky and Terlaky [42] related to symmetric, linearly constrained convex quadratic primal-dual problems. In the case of the PSD matrices P and Q we used the generated "TR_PSD" matrices.

Our aim was to generate $P_*(\kappa)$-matrices with specific values κ in order to analyse the influence of the size of κ on the practical iteration numbers. We generated $P_*(\kappa)$-matrices with the scaling technique shown in [5] from Theorem 3.5. Furthermore, the scaling technique was mentioned in [43], as well. We defined U and V as matrices having ones on the diagonal except for one element at the same index, which was $\sqrt{1+4\bar{\kappa}}$ in the case of matrix U and $\frac{1}{\sqrt{1+4\bar{\kappa}}}$ in the case of matrix V, where $\bar{\kappa}$ is the desired value of κ for the new modified matrix. Note that the elements that differ from one in the matrices U and V have the same index. Finally, the new matrix is obtained by computing $\bar{M} = UMV$.

Table 2
PC IPA on matrices with $\bar{\kappa} = 0$ and $\bar{\kappa} = 10000$.

Name of matrix	Size	$\bar{\kappa}=0$				$\bar{\kappa}=10000$			
		φ_1	φ_2	φ_4	$\psi_{1,5}$	φ_1	φ_2	φ_4	$\psi_{1,5}$
TR_PSD_1_2	27	**7**	**7**	8	10	11	11	**9**	13
TR_PSD_2_2	32	9	**8**	**8**	12	13	12	**11**	15
TR_PSD_3_2	41	8	8	**7**	11	12	12	**11**	15
TR_PSD_4_2	43	9	**8**	**8**	12	12	12	**10**	16
TR_PSD_5_2	46	8	8	**7**	11	12	12	**11**	15
TR_PSD_6_2	47	**7**	**7**	8	11	**11**	**11**	**11**	15
TR_PSD_7_2	48	**8**	**8**	**8**	12	12	12	**11**	15
TR_PSD_8_2	48	**7**	**7**	8	11	**11**	**11**	**11**	15
TR_PSD_9_2	48	8	8	**7**	11	12	12	**10**	16
TR_PSD_10_2	49	**8**	**8**	**8**	11	12	14	**10**	17
TR_PSD_11_2	50	9	**8**	**8**	11	12	12	**10**	16
TR_PSD_12_2	50	**8**	**8**	**8**	12	12	14	**10**	17
TR_PSD_13_2	56	7	7	**6**	8	11	11	**9**	11
TR_BS_1_2	59	8	8	**7**	11	12	12	**9**	14
TR_BS_2_2	68	8	8	**7**	10	12	12	**10**	12
TR_BS_3_2	73	8	8	**7**	9	13	13	**11**	14
TR_BS_4_2	74	8	8	**7**	10	12	12	**11**	15
TR_BS_5_2	75	8	8	**7**	11	**14**	15	17	19
TR_BS_6_2	80	**8**	**8**	**8**	11	11	11	**10**	14
TR_BS_7_2	81	**8**	**8**	**8**	12	12	12	**10**	15
TR_BS_8_2	84	**8**	9	**8**	12	14	**13**	14	19
TR_BS_9_2	88	8	8	**7**	11	11	12	**10**	15
TR_BS_10_2	89	**8**	**8**	**8**	12	12	12	**11**	15
TR_BS_11_2	89	**8**	**8**	**8**	10	13	**12**	**12**	16
TR_BS_12_2	90	**8**	**8**	**8**	12	12	12	**9**	13
TR_BS_13_2	90	9	9	**8**	12	13	13	**12**	16
TR_BS_14_2	94	**8**	**8**	**8**	11	**13**	**13**	**13**	15
TR_BS_15_2	95	8	8	**7**	10	**12**	**12**	**12**	14
TR_BS_16_2	95	8	8	**7**	10	13	13	**12**	15
TR_BS_17_2	96	8	8	**7**	10	13	13	**12**	15
TR_BS_18_2	97	**8**	**8**	**8**	11	13	12	**11**	15
TR_PSD_14_2	97	9	9	**8**	13	14	13	**12**	15
TR_BS_19_2	102	8	8	**7**	10	13	13	**12**	14
TR_BS_20_2	103	8	8	**7**	10	13	13	**11**	14
TR_BS_21_2	124	**9**	10	**9**	14	15	15	**14**	17
TR_BS_22_2	140	9	9	**8**	14	**14**	**14**	15	16
TR_BS_23_2	145	11	10	**9**	16	15	15	**14**	19
TR_BS_24_2	146	10	10	**9**	14	**15**	**15**	**15**	18
TR_BS_25_2	153	**9**	10	**9**	14	15	**14**	**14**	17

With the scaling technique defined above, we generated 39 test matrices with each of the following values $\bar{\kappa} = 0, 10, 100, 1000, 10000$. Due to the obtained similar results, we show the results only for $\bar{\kappa} = 0$ and $\bar{\kappa} = 10000$. We chose $\mathbf{x} = \mathbf{s} = \mathbf{e}$ as starting points, where $\mathbf{e}$ is the vector of all ones, and we generated the right-hand side with $\mathbf{q} = -M\mathbf{e} + \mathbf{e}$, ensuring the strict feasibility of the test LCP problems.

We compared the PC IPA based on the AET functions φ_1, φ_2 and φ_4 with the PC IPA based on the kernel function $\psi_{p,\sigma}(t) = \frac{t^{p+1}-1}{p+1} + \frac{e^{\sigma(1-t)}-1}{\sigma}$, $p \in [0,1]$, $\sigma \geq 1$, see [44,45]. In our implementation we used $\psi_{1,5}(t)$.

The results can be seen in Table 2 in the case of PSD matrices and matrices generated with the scaling technique with $\bar{\kappa} = 10000$. In Table 2 the first row contains the value of $\bar{\kappa}$ for the matrices used for the test cases, and the first column contains the names of the matrices that were used for testing. In the second column the sizes of the matrices can be found. Finally, in the last columns the numbers of iterations are shown in the case of the PC IPAs based on different search directions. Note that the boldface letter means that it represents the smallest number of iterations for a given size and matrix.

It can be seen that on this test set of problems the PC IPA based on the kernel function has the worst iteration numbers in most of the cases. Furthermore, the PC IPA based on the function φ_4 had the best performance in most of the cases. It should be mentioned that we obtained promising results even in the case when $\bar{\kappa} = 10000$. This might be due to the fact that these matrices can be scaled to matrices with $\bar{\kappa} = 0$. Hence, in some cases the value κ is not as significantly influential as expected from theoretical analyses. It would be interesting to generate another test set of problems, where the matrices cannot be scaled to ones with $\bar{\kappa} = 0$.

We showed the changes of the local κ in each iteration through the algorithm based on the function φ_4. The chosen matrix for this purpose was "TR_PSD_1", since the maximum local κ was the largest for this one. It is easy to see that even in the case when the value $\bar{\kappa}$ of the matrix is 10000, the maximum of the local κ was less than 250. Furthermore, the local κ is the largest in the first iteration, and then it reaches 0 after some iterations. Fig. 1 shows the changes of $\kappa(\Delta x)$ and the decrease of the duality gap.

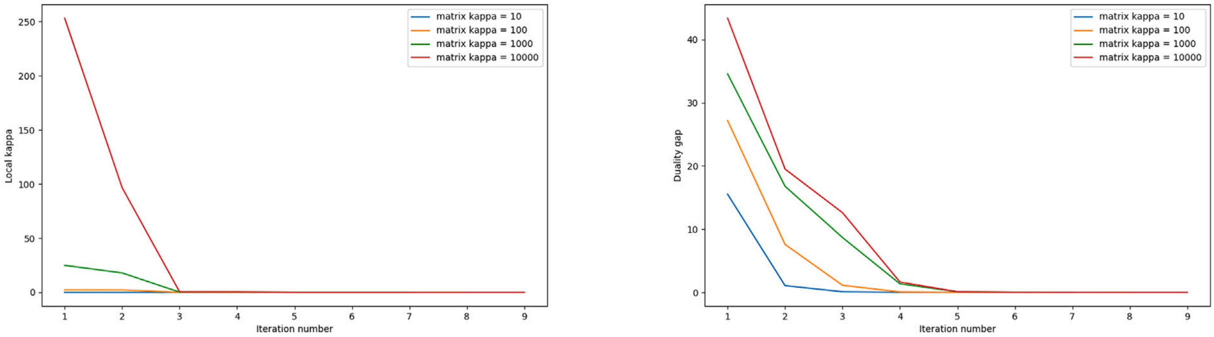

Fig. 1. The changes in $\kappa(\Delta x)$ and the duality gap in the case of TR_PSD_1.

6. Conclusions and further research

We introduced the class of AET functions with bounded proportional growth rate in order to define search directions in the case of PC IPAs for $P_*(\kappa)$-LCPs. In this way, we introduced a new PC IPA for solving $P_*(\kappa)$-LCPs. Several PC IPAs, see for example [16,38], are special cases of the PC IPA proposed in this paper. The main difference between those PC IPAs and ours is that the function φ is input data for our PC IPA. We also showed that our PC IPA using any member φ of this new class of AET functions has polynomial iteration complexity in the size of the problem, the starting point's duality gap, the accuracy parameter and the parameter κ. In Theorem 4.6 and Corollary 4.1 we provided values and bounds for the parameters τ and θ for which the PC IPA is well defined and the best theoretical complexity result can be obtained. These bounds on τ and θ depend on the selection of the functions φ from the new class. The parameters (c_2 and c_3) related to the selection of φ occur in the complexity result in Theorem 4.6 and Corollary 4.1. Examples 4.1 and 4.2 show that there is a possibility to find different settings of parameters for which the algorithm is well defined and for which the polynomial complexity can be proven.

Based on our numerical experiments we observed that in the case of matrices with larger κ obtained by rescaling the test set problems, there was no significant increase in the number of steps. This indicates that in some cases the value κ is not as significantly influential as expected from theoretical analyses. It would be interesting to generate another test set of problems, where the matrices cannot be scaled to ones with $\bar{\kappa} = 0$. Our aim is to test our PC IPA on $P_*(\kappa)$-LCPs with these types of matrices, as well. As a further research topic it would also be interesting to extend the presented PC IPA in a similar way that Illés et al. did in [46], for general LCPs.

CRediT authorship contribution statement

Tibor Illés: Writing – review & editing, Supervision, Methodology, Investigation, Funding acquisition, Conceptualization. **Petra Renáta Rigó:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Funding acquisition, Conceptualization. **Roland Török:** Writing – review & editing, Validation, Software, Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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