Corvinus Economics Working Papers

02/2026

Péter Bayer and Mánuel László Mágó

Uncertain participation

https://unipub.lib.uni-corvinus.hu/13243

Corvinus University of Budapest

Uncertain participation*

Péter Bayer[†]   Mánuel László Mágó[‡]

August 26, 2026

Abstract

We introduce participation uncertainty to the theory of cooperative TU-games. Under participation uncertainty, each coalition's effective value is an expected value taken over the (full) value of its subsets. We identify 'constant-marginal' participation as a key property for various economic problems to be well-behaved. This class of uncertainty environments is closed for mixing and composition. Under this property, solution concepts are easily calculate, two-sided markets clear and are insurable, and voting power under uncertain participation can be characterized and efficiently computed.

Keywords: cooperative games; participation uncertainty; matching markets; voting power; overselling

JEL codes: C71, C72, D51, D72, D81

1 Introduction

Uncertainty of participation is ubiquitous in economic problems. Buyers and sellers renege on deals which is a primary source of tension and friction in the market; MPs unexpectedly vote against their parliamentary coalitions with consequences on policy and political stability;[1] nations or organizations pull out of multilateral agreements, disrupting coordination on a global scale. In each case, the realized

*We thank Ata Atay and Michel Grabisch for helpful discussions and comments. Péter Bayer gratefully acknowledges financial support from the grant PID2023-147183NB-I00 and the Agencia Estatal de Investigación (AEI) through the Severo Ochoa Program Grant CEX2024-001476-S, both funded by MCIN/AEI/10.13039/501100011033.

†Universitat Autònoma de Barcelona and Barcelona School of Economics. E-mail: peter.bayer@uab.cat.

‡Corvinus University of Budapest, Institute of Economics. E-mail: manuel.mago@uni-corvinus.hu

[1] As a salient example, in 2017, US Senator John McCain cast a pivotal vote against his own party's planned repeal of the Affordable Healthcare Act.

value of a coalition differs from its full potential as the available market surplus, voting power, or total budget for a public project falls short of the coalition's nominal value. For economic analysis focusing on economic outcomes under uncertainty rather than optimal behavior, these interactions can be captured by the theory of stochastic cooperative (TU-)games.

Despite stochastic TU-games having a long-standing literature with various ways of introducing uncertainty in the basic framework, a general model of participation uncertainty has not yet been attempted.[2] Partial participation has been first raised by the interpretations of the multilinear extension (MLE) of TU-games (Owen, 1972) as the expected value of the grand coalition under player-wise independent participation.[3] Extant models of stochastic TU-games are, in a sense, too general to cleanly capture participation uncertainty, whereas, we argue that relying on the MLE's assumption of player-wise independence is not general enough.

In our paper, we build an explicit model of participation uncertainty. The key idea is that the ex post value of a realized coalition is assumed to be independent of the broader coalition that was intended to form; if, out of 50 senators in one's party, only 49 turns out to vote 'yes', the realized worth of that coalition is equal to the worth of that 49 senator coalition. Thus, a coalition's (expected) value under uncertain participation is the sum of its subsets' values weighted by the probability of that exact subset participating. By this structure, taking the probabilistic extension of a TU-game becomes a linear transformation of that game. The transformation matrix, which we call the *participation kernel* contains all rel-

[2]In Charnes and Granot (1976, 1977) and Granot (1977), coalitional values are independent random variables and players are risk neutral. Suijs and Borm (1999) and Suijs et al. (1999) consider a similar independent setup with more general risk preferences. In Bossert et al. (2005) and further developed by Habis and Herings (2011), coalitional values are determined by a randomly drawn state of the world, allowing for general correlational structures. While most works focus on binding agreements between coalitional members before the uncertainty is resolved, Habis and Herings (2011) focuses on agreements that are self-enforcing in the sense that players stick to it once the uncertainty is resolved. Ieong and Shoham (2008) considers an epistemic extension, where players hold a common prior as to the states of the world that may be realized but are able to distinguish different information sets, a framework generalized by Grabisch and Lorenzini (2025) for non-additive beliefs. More recently Borkotokey et al. (2023) discusses solutions to problems where one coalition forms and realizes its worth according to a random draw from all possible coalitions. See Grabisch and Lorenzini (2026) for a recent survey.

[3]Branches of literature considers differing participation levels known as multi-choice games (Faigle and Kern, 1992; Hsiao and Raghavan, 1993; Klijn et al., 1999; Calvo and Santos, 2000; Peters and Zank, 2005; Grabisch and Lange, 2007) and imperfect participation through the theory of fuzzy games (Butnariu, 1978; Aubin, 1981; Mareš and Vlach, 2001; Mallozzi et al., 2011).

evant statistical information about the uncertainty structure of participation; for any given coalition it specifies a distribution over its subsets that could form in its stead. Our paper's technical innovations amount to using tools of linear algebra and polynomial extensions of set functions (Lovász, 1983) to classify uncertainty structures and evaluate their economic implications.

Our results reveal that a single intuitive property of the uncertainty environment is decisive for the behavior of the system. If players' or coalitions' probability of participation does not depend on the coalition they are supposed to be joining, the problem is remarkably well-behaved. The class of uncertainty environments that satisfies this property, which we call the class of *constant-marginal* participation kernels has three key properties: (1) it has attractive algebraic properties that become relevant for its interpretation as an uncertainty environment; (2) probabilistic games under such transformations inherit the original game's key properties; (3) solutions of the probabilistic game are easily computed from the original game's solutions. Extending the base model, we show that under this property, probabilistic markets (which we model as probabilistic extensions of assignment games) clear in the Arrow-Debreu sense and are insurable, a central component of economic analysis of stochastic markets. Such kernels are useful in calculating political power of parliamentary coalitions with uncertainty in party discipline and, in an extension with dynamic participation, we show that participation fluctuation accounts for price fluctuation in markets.

While the constant-marginal structure is restrictive compared to a completely arbitrary participation structure, it is far less restrictive than imposing player-wise independent participation. Notably, the class includes comonotonic participation structures, in which participation is decided by meeting a hierarchical or merit-based threshold. In fact, one characterization of the constant-marginal class is the class that obtains as mixtures or compositions of independent and comonotonic uncertainty environments. There exist of course market structures that are incongruous with the constant-marginal assumption; for instance, if there is agglomeration or congestion in the market, individuals' participation likelihood is increasing or, respectively, decreasing with coalition size. We show that such markets generally have an empty ex-ante core and are thus liable for fragmentation; groups of buyers or sellers could leave to form an exclusive market on their own, changing the uncertainty environment in a way advantageous to them.

This paper proceeds as follows. In Section 2, we define TU-games under uncertain participation, characterize constant-marginal kernels by their algebraic properties, and present the game theoretic implications. In Section 3, we discuss matching market games under uncertain participation. Section 4 shows extensions to dynamic participation, voting in parliamentary democracies, and non-cooperative games. Section 5 concludes.

2 Participation kernels in TU-games

Let $N = \{1, \dots, n\}$ be a finite set of players and let $v \colon 2^N \to \mathbb{R}$ be given such that $v(\emptyset) = 0$. In other words, the double (N, v) is a TU-game with $v(T)$ being the total value that coalition $T \subseteq N$ is able to achieve.

Definition 1 (Games with uncertain coalitional participation)**.** Let $P \colon 2^N \times 2^N \to [0, 1]$ be a lower diagonal row-stochastic kernel called the *participation kernel*.[4] Then, for game (N, v), we define v_P as

$$v_P(T) = \sum_{S \subseteq T} P(S, T) v(S),$$

for all $T \subseteq N$.[5]

For a coalition T intending to form, $P(S, T)$ is the probability that S forms instead. The value of the intended coalition T under uncertain participation is calculated as taking the expected value of the possible coalitional of the subsets of T.

Expressing $P(S, T)$ as a $2^N \times 2^N$ matrix and writing v as a 2^N vector, it is clear that $v_P = Pv$ where the multiplication on the right-hand side is matrix multiplication. Thus the operation in Definition (1) is a linear transformation.

With this in mind we can define two operations for uncertainty environments as matrix operations. Mixing two kernels P and P' in the sense of second-order uncertainty, becomes the convex combination $\alpha P + (1 - \alpha) P'$ for $\alpha \in [0, 1]$, whereas composing them, in the sense of two uncertainty environments acting together

[4] Lower diagonal with respect to set inclusion, that is, $P(S, T) = 0$ for all $S \not\subseteq T$, whereas row-stochasticity means $\sum_{S \subseteq N} P(S, T) = 1$ which becomes $\sum_{S \subseteq T} P(S, T) = 1$ under the lower diagonal property.

[5] For most of this paper, we assume risk neutral evaluation by the agents or planners evaluating the coalitions under uncertainty.

becomes the matrix multiplication PP'. As matrix multiplication is not commutative, we generally have $(v_P)_{P'} = P'Pv \neq PP'v = (v_{P'})_P$.

As TU-games of player set N form a vector space in 2^N, the transformation to games of uncertain participation is a linear property.

Lemma 1. *Let two games (N, v), (N, w), two constants $\alpha, \beta \in \mathbb{R}$, and a participation kernel P be given. Then, for every $S \subseteq N$ we have $(\alpha v + \beta w)_P(S) = \alpha v_P(S) + \beta w_P(S)$.*

Using the fact that the set of participation kernels is convex, we also get linearity in participation kernels. Thus, unlike the composition of uncertainty environments, mixing is well-behaved.

Lemma 2. *Let a game (N, v), two participation kernels P and P', and a constant $\alpha \in [0, 1]$ be given. Then, for every $S \subseteq N$ we have $v_{\alpha P+(1-\alpha)P'}(S) = \alpha v_P(S) + (1 - \alpha)v_{P'}(S)$.*

An elementary participation structure that turns out to be an important building block is the *Dummy* kernel. For $R \subseteq N$ let D_R denote the following:

$$D_R(S, T) = \begin{cases} 1 & \text{if } S = R \cap T, \\ 0 & \text{otherwise.} \end{cases}$$

The Dummy kernel of set of players R puts all probability of a coalition T to the largest subset of T within R. Applying D_R to v returns $v_{D_R} = D_R v = v|_R$, the restricted game to the subset of players R.[6]

2.1 Constant-marginal kernels

To obtain our results we introduce some structure on our kernels. For a kernel P and a coalition T let $P_S(T) = \sum_{R:S\subseteq R} P(R, T)$ denote the marginal participation probability of group S in coalition T. For players, we can interpret $P_i(T) = \sum_{R:i\in R} P(R, T)$ as player i's marginal probability of participation in coalition T.

Definition 2. We call P a *constant-marginal* kernel if we have $P_S(T) = p_S$ for all $S \subseteq T \subseteq N$.

[6] In the literature, the subgame (R, v_R) is defined for $v_R(S) = v(S)$ for $S \subseteq R$, reducing the player set to R. The restricted game $(N, v|_R)$, embeds it into 2^N, treating all players outside of R as null players.

Constant-marginal kernels are kernels where players' and groups' participation probabilities do not depend on the coalition they are intending to join. This is an intuitive property, containing many important classes of kernels, and it turns out to be crucial for our results. Constant-marginal kernels also include important special classes of kernels that are derived from explicit continuous extensions of games.

Definition 3. Fix individual participation probabilities $(p_i)_{i\in N}$. Then let P be called the

- Owen kernel (independent participation), if for every $S, T \subseteq N$

$$O(S,T) = \begin{cases} \prod_{i\in S} p_i \prod_{j\in T\setminus S}(1-p_j) & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

- Lovász kernel (comonotic participation), if for every $S, T \subseteq N$

$$L(S,T) = \left(1 - p_{o(n)}\right) \mathbb{1}\{S = \emptyset\} + \sum_{k=1}^{n} \left(p_{o(k)} - p_{o(k-1)}\right) \mathbb{1}\{S = \{\{o(k), \dots, o(n)\} \cap T\}\}.$$

  where $o(k)$ is an ordering of players such that $p_{o(1)} \leq p_{o(2)} \leq \ldots \leq p_{o(n)}$ and $p_{o(0)} = 0$.

Given the individual participation probabilities, the Owen kernel is simply the participation kernel that is characterized by players participating independently, no matter what coalition they are supposed to be joining. It is derived from the multilinear extension (MLE) by Owen (1972), used in calculating and estimating solution concepts of TU-games. The MLE of TU-game (N, v) is defined as

$$\text{MLE}(v)(p_1, \dots, p_n) = \sum_{S\subseteq N} \prod_{i\in S} p_i \prod_{j\in N\setminus S} (1-p_j) v(S),$$

which corresponds to the value of the grand coalition $v_O(N)$ where O is the Owen-kernel as defined above. Other $v_O(T)$ values follow by restricting the set of players and applying the same transformation as doing so preserves the independent participation of players.

The Lovász kernel captures comonotonic player participation. Players are ordered from most able to participate to least able (allowing for equalities), creating a hierarchy of participation. If any player participates, all players on the same or higher level of the hierarchy participates as well. One can imagine p_i to measure the players' aptitudes on a 0-1 scale. A common threshold, $\lambda \sim \text{Uni}[0, 1]$ is drawn and any player whose aptitude exceeds λ participates. The Lovász kernel is derived from the Lovász extension of set functions (Lovász, 1983), defined as

$$\mathrm{LE}(v)(p_1, \dots, p_n) = \mathbb{E}_\lambda\big(v(\{i\colon p_i \geq \lambda\})\big),$$

with $\lambda \sim \mathrm{Uni}[0,1]$. The Lovász extension corresponds to the value of the grand coalition, $v_L(N)$ where L is the Lovász kernel defined above. Other $v_L(T)$ values follow by restricting the set of players and applying the same transformation as doing so preserves the comonotonic participation structure with the same hierarchy projected down to T.[7]

In a sense, the Owen kernel and the Lovász kernel are two extremes. Given the same marginal player participation rates $(p_i)_{i\in N}$, the Owen kernel is the least correlated structure, while the Lovász kernel is the most correlated. Notice that Dummy kernels fall into both categories due to the fact that they correspond to situations where all the marginal player participation rates are 0 or 1.

Example 1. Let $N = \{1, 2\}$ and consider the following kernel:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-p_1 & p_1 & 0 & 0 \\ 1-p_2 & 0 & p_2 & 0 \\ 1-p_1-p_2+p_{12} & p_1-p_{12} & p_2-p_{12} & p_{12} \end{pmatrix},$$

where the order of rows and columns is $(\emptyset, \{1\}, \{2\}, \{1,2\})$ and with p_1, p_2, and p_{12} chosen such that all entries are probabilities.

Setting $p_{12} = p_1 p_2$ returns the Owen kernel, and letting $p_{12} = \min\{p_1, p_2\}$ returns the Lovász kernel.

Constant-marginal kernels have two equivalent characterizations that turn out to be very convenient. First we show that these kernels share a common basis, the set of Dummy kernels.

Theorem 1 (Constant-marginal kernels in Dummy basis)**.** *The participation kernel P is constant-marginal if and only if*

$$P = \sum_{S\subseteq N} \alpha_S(P) D_S,$$

with $\alpha_S(P) = P(S, N)$.

[7] For prior applications of the Lovász extension in cooperative games see Algaba et al. (2004); Casajus (2021).

Whenever we use them as coordinates, we write elements of the last row of P as $\alpha_S(P)$. We omit the argument P whenever the kernel is unambiguous.

Theorem 1 is our most important technical result. It leverages the fact that the last row of the participation kernel P, the distribution of subsets of the grand coalition, pins down all other probabilities under the constant-marginal property:

$$P(S,T) = \sum_{R:R\cap T=S} P(R,N) = \sum_{R:R\cap T=S} \alpha_R.$$

The rest of the proof, like all other proofs, are shown in the appendix.

As an immediate consequence, we get that games under uncertain participation with constant-marginal kernels can be expressed in restricted-game basis, a property we exploit for all our game theoretical results. That is, for any constant-marginal kernel P we have

$$v_P = Pv = \left(\sum_{S\subseteq N} \alpha_S D_S\right) v = \sum_{S\subseteq N} \alpha_S(D_S v) = \sum_{S\subseteq N} \alpha_S v|_S, \tag{1}$$

by the linearity of kernels and games (Lemmas 1 and 2). It follows that the set of constant-marginal kernels is convex:

Corollary 1. *For a game (N,v), two constant-marginal kernels P and P', and $\alpha \in [0,1]$, the kernel $\alpha P + (1-\alpha)P'$ is constant-marginal, and we have*

$$v_{\alpha P+(1-\alpha)P'} = \alpha v_P + (1-\alpha) v_{P'} = \sum_{S\subseteq N} \big(\alpha\alpha_S(P) + (1-\alpha)\alpha_S(P')\big) v|_S.$$

Finally, due to the fact that Dummy kernels are both Owen kernels and Lovász kernels, we get the following corollary:

Corollary 2. *The set of constant-marginal kernels equals the convex hull of Owen kernels, the convex hull of Lovász kernels, and the convex hull of the set of all Owen kernels and Lovász kernels.*

As such, the class of constant-marginal kernels is the class of uncertain participation structures that we get by mixing independent and/or hierarchical sources of uncertainty.

The second characterization shows that all constant-marginal kernels share an eigenspace as well, the set of unanimity games.

Theorem 2 (Constant-marginal kernels spectral decomposition)**.** *The participation kernel P is constant-marginal if and only if*

$$P = U\Lambda_P U^{-1},$$

where $U = (u_S)_{S\subseteq N}$ with u_S being the unanimity game for coalition S, whereas $\Lambda_P = \mathrm{diag}(\lambda_S(P))_{S\subseteq N}$ with $\lambda_S(P) = P(S,S)$.[8]

As before, we omit the argument from $\lambda_S(P)$ whenever the participation kernel is unambiguous.

Theorem 2 makes use of the fact that lower-diagonal matrices have their eigenvalues in their main diagonal, implying $\lambda_S = P(S,S) = p_S$, and that the constant-marginal property implies that each unanimity game u_S is an eigenvector of P with the corresponding eigenvalue p_S. As unanimity games are linearly independent, we get the spectral decomposition shown in the statement.

As an immediate corollary, we get that the class of constant-marginal kernels is commutative with respect to matrix multiplication and the product is itself a constant-marginal kernel. Even stronger, any kernel that commutes with all constant-marginal kernels is itself constant-marginal.

Corollary 3. *Participation kernel P is constant marginal if and only if for every constant-marginal kernel P' we have*

$$PP' = P'P.$$

Furthermore, for every pair of constant-marginal kernels P and P' we have that PP' is constant-marginal with $\Lambda_{PP'} = \Lambda_P\Lambda_{P'}$.

Together, Corollaries 1 and 3 mean that the class of constant-marginal kernels is the one we get by taking mixtures of composites of independent or comonotonic participation environments and the class is closed for both operations.

Theorems 1 and 2 give two powerful algebraic characterizations, one making use of the fact that the last row of P pins it down under the constant-marginal property, the other making use of the fact that the main diagonal pins it down as well. Ultimately, these characterizations are equivalent due to the fact that

[8]The unanimity game for S is given by $u_S(R) = 1$ if $S \subseteq R$ and $u_S(R) = 0$ otherwise.

$(P(S,N))_{S\subseteq N} = (\alpha_S)_{S\subseteq N}$ is the Möbius inverse of $(P(S,S))_{S\subseteq N} = (\lambda_S)_{S\subseteq N}$, that is:

$$\alpha_S = \sum_{T:S\subseteq T} (-1)^{|T|-|S|}\lambda_T; \qquad \text{or, equivalently:} \qquad \lambda_S = \sum_{T:S\subseteq T} \alpha_T.$$

To conclude this subsection, we define two more constant-marginal kernels that obtain as extrema of minimizing or maximizing $v_P(N)$ under the condition that P is a constant-marginal kernel and with given player participation probabilities.

Definition 4. Fix individual participation probabilities $(p_i)_{i\in N}$. Then, a participation kernel is called

- optimistic if
$$P^+(S,T) = \sum_{R\subseteq N} \alpha_R^+ \, \mathbb{1}\{R\cap T = S\},$$
where $\alpha^+ \in \arg\max \left(\sum\limits_{R\subseteq N} \alpha_R v(R) \colon \alpha_R \geq 0, \sum\limits_{R\subseteq N} \alpha_R = 1, \sum\limits_{R:i\in R} \alpha_R = p_i \right)$.
- pessimistic if
$$P^-(S,T) = \sum_{R\subseteq N} \alpha_R^- \, \mathbb{1}\{R\cap T = S\},$$
where $\alpha^- \in \arg\min \left(\sum\limits_{R\subseteq N} \alpha_R v(R) \colon \alpha_R \geq 0, \sum\limits_{R\subseteq N} \alpha_R = 1, \sum\limits_{R:i\in R} \alpha_R = p_i \right)$.

Much like Owen and Lovász kernels, the kernels of Definition 4 come from polynomial extensions of set functions. The optimistic kernel corresponds to the concave closure of v, whereas the pessimistic kernel corresponds to the convex closure.

Optimistic and pessimistic kernels are relevant for robust design. Assuming that we have information on the players' individual participation rates, the optimistic (respectively, pessimistic) kernel is the correlation structure that accomplishes the highest (lowest) expected value of the game under uncertain participation while preserving the constant-marginal property. It is a known result that the optimistic (pessimistic) kernel coincides with the Lovász-kernel for convex (concave) games (Lovász, 1983). The fact that these are indeed constant-marginal follows from the Dummy decomposition of Theorem 1.

2.2 Game theoretical properties

By Theorem 1, the transformation by a constant-marginal kernel amounts to taking a convex combination of restricted games (as stated in 1). Any linear property of TU-games that is preserved for restricted games are preserved in the game of uncertain participation under the constant-marginal property. These include monotonicity, sub-/superadditivity, as well as convexity/concavity as we show in the next series of Lemmas.

Lemma 3. *If (N, v) is monotonic and P is a constant-marginal kernel, then (N, v_P) is also monotonic.*

Lemma 4. *If (N, v) is a subadditive (superadditive) game and P is a constant-marginal kernel, (N, v_P) is also additive subadditive (superadditive).*

Lemma 5. *If (N, v) is convex (concave) and P is a constant-marginal kernel, then (N, v_P) is also convex (concave).*

The same logic carries over to linear solution concepts.

Proposition 1. *Let (N, v) be a game, P be a constant-marginal kernel, and f a point-valued linear solution concept. Then*

$$f(N, v_P) = \sum_{S \subseteq N} \alpha_S f(N, v|_S).$$

Furthermore, if f gives null players 0*, then*

$$f_i(N, v_P) = \sum_{S \subseteq N:\, i \in S} \alpha_S f_i(S, v|_S).$$

Proposition 1 gives a straightforward calculation of the solutions of games under uncertain participation as convex combinations of the players' values under the restricted games. It highlights that the constant-marginal property introduces no additional computational complexity to evaluate the players' worth. Note that the most commonly used solutions, the Shapley value and Banzhaf value satisfy both linearity and the null player property. Our result is a generalized version of Borkotokey et al. (2023)'s results on the expected Shapley value, explicitly identifying the probabilistic TU-game whose Shapley value equals the expected Shapley value.

Notably, there is no analogous result to cover the core due to the fact that a game with a nonempty core may have restrictions with empty cores. However, totally balanced games, games where every restriction has a nonempty core, have an analogous result, owing to the Minkowski-superadditivity of the core.[9]

Proposition 2. *Let (N, v) be a totally balanced game and P be a constant-marginal kernel. Then v_P is totally balanced, and*

$$\sum_{S \subseteq N} \alpha_S \mathrm{Core}(N, v|_S) \subseteq \mathrm{Core}(N, v_P),$$

where the multiplication on the left hand side is scalar multiplication and the summation is the Minkowski sum. The statement holds with equality for convex games.

3 Markets under uncertain participation

In this section we show how results on games with uncertain participation apply to two-sided markets. We discuss the properties of partial equilibrium models and insurance against participation uncertainty. Matching market games, a subclass of assignment games, are essentially discretized partial equilibrium models. Uncertainty in participation may effect both the supply and the demand side of the market, a feature that is readily captured by the participation kernels of Section 2. As games of this class are totally balanced, we can apply Proposition 2 and interpret the core elements of uncertain participation games as state-dependent market equilibria.

If markets are affected by participation uncertainty, risk-aversion of market actors opens the possibility for Pareto-improvement through insurance. The nuance of implementing such an insurance comes from the fact that we impose that, in the goods-dimension, trades must be balanced for all states of the world. Once again, the total balancedness of assignment games can be leveraged, guaranteeing that in each state of the world, characterized by the set of participating buyers and sellers, there exists a matching between buyers and sellers that is ex post implementable. If participation is governed by a constant-marginal kernel, the contract implements an element of the ex ante core, and is thus acceptable for market actors.

[9]For two games (N, v) and (N, w) we have $\mathrm{Core}(N, v) + \mathrm{Core}(N, w) \subseteq \mathrm{Core}(N, v + w)$ where addition on the left hand side is the Minkoswki-sum.

Neither property applies in general if the participation kernel is not constant-marginal. For instance, if individual participation rate depends on coalition size (such as congestion), the ex ante core is empty, we get *market fragmentation*.

3.1 Probabilistic market equilibria

Consider a matching market game with disjoint sets of buyers B and sellers S with $B \cup S = N$. The game is characterized by the buyers' benefits $(b_i)_{i \in B}$ and the sellers' costs $(c_j)_{j \in S}$. With assignment matrix $A = (a_{ij})_{i \in B, j \in S}$ such that $a_{ij} = \max\{0, b_i - c_j\}$, the matching market game is given as

$$v(T) = \max\{0, \max_{(i_1,j_1),\ldots,(i_k,j_k) \in (B \cap T) \times (S \cap T)} \{a_{i_1 j_1} + \ldots + a_{i_k, j_k}\}\} \tag{2}$$

where $k = \min\{|B \cap T|, |S \cap T|\}$ and $\{i_1, j_1\}, \ldots, \{i_k, j_k\}$ are all disjoint. In words, (2) expresses that the value of coalition T equals the maximum surplus possible generated by the exchanges between the buyers and sellers within T.

We rely on the classic result of Shapley and Shubik (1971):

Theorem 3. *(Shapley and Shubik, 1971) Every market game (N, v) of the form (2) is totally balanced. Furthermore, every core element $x \in Core(N, v)$ is uniquely characterized by a market price $\pi \geq 0$ such that*

$$x_i = \max\{0, b_i - \pi\} \qquad i \in B,$$
$$x_j = \max\{0, \pi - c_j\} \qquad j \in S.$$

Now consider the case of uncertainty as characterized by participation kernels. We can attach the following friction-free interpretation: negotiation takes place across two periods. In the first period, a set of buyers and sellers $T \subseteq N$ is engaged in a pre-negotiation phase. The goods do not get transferred from sellers to buyers. In the second period, only a subset $R \subseteq T$ participates in the final negotiation phase, according to distribution $P(\cdot, T)$. Each remaining player, buyer and seller, may freely renegotiate earlier deals. Only the second period deals are executed. Under these assumptions, the expected surplus generated by the trades of each coalition T is indeed

$$v_P(T) = \sum_{R \subseteq T} P(R, T) v(R).$$

Because the only source of surplus comes from successful exchanges between buyers and sellers, neither the surplus nor the players' incentives in the final negotiation period are affected by whether money changes hands in the pre-negotiation period (e.g., a hotel room reservation fee), or whether non-participation penalties are imposed, as long as they are transfers between members of T and non-participation is exogenous (e.g., a tourist fails to turn up for his reservation due to illness). Renegotiation penalties may change the players' incentives in the final stage so we abstract away from them by assuming they are small enough that the maximal assignments are achieved in each sub-coalition R. Finally, we assume there is no "overselling", that is, players cannot agree on multiple binding commitments to hedge against or exploit no-shows (e.g., the hotel overbooking the same room). Overselling (or buying) without a binding commitment is allowed during pre-negotiation. We briefly explore the case of overselling in Subsection 4.2 While this structure makes the pre-negotiation phase seem superfluous, its purpose is to show that our results are robust to money exchanges and cheap talk prior to the resolution of the uncertainty in participation.

Proposition 2 applies to matching market games the following way:

Corollary 4. *Let a market game (N, v) of the form (2) and a constant-marginal kernel P be given. Then, the probabilistic market game (N, v_P) is totally balanced. For every vector of payoff allocations $(x^T)_{T \subseteq N} \in Core(T, v|_T)$ for all $T \subseteq N$, the allocation $(x_i)_{i \in N}$ defined by $x_i = \sum_{T \subseteq N} P(T, N) x_i^T$ is in the core of (N, v_P).*

Corollary 4 implies that, as long as the uncertainty environment is given by a constant-marginal kernel, well-known results of equilibrium under uncertainty apply. To see this, consider that matching market games of the form (2) lend themselves to the classical economic interpretation of a partial equilibrium. For a game (N, v), parameterized by Buyers B, Sellers S, valuations $(b_i)_{i \in B}$ and costs $(c_j)_{j \in S}$, we can define demand and supply curves $Q_d, Q_s \colon \mathbb{R}_+ \to \mathbb{N}$ as

$$Q_d(\pi) = |\{i \in B \colon b_i \geq \pi\}|,$$
$$Q_s(\pi) = |\{j \in S \colon c_j \leq \pi\}|.$$

A price π^* is an equilibrium price if $Q_d(\pi^*) = Q_s(\pi^*)$.

Consider the state space $\Omega = \{T \subseteq N\}$ where the probability of each state is $P(T, N)$. In state T, corresponding to the set of participating players, the effective demand and supply curves become

$$Q_d^T(\pi) = |\{i \in B \cap T \colon b_i \geq \pi\}|,$$
$$Q_s^T(\pi) = |\{j \in S \cap T \colon c_j \leq \pi\}|.$$

A price π^T is thus an equilibrium of the *submarket* corresponding to participating players T if $Q_d^T(\pi^T) = Q_s^T(\pi^T)$. The distribution over prices $(\pi^T)_{T \subseteq N}$ with probabilities $(P(T,N))_{T \subseteq N}$ characterizes the Arrow-Debreu equilibrium in this market. The main structural property of this uncertain matching market environment is that the demand and supply curves are nested by the states of the world, that is, for each $T' \subseteq T \subseteq N$ and π we have $Q_d^{T'}(\pi) \leq Q_d^T(\pi) \leq Q_d(\pi)$ and $Q_s^{T'}(\pi) \leq Q_s^T(\pi) \leq Q_s(\pi)$. It follows that, ceteris paribus, more uncertainty on the demand (supply) side is comparable to a leftward shift of the demand (supply) curve.

By Theorem 3 every core element of every restricted game $v|_T$ is characterized by a submarket equilibrium price π^T. Together with Corollary 4, we get the following:

Corollary 5. *Consider a matching market game (N, v) with participation uncertainty according to a constant-marginal kernel P. Then:*

- *For each state T there exists a competitive equilibrium price π^T.*
- *There is an Arrow-Debreu equilibrium with state-dependent prices $(\pi^T)_{T \subseteq N}$.*
- *The set of Arrow-Debreu allocations is a subset of the core of (N, v_P).*

By Corollary 5, the ex ante market game clears in equilibrium in the sense that no coalition of players T can leave and form a submarket with a more favorable uncertainty or exchange environment.

We summarize our results' predictions on probabilistic matching markets by the following claim:

Claim 1. *If the probabilistic market game given by $(b_i)_{i \in B}$, $(u_i)_{i \in S}$ in uncertainty environment described by the constant-marginal kernel P under some fixed pricing scheme π^T that assigns core prices to the emerging participating coalitions $T \subseteq N$, the realized prices is given by a random variable Π for which*

$$Pr(\Pi = \pi) = \sum_{T : \pi^T = \pi} \alpha_T.$$

That is, the uncertain participation environment induces a natural price fluctuation where observed market price follows realized participation.

We note that the structure provided by constant-marginal kernels is essential: even though the matching market game is totally balanced, meaning that every submarket clears for any realization of the set of participating players, the probabilistic matching market game could have an empty core, leading to the fragmentation of the market. We discuss this explicitly in Subsection 3.3.

3.2 Probabilistic matching market games with insurance

We now turn to whether the uncertainty of participation in matching market games can be resolved through insurance. We first define insurability of an uncertain matching market.

Definition 5. For a matching market game (N, v) and participation kernel P, an *insurance* is a double $(\mathbf{y}, \mathbf{g})$, whose components are

1. state-dependent cash transfers $\mathbf{y} = (y^T)_{T\subseteq N}$ with $y^T \in \mathbb{R}^N$ for each T such that

$$\sum_{T\subseteq N} \alpha_T \sum_{i\in N} y_i^T = 0 \qquad \text{(cash-neutral in expectation).}$$

2. state-dependent good transfers $\mathbf{g} = (g^T)_{T\subseteq N}$ with $g^T \in \{0,1\}^N$ for each T such that

$$\sum_{i\in N\setminus T} g_i^T = 0 \qquad \text{(non-participants do not trade) and}$$
$$\sum_{i\in B\cap T} g_i^T = \sum_{i\in S\cap T} g_i^T \qquad \text{(good-neutral in each state).}$$

Cash transfers may be negative or positive, corresponding to a conditional purchase or selling of a good. Good transfers are state-dependent obligations to either relinquish the good ($g_i^T = 1$ for seller $i \in S\cap T$), take ownership of the good ($g_i^T = 1$ for buyer $i \in B\cap T$), or remain passive ($g_i^T = 0$).

As market outcomes are core elements, we are interested in whether an insurance can implement any ex ante core element with certainty. A market game under uncertain participation (N, v_P) is called *insurable* if there exists an insurance and a core allocation $x \in \text{Core}(N, v_P)$ such that for every $T \subseteq N$ we have

$$\begin{aligned} y_i^T + g_i^T b_i &= x_i \qquad \text{for } i \in B, \\ y_i^T - g_i^T c_i &= x_i \qquad \text{for } i \in S. \end{aligned}$$

Insurances allow players to hedge against participation risk and the resultant availability of goods and variation of prices. Since g_i^T is a binary variable due to the indivisibility of the good, the realized monetary transfers must also come from a binary set of possible transfers: premia and payouts. After the state of the world is realized, characterized by the set of participating players T, a buyer is either assigned a good in exchange for a premium (a negative transfer), or awarded monetary payout after failing to buy. Sellers either get a high compensation and have to part with the good, or receive a compensation for failing to sell and keep the good. Notably, while cash transfers only have to be net zero in expectation, we do not allow disposal of excess goods or acquiring more goods from outside sources under any state of the world, hence purchases and sales have to be balanced in each state of the world.

Insurances of the form in Definition 5 guarantee certainty equivalents expressed in cash terms. Thus, the insurer in expectation makes 0 profits. Naturally, it follows that if individuals are strictly risk averse, there exists a Pareto-improving insurance contract if and only if the probabilistic market game is insurable.

We also note that offering a cash compensation for failing to participate could change the risk environment due to moral hazard. With strictly risk averse players, the insurer can overcome this by charging a marginal non-participation malus; a slightly lower payoff than x_i if $i \notin T$. This is enough to make "strategic absence" weakly dominated without altering the outcomes significantly. In this case the cash component of the insurance contract would have three possible realizations instead of two (as a player may participate and still be unable to buy/sell, in which case he or she would indeed receive the intended payout x_i as he/she does not need to be penalized).

We next characterize an insurance scheme that implements submarket transfers of goods while guaranteeing certainty equivalents in payoffs across all states.

Definition 6. Let a market game (N, v) of the form (2) and a participation kernel P be given. Take $(x^T)_{T \subseteq N} \in \times_{T \subseteq N} \text{Core}(T, v|_T)$ and the associated vector of shadow prices $(\pi^T)_{T\subseteq N}$ and define *expected personalized prices*

$$\pi_i = \sum_{T\subseteq N} \alpha_T \left(\mathbb{1}_T(i)\min\{b_i,\pi^T\} + (1-\mathbb{1}_T(i))b_i\right) \qquad \text{for } i\in B,$$
$$\pi_i = \sum_{T\subseteq N} \alpha_T \left(\mathbb{1}_T(i)\max\{c_i,\pi^T\} + (1-\mathbb{1}_T(j))c_i\right) \qquad \text{for } i\in S.$$

The *submarket insurance* associated to the probabilistic market game (N, v_P) and submarket outcomes $(x^T)_{T\subseteq N}$ is then defined as follows: If $i\in B$, then

$$y_i^T = \begin{cases} x_i & \text{if } i\notin T \text{ or } b_i<\pi^T \\ -\pi_i & \text{if } i\in T \text{ and } b_i\geq \pi^T \end{cases} \quad \text{and} \quad g_i^T = \begin{cases} 0 & \text{if } i\notin T \text{ or } b_i<\pi^T \\ 1 & \text{if } i\in T \text{ and } b_i\geq \pi^T \end{cases},$$

whereas if $i\in S$, then

$$y_i^T = \begin{cases} x_i & \text{if } i\notin T \text{ or } c_i>\pi^T \\ \pi_i & \text{if } i\in T \text{ and } c_i\leq \pi^T \end{cases} \quad \text{and} \quad g_i^T = \begin{cases} 0 & \text{if } i\notin T \text{ or } c_i>\pi^T \\ 1 & \text{if } i\in T \text{ and } c_i\leq \pi^T \end{cases},$$

Where $x_i = \sum_{T\subseteq N}\alpha_T x_i^T$.

The submarket insurance of Definition 6 implements the core outcomes x as follows: If a player fails to participate or a state of the world realizes in which they do not trade, they receive the payment x_i. If the player participates and trades, they trade with the insurer for the expected personalized price of π_i (paid by buyers and received by sellers). The calculation of the expected personalized price accounts for both non-participation and the case of no trade even in the case of participation. With knowledge of the shadow prices π^T, the insurer is able buy from sellers with costs lower than π^T and sell to buyers with valuations higher than it, leading to a net 0 of the good in each state of the world. As participating buyers' valuations are necessarily higher than the participating sellers' costs, the insurer realizes a positive cash-flow after each good that passes through it and is thus able to pay to non-trading players.

Proposition 3. *If P is a constant-marginal kernel, the probabilistic extension of any market game (N, v) is insurable by any submarket insurance contract.*

As each probabilistic market game has a core element decomposable across submarkets, the proof of Proposition 3 just amounts to showing that the submarket insurance of Definition 6 satisfies the Definition 5.

We showcase Corollary 4 and Proposition 3 in an example:

Example 2. Let $B = \{1, 2\}$ and $S = \{3, 4\}$ with $(b_1, b_2) = (12, 14)$ and $(c_3, c_4) = (6, 10)$. Then, the core of (N, v) is characterized by the set of possible market prices $[10, 12]$. For each $\pi \in [10, 12]$, generically, two trades take place. The total surplus is $v(N) = 14 + 12 - 6 - 10 = 10$. For the possible non-trivial submarkets, we have the following values and core-inducing shadow prices:

$$\begin{array}{llll}
v(\{1,3\}) = 6 & \pi^{\{1,3\}} \in [6,12] & v(\{1,2,3\}) = 8 & \pi^{\{1,2,3\}} \in [12,14] \\
v(\{1,4\}) = 2 & \pi^{\{1,4\}} \in [10,12] & v(\{1,2,4\}) = 4 & \pi^{\{1,2,4\}} \in [12,14] \\
v(\{2,3\}) = 8 & \pi^{\{2,3\}} \in [6,14] & v(\{1,3,4\}) = 6 & \pi^{\{1,3,4\}} \in [6,10] \\
v(\{2,4\}) = 4 & \pi^{\{2,4\}} \in [10,14] & v(\{2,3,4\}) = 8 & \pi^{\{2,3,4\}} \in [6,10]
\end{array}$$

Let P be an Owen kernel with parameters $(1/2, 1/2, 1, 1)$, that is, buyers show up independently with 50% probability and sellers show up with certainty. The probabilistic market game is then the following:

$$\begin{array}{lll}
v_P(\{1,3\}) = 3 & v_P(\{1,2,3\}) = 5.5 & v_P(\{1,2,3,4\}) = 6 \\
v_P(\{1,4\}) = 1 & v_P(\{1,2,4\}) = 3.5 & \\
v_P(\{2,3\}) = 4 & v_P(\{1,3,4\}) = 3 & \\
v_P(\{2,4\}) = 2 & v_P(\{2,3,4\}) = 4 &
\end{array}$$

On top of their own show-up risk, buyers face risk from each other as each buyer's presence lowers the bargaining power of the other. Since seller 3 is more efficient, seller 4 only trades if both buyers show up. Seller 3 only faces the risk of congestion on the seller side; if only one buyer shows, seller 4's mere presence eliminates the possibility of charging a high market price.

For each restricted game of v, take the core element defined by the midpoint price. For instance, $\pi^{\{1,3,4\}} = 1/2 * (6 + 10) = 8$, giving $x_1^{\{1,3,4\}} = x_3^{\{1,3\}} = 2$ and $x_4^{\{1,3,4\}} = 0$. Then, we have

$$\begin{aligned}
x_1 &= 1/4 * (x_1^{\{1,3,4\}} + x_1^{\{1,2,3,4\}}) = 1/4 * (4 + 1) = 1.25 \\
x_2 &= 1/4 * (x_2^{\{2,3,4\}} + x_2^{\{1,2,3,4\}}) = 1/4 * (6 + 3) = 2.25 \\
x_3 &= 1/4 * (x_3^{\{1,3,4\}} + x_3^{\{2,3,4\}} + x_3^{\{1,2,3,4\}}) = 1/4 * (2 + 2 + 5) = 2.25 \\
x_4 &= 1/4 * (x_4^{\{1,3,4\}} + x_4^{\{2,3,4\}} + x_4^{\{1,2,3,4\}}) = 1/4 * (0 + 0 + 1) = 0.25
\end{aligned}$$

The allocation $(1.25, 2.25, 2.25, 0.25)$ is indeed in the core of the game (N, v_P). The expected personalized prices are

$$\begin{aligned}
\pi_1 &= 1/4 \left(\pi^{\{1,3,4\}} + \pi^{\{1,2,3,4\}}\right) + 1/2 * b_1 = 1/4 * (8 + 11) + 1/2 * 12 = 10.75 \\
\pi_2 &= 1/4 \left(\pi^{\{2,3,4\}} + \pi^{\{1,2,3,4\}}\right) + 1/2 * b_2 = 1/4 * (8 + 11) + 1/2 * 14 = 11.75 \\
\pi_3 &= 1/4 * \left(\pi^{\{1,3,4\}} + \pi^{\{2,3,4\}} + \pi^{\{1,2,3,4\}} + c_3\right) = 1/4 * (8 + 8 + 11 + 6) = 8.25 \\
\pi_4 &= 1/4 * \pi^{\{1,2,3,4\}} + 3/4 * c_4 = 1/4 * 11 + 3/4 * 10 = 10.25
\end{aligned}$$

The submarket insurance contract is thus the following: If neither buyer shows up, players receive the transfer $(1.25, 2.25, 2.25, 0.25)$ and no goods change hands with the insurer realizing net of -6. If only buyer 1 shows, the transfer is $(-10.75, 2.25, 8.25, 0.25)$, seller 3 gives the good to buyer 1 and the insurer receives net 0. If only buyer 2 shows, the transfer is $(1.25, -11.75, 8.25, 0.25)$ and seller 3 gives the good to buyer 2, the insurer again receives net 0. If both buyers participate, the transfer is $(-10.75, -11.75, 8.25, 10.25)$ sellers give up their goods and the buyers receive them with the insurer netting 6, the total gains of trade.

We summarize the implications of this section in the following claim.

Claim 2. *Implementing insurance in two-sided markets is possible if the risk environment is described by a constant-marginal kernel P. Doing so induces effective price discrimination, where personalized prices π_i take into account i's own participation risk and the effect of other's participation risk on i.*

3.3 Uncertain markets with agglomeration and congestion

While constant-marginal kernels are powerful in modeling participation and absences due to player- and coalition-specific factors, they rule out many economically relevant sources of uncertainty. Of special importance is participation due to group size. Two such forces are known in markets, *agglomeration*, in which larger group sizes beget more market participation due to broader opportunities, and *congestion*, in which larger group sizes beget less market participation due to crowding (Ellison and Fudenberg, 2003; Rochet and Tirole, 2003). For this paper, agglomeration and congestion are interpreted in reduced form, as by-products of the physical or digital organization of the market rather than strategically. However, the model can be extended to include a micro-foundation.

We call kernels where participation probabilities depend only on group size, 'homogeneous', defined as follows;

Definition 7 (Homogeneous kernels)**.** Participation kernel P is called *homogeneous* if there exists a function $G\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ such that for every $R \subseteq T \subseteq N$ we have

$$P(R, T) \propto G(r, t), \tag{3}$$

with $|R| = r$ and $|T| = t$. The value $f(r, t)$ is called the *participation intensity* of r-sized groups in the coalition with size t.

Homogeneous kernels of the form (3) naturally satisfy

$$P(R,T) = \frac{H(r,t)}{h(t)}$$

with $H(t) = \sum_{k=0}^{t} \binom{t}{k} H(k,t)$, thus, P is exactly characterized by the participation intensity function $H(\cdot,\cdot)$.

For a tractable model of the two group-size effects, agglomeration and congestion, we can study the following *endogenous Owen* family:[10]

$$H(r,t) = p(t)^r (1 - p(t))^{t-r}, \tag{4}$$

which leads to $H(t) \equiv 1$. Thus, $h(r,t)$ becomes the participation probability of any subcoalition with size r given group size t, and $p(t)$ is interpretable as the individual's participation probability depending on group size t. Let $\alpha > 0$ and $\gamma > 0$ measure the strength of agglomeration and congestion, respectively, and let $p_0 \in [0,1]$ be a scaling parameter (measuring baseline participation). Then, we specify the following parametric functional form:

$$p(t) = \frac{p_0}{\exp\left(\alpha^2/(2\gamma)\right)} \exp\left(\alpha t - \frac{1}{2}\gamma t^2\right). \tag{5}$$

Under (5), individual participation probability is hump-shaped with a maximum in $t^* = \alpha/\gamma$. The ratio α/γ determines the group size maximizing participation, whereas the absolute sizes determine how responsive individuals are to the two effects. With α/γ kept constant, as $\alpha, \gamma \to 0$, participation probability for any group size tends to the constant function p_0, and as $\alpha, \gamma \to \infty$, participation probability tends to zero for every group size other than t^*. By the Owen construction (4), for any coalition T with $|T| = t$, the expected number of participants is equal to $e(t) = tp(t)$.

Given these effects, the resultant participation kernel p is not a constant-marginal kernel. As such, Corollary 5 does not apply and instead of a straightforward realization of the Arrow-Debreu equilibrium, we see *market fragmentation*, as shown in the following example.

Example 3. Consider a simple market game with $n = 2m$ players, m buyers and m sellers, with $b_i = 2$ for each buyer and $c_j = 0$ for each seller. The market game

[10] We could study the endogenous Lovász kernel, which, under homogeneity, would amount to every player participating or no player participating with some probability $p(t)$ as function of group size and arrive at a similar conclusion regarding market fragmentation.

capturing this is

$$v(T) = 2\min\{|T \cap S|, |T \cap B|\},$$

specifically, for balanced coalitions T_k for $k \in \{0, \ldots, m\}$, that satisfy $|T_k \cap S| = |T_k \cap B| = k$, the value is simply $v(T_k) = 2k$. It is clear that $x_i = 1$ for all $i \in N$ is a core element of (N, v).

Now consider an endogenous Owen participation kernel p of the form (4)-(5) and let $k^* = \alpha/\gamma$. Then, $v_P(T_k) = \mathbb{E}(\min\{X, Y\})$, with $X, Y \sim \text{Binom}(k, p(k))$ independent, which has no close-form expression but can be approximated by the expected minimum of two normal distributions, giving:

$$v_P(T_k) \approx 2\left(kp(k) - \sqrt{\frac{kp(k)(1-p(k))}{\pi}}\right),$$

for k large.

Then, taking per capita values, we have $\frac{v_P(T_k)}{2k} \approx p(k) - \sqrt{\frac{p(k)(1-p(k))}{\pi k}}$, which is maximized by the nearest integer to $\hat{k} \approx k^* + \frac{1}{2\gamma k^*} = k^* + \frac{1}{2\alpha}$. Hence, each balanced coalition of k^* buyers and k^* sellers is able to achieve an average value of $x_i^* \approx p(k^*)$ for all its members, and thus all players have a "claim" of approximately $p(k^*)$. However, the value of the grand coalition is lower at $p(m) < p(k^*)$ due to congestion, meaning the core of (N, v_P) is empty. Congestion leads to the market fragmenting, but agglomeration prevents complete dissolution. The result is an efficient outcome of approximately equal-sized markets of $k^* = \alpha/\gamma$ buyers and sellers.

The following claim summarizes the main points of this subsection.

Claim 3. *The ex ante probabilistic market may not clear if the uncertainty environment is given by a participation kernel P that does not satisfy the constant-marginal property. If P reflects congestion, a subset of players may change the uncertainty environment favorably by leaving, leading to a fragmentation of the market.*

4 Extensions and applications

The framework developed in the previous sections provides a general and flexible way to extend the standard model of cooperative games in a way to allow player participation to be uncertain. In this section, we explore the scope of the participation-kernel approach by considering several extensions of our baseline model and illustrating its feasibility in an application. The extensions differ in the source and timing of participation uncertainty, but share the same basic idea, namely that participation kernels provide a flexible description of how individual and joint participation behavior affects the outcome of the underlying game.

We first introduce dynamics into the participation environment. This allows participation shocks to persist over time and generates corresponding dynamics in the value of the players and in the core of the resulting uncertain game. We then consider a setting in which participation uncertainty is resolved only after agents make binding commitments. This provides a natural framework for studying overselling, where firms may commit to serving more customers than they will ultimately be able to serve. We next apply the framework to voting blocs, where participation uncertainty can arise from both attendance and deviations from a collective voting position. Using roll-call data, we illustrate how empirically calibrated individual participation probabilities can be combined with alternative dependence structures to quantify the probability that a political bloc reaches a prescribed voting threshold. Finally, we extend the framework to a non-cooperative setting and propose an equilibrium concept that incorporates uncertain participation.

These extensions and applications illustrate the breadth of the participation-kernel framework. The same representation of participation uncertainty can be used to study temporal dependence, commitment under uncertainty, collective voting, and strategic interaction. At the same time, the applications highlight a central feature of the framework, namely that the consequences of participation uncertainty depend not only on individual participation, but also on how participation events are jointly related.

4.1 Dynamic participation

While our model is strictly static, the pricing results in Subsection 3.1 point towards a model of stochastic price processes. If the same probabilistic market game is played repeatedly under an identically drawn participation kernel, the limiting distribution of the price realizations follows the distribution generated by the kernel.

A natural extension in this vein is dynamic participation, expressed by a sequence of participation kernels, $(P_t)_{t=0}^{\infty}$, with $t = 0, 1, \dots$ representing the time variable. We focus on processes that obtain as convex combinations of past participation *tendencies* to capture inertia as well as past participation *realizations* to model past participation as predictive of future participation. This is motivated by findings in behavioral finance showing considerable inertia in stock market participation decisions (Vissing-Jørgensen and Attanasio, 2003; Bilias et al., 2010; Briggs et al., 2021).

For $t \geq 0$, let $R_t \subseteq 2^N$ denote the *realized* participating coalition in period t, drawn from the distribution $P_t(\cdot, N)$. If participation is stationary, an ARMA(p, q) sequence of participation kernels is given as

$$P_t = \sum_{k=1}^{p} \phi_k P_{t-k} + \sum_{\ell=0}^{q} \theta_\ell D_{R_{t-1-\ell}}, \tag{6}$$

with $\sum_{k=1}^{p} \phi_k + \sum_{\ell=0}^{q} \theta_\ell = 1$. Since the Dummy kernel D_R is constant-marginal for any realization R, by Corollary 1, if P_0 is a constant-marginal kernel, so is every P_t for $t \geq 1$. If (N, v) is totally balanced, then Proposition 2 guarantees that the grand coalition does not disband (or, for matching market games, the market does not fragment) in any period, hence repeating the coalition draw from the same maximal set N is a natural assumption in this environment.

We note that the stationarity of (6) may be generalized by introducing player- or coalition-dependent drift by a long-run (constant-marginal) kernel P_∞. Then, the dynamic process with drift is

$$P_t = (1 - \eta) \left(\sum_{k=1}^{p} \phi_k P_{t-k} + \sum_{\ell=0}^{q} \theta_\ell D_{R_{t-1-\ell}} \right) + \eta P_\infty, \tag{7}$$

where $\eta \in [0, 1]$ captures the speed by which the limiting distribution P_∞ attracts the process. Decomposing the $P_\infty = \sum_{T \subseteq N} \omega_T D_T$ as per Proposition 1 reveals the coalition-specific drift; for $T \subseteq N$, $\eta\omega_T$ is the drift experienced by coalition T. For $\eta > 0$, the process in (7) is a form of transient dynamics from an initial state where participation is governed by P_0 to a long-run participation dynamic characterized by P_∞. As it is still a convex combination, as long as P_0 and P_∞ are constant-marginal, so is every P_t for $t \geq \infty$.

Leveraging the convex structure once more we can apply Proposition 1.

Proposition 4. *Let (N, v) be a game, P_0 and P_∞ constant-marginal, and f a point-valued linear solution concept. Suppose the game of uncertain participation* *is played repeatedly and participation follows the process described in (7). Then, for every $t \geq 0$, the value of player i satisfies*

$$f_i(P_t v) = (1-\eta)\left(\sum_{k=1}^{p} \phi_k f_i(P_{t-k} v) + \sum_{\ell=0}^{q} \theta_\ell f_i(D_{R_{t-1-\ell}} v)\right) + \eta f_i(P_\infty v). \tag{8}$$

The stochastic dynamics of the participation kernel are inherited exactly by any linear point-valued solution concept. In particular, the ex-ante value of player i evolves according to the same autoregressive structure as participation, with realized participation outcomes entering as shocks. When $\eta > 0$ we get attraction to the long-run value $f_i(P_\infty v)$.

By Proposition 4, observed fluctuations in players' worth (power, returns, shared costs) can be explained by uncertainty in participation and are likewise predicted by past realizations.

If, furthermore, (N, v) is totally balanced, we can use the Minkowski superadditivity of the core and Proposition 2 to get the analogous result for it as well.

Proposition 5. *Let (N, v) be a totally balanced game and P_0 and P_∞ be constant-marginal participation kernels. Suppose the game of uncertain participation is played repeatedly and participation follows the process described in (7). Then, for every $t \geq 0$ we have*

$$\mathrm{Core}(N, P_t v) \supseteq (1-\eta)\left(\sum_{k=1}^{p} \phi_k \mathrm{Core}(N, P_{t-k} v) + \sum_{\ell=0}^{q} \mathrm{Core}(N, D_{R_{t-1-\ell}}) + \eta \ \mathrm{Core}(N, P_\infty v)\right) \tag{9}$$

To sharpen Proposition 5 for large markets, we can rely on Edgeworth's limit theorem (Aumann, 1964). As the number of buyers and sellers increases and the generated supply and demand curves approach continuous lines, the set of core outcomes converges to the set of competitive equilibria. If supply and demand intersect transversally under every submarket (which holds for appropriately chosen "replica participation kernels"), the equilibrium shrinks to a single point and (9) holds with equality for the single core element on the left hand side and the convex combination of single core elements on the right hand side. As such, the surplus of each player will satisfy the same stochastic properties in the large n limit as the dynamic participation process. Given that a player's surplus is equal to the difference between his valuation, a constant in this model, and the prevailing market price, the stochastic process corresponding to price also inherits the properties of the participation process.

4.2 Ex ante commitment: towards a model of overselling

Overselling, the selling of goods and services in quantities that exceed actual supply, is a direct consequence of uncertainty in the market. Possible cancellations from the buyers' side induce sellers to oversell volatile or perishable goods in advance to maximize expected profits. Sellers' incentives do not change by whether cancellations are reimbursed. Crucially, these (at least partially) binding commitments are made prior to the resolution of the uncertainty environment. This is unlike our market game models in Section 3, where binding agreements are made after the uncertainty is resolved.

In this section we build a market game of overselling. We provide a proof of concept that participation kernels may be used in this application. However, the game transformation is substantially different from the linear structure we establish for our main results, so a detailed analysis is beyond our current focus. Our proposed approach to merge the literature of assignment games, focusing on market outcomes, with the model of optimal overselling for a monopolist (Rothstein, 1971). Consumer incentives, for instance via uncertainty in values at the time of purchased (Ely et al., 2017), only enter through the induced participation kernel.

To account for non-realization of binding agreements, we introduce an alternative way to incorporate uncertainty in a market game. The game primitives are identical, we take a set of Buyers B with valuations $(b_i)_{i\in B}$, Sellers S with costs $(c_i)_{i\in S}$, inducing assignment matrix $A = (a_{ij})_{i\in B, j\in S}$ such that $a_{ij} = \max\{0, b_i - c_j\} = v(\{i,j\})$. For $T \subseteq N = B \cup S$, let $M(T)$ denote the set of assignments using only elements of T. Then, for $m = \{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_k, j_k\}\} \in M(T)$ the expected value of the match is

$$\tilde{v}_P^m(T) = \sum_{R\subseteq T} P(R,T)\big(v(\{i_1, j_1\} \cap R) + \ldots + v(\{i_k, j_k\} \cap R)\big). \tag{10}$$

To express commitment before the revelation of the state of the world R, in (10), we take the value of each matched pair of buyers and sellers intersected with R; if both the buyer and seller are members of R, the deal goes through and the surplus from the match enters the sum at full value, weighted by the probability of R, otherwise one of them does not show and the deal is annulled and the surplus of the match is lost in that state. Then, the *probabilistic market game with ex ante commitment*, $\tilde{v}_P$ is defined by taking the largest possible expected surplus for each coalition as follows:

$$\tilde{v}_P(T) = \max_{m \in M(T)} \tilde{v}_P^m(T).$$

To account for the possibility of overselling, we have to include a second ingredient, the ability for sellers to be matched with multiple buyers, creating edges of the type $\{i_1, \ldots, i_k, j\} \equiv I \cup \{j\}$ with $\{i_1, \ldots, i_k\} = I \subseteq B$ and $j \in S$. Let the value of such an edge be defined as $v(I \cup \{j\}) = \max_{i \in I} a_{ij}$, that is, if multiple buyers show, the pairwise match with the highest surplus is executed. The surplus is once again invariant to refunds or any other types of compensation between the seller and the buyers who do not get serviced. Letting $\hat{M}(T)$ be the set of such disjoint many-to-one matchings in set T, for $m = \{I_1 \cup \{j_1\}, \ldots, I_k \cup \{j_k\}\} \in \hat{M}(T)$, the expected value of such a matching is given as:

$$\hat{v}_P^m(T) = \sum_{R \subseteq T} P(R,T)\Big(v\big((I_1 \cup \{j_1\}) \cap R\big) + \ldots + v\big((I_k \cup \{j_k\}) \cap R\big)\Big). \quad (11)$$

Then, the *probabilistic market game with ex ante commitment and overselling* $\hat{v}_P$ is defined as

$$\hat{v}_P(T) = \max_{m \in \hat{M}(T)} \hat{v}_P^m(T).$$

We demonstrate with an example.

Example 4. Let $B = \{1,2\}$ and $S = \{3,4\}$ with $(b_1, b_2) = (14, 12)$ and $(c_3, c_4) = (4, 10)$. Let P again be an Owen kernel with parameters $(1/2, 1/2, 1, 1)$, that is, buyers show up independently with 50% probability and sellers show up with certainty. Then, the probabilistic market game with ex ante commitment is the following:

$$\begin{array}{lll} \tilde{v}_P(\{1,3\}) = 5 & \tilde{v}_P(\{1,2,3\}) = 5 & \tilde{v}_P(\{1,2,3,4\}) = 6 \\ \tilde{v}_P(\{1,4\}) = 2 & \tilde{v}_P(\{1,2,4\}) = 2 & \\ \tilde{v}_P(\{2,3\}) = 4 & \tilde{v}_P(\{1,3,4\}) = 5 & \\ \tilde{v}_P(\{2,4\}) = 1 & \tilde{v}_P(\{2,3,4\}) = 4 & \end{array}$$

In this example, the optimal matches in the grand coalition $\{\{1,3\},\{2,4\}\}$ and $\{\{1,4\},\{2,3\}\}$ return the same expected value, a property which in general is not true for participation kernels, even Owen kernels. On top of independence, this needs equality of marginal participation probability within buyers and within sellers.

Now evaluate the case of overselling. For coalitions that contain no more than one buyer, the value is identical. For coalitions $T \supseteq \{1,2,3\}$, the match $\{1,2,3\}$ gives the following expected surplus:

$$\hat{v}_P^{\{1,2,3\}}(T) =$$
$$P(\{3,4\},T)v(\{3\}) + P(\{1,3,4\},T)v(\{1,3\}) + P(\{2,3,4\},T)v(\{2,3\})$$
$$+ P(\{1,2,3,4\},T)v(\{1,3\}) = \frac{1}{4}0 + \frac{1}{4}10 + \frac{1}{4}8 + \frac{1}{4}10 = 7.$$

For coalitions $T \supseteq \{1,2,4\}$, the match $\{1,2,4\}$ gives the expected surplus:

$$\hat{v}_P^{\{1,2,4\}}(T)$$
$$= P(\{3,4\},T)v(\{4\}) + P(\{1,3,4\},T)v(\{1,4\}) + P(\{2,3,4\},T)v(\{2,4\}) +$$
$$P(\{1,2,3,4\},T)v(\{1,4\}) = \frac{1}{4}0 + \frac{1}{4}4 + \frac{1}{4}2 + \frac{1}{4}4 = 2.5.$$

So, in coalitions with one seller and two buyers, both sellers will oversell to extract surplus from both buyers, as $\hat{v}_P(T) > \tilde{v}_P(T)$ for such coalitions. Notably, $\hat{v}_P^{\{1,2,3\}}(N) = \hat{v}_P(N) > \tilde{v}_P(N)$, meaning that in the presence of overselling, the match $\{1,2,3\}$ blocks both matchings with all four players, forcing seller 2 off the market. This outcome is more efficient than the full matching as it maximizes the probability of trade for the more efficient seller.

For the overselling outcome to be compatible with market incentives, seller 1 has to set a price lower than $\hat{\pi} = 10$, otherwise seller 2 could match it, meaning that his or her maximal expected profit is slightly below $3/4 * 6 = 4.25$. Under the full matchings, the mid-point price is 11, in which case seller 1's profit would amount to $1/2 * 7 = 3.5$, meaning that overselling is incentive-compatible. However, if the probability of buyer participation rises, the benefit of the extra security gained from overselling decreases, while the cost of lowered prices increases; indeed under certain participation, there is no incentive to oversell.

4.3 Voting blocs with uncertain voting discipline

Voting blocs provide a natural application of participation kernels. In cooperative voting games, a coalition is successful whenever it contains enough members to reach a prescribed quota. In practice, however, members of a political bloc may be absent, abstain, or vote against the collective position. The resulting uncertainty is particularly relevant when the bloc is close to the voting threshold. Participation kernels allow us to introduce this uncertainty while retaining the cooperative-game structure.

This application also connects our framework to the literature on voting power. The multilinear extension of Owen (Owen, 1972), which is the foundation of our Owen kernel, corresponds to independent participation. Independence is a natural benchmark, but empirical work has questioned whether standard voting-power calculations based on such assumptions adequately describe observed voting behavior, see in particular Gelman et al. (2002) and Gelman et al. (2004). Our framework allows us to hold the individual participation probabilities fixed while varying their dependence structure. This makes it possible to assess how much the reliability of a voting bloc depends on coordination in participation and voting discipline, rather than only on the individual reliability of its members.

Consider a simple voting game with player set N with $|N| = n$ and quota $q \leq n$. The characteristic function is given by

$$v(S) = \mathbb{1}\{|S| \geq q\}.$$

Thus, a coalition generates value if and only if it contains enough effective voters to pass the proposal. Under uncertain participation, its value becomes

$$v_P(T) = \sum_{S \subseteq T} P(S, T) v(S),$$

which is the probability that the effective voting coalition reaches the quota when the formal coalition is T, given the participation kernel P. Throughout this section, we restrict attention to constant-marginal kernels and order the individual participation probabilities as $p_1 \leq \ldots \leq p_n$.

The different kernels considered below should be interpreted as different assumptions about the dependence structure of participation. Under the Owen kernel, participation events are independent, conditional on the individual probabilities $(p_i)_{i \in N}$. Hence,

$$v_O(N) = \sum_{\substack{T: T \subseteq N \\ |T| \geq q}} \prod_{i \in T} p_i \prod_{j \in N \setminus T} (1 - p_j),$$

which is a tail probability of a Poisson binomial distribution, where the success probabilities are $(p_i)_{i \in N}$ and can therefore be evaluated efficiently for even large number of players.

At the opposite extreme, the Lovász kernel assumes maximal positive dependence between participation events. It can be represented by a common random threshold. Players are ordered according to their participation probabilities, and a player participates whenever their probability exceeds this common threshold. For the voting game above, the bloc succeeds whenever the pivotal member participates, giving

$$v_L(N) = p_{n-q+1}.$$

The optimistic and pessimistic kernels provide the corresponding extremal benchmarks. Holding the marginal participation probabilities fixed, the optimistic kernel gives the largest possible success probability of the whole bloc over admissible joint participation structures, while the pessimistic kernel gives the smallest. Exact computation requires a linear program with exponentially many variables. For the empirical application below, we therefore use a grouping procedure to obtain bounds on the two extremal values. In our calculations below we used up to a third order grouping. The first order bounds are computationally trivial:

$$\min\left\{1, \frac{np_1}{q}\right\} \leq v_{P^+}(N) \leq \min\left\{1, \frac{\sum_{i\in N} p_i}{q}\right\},$$

$$\max\left\{0, \frac{\sum_{i\in N} p_i - q + 1}{n-q+1}\right\} \leq v_{P^-}(N) \leq \max\left\{0, \frac{np_n - q + 1}{n-q+1}\right\}.$$

Our framework also allows dependence to differ across sources of participation uncertainty. In particular, we can combine an Owen kernel for attendance with a Lovász kernel for conditional voting discipline. This composition is useful when absence is viewed as primarily idiosyncratic, while departures from the bloc position are allowed to reflect common political or ideological forces. If the Owen kernel is parametrized using probabilities $s_1, \ldots, s_n$, while the Lovász kernel is parametrized by ordered probabilities $d_1 \leq \ldots \leq d_n$, then

$$v_{OL}(N) = \sum_{k=q}^{n} (d_{n-k+1} - d_{n-k}) \left(\sum_{\substack{T: T\subseteq S_k \\ |T|\geq q}} \prod_{i\in T} s_i \prod_{j\in S\setminus T} (1 - s_j) \right),$$

where $S_k = \{n-k+1, \ldots, n\}$. The inner probability is again a tail probability of a Poisson binomial distribution, evaluated only over the k most reliable players and weighted by the Lovász coefficients. Consequently, the composition inherits the computational tractability of the Owen kernel while allowing richer dependence structures.

We illustrate these ideas using roll-call data from the 115th United States House of Representatives obtained from the Voteview Congressional Roll-Call Votes Database (Lewis et al., 2026). We focus on the Republican attempt to repeal the Affordable Care Act. This vote provides a natural case study because the Republican Party held a narrow majority and the number of Republicans required for passage without Democratic support was close to the size of the Republican caucus. At the time of the vote, the House contained $n = 238$ Republican representatives, and the Republican bloc required at least $q = 216$ affirmative votes to pass the legislation without Democratic support. We therefore take the 238 Republicans serving at the time of the vote as the player set and calibrate representative-specific participation probabilities using their observed voting behavior throughout the entire 115th Congress.

For each roll call, we define the Republican bloc position as the modal Republican vote, namely the vote choice receiving the largest number of Republican votes. A representative is classified as supporting the bloc whenever their vote coincides with this position. A deviation can therefore arise either because the representative is absent or because they are present, but they vote against the bloc. Let N_i denote the number of roll calls in which representative i was expected to vote, P_i the number of roll calls in which they were present, and R_i the number of roll calls in which they were present but voted against the modal Republican position. We obtain

$$s_i = \Pr(i \text{ is present}) = \frac{P_i}{N_i},$$

$$d_i = \Pr(i \text{ follows the bloc} \mid i \text{ is present}) = \frac{P_i - R_i}{P_i},$$

and hence the reduced-form probability of effective participation

$$p_i = s_i d_i = \frac{P_i - R_i}{N_i}.$$

The probabilities $(p_i)_{i \in N}$ measure the overall reliability of each representative as a member of the voting bloc. Importantly, the decomposition into $(s_i)_{i \in N}$ and $(d_i)_{i \in N}$ also allows us to distinguish between two sources of uncertainty. One natural specification is to treat attendance as approximately independent across representatives while assuming deviations from the party position to be positively dependent. This motivates comparing Owen and Lovász kernels using the calibrated $(p_i)_{i \in N}$ values as marginal probabilities and, in particular, the composition of the Owen and Lovász kernels, where attendance is modeled through the Owen kernel using $(s_i)_{i \in N}$ and the conditional voting discipline through the Lovász kernel using $(d_i)_{i \in N}$.

Table 1: Probability of reaching the voting quota under alternative participation kernels

Participation structure	Probability of reaching the quota
Empirical same-size benchmark	0.8593
Owen kernel	0.6221
Lovász kernel	0.8427
Owen-Lovász composition	0.8887
Optimistic kernel	[0.9462, 1.0000]
Pessimistic kernel	[0.0749, 0.4014]

Table 1 summarizes our results. The differences across participation structures are substantial. Under independent participation, represented by the Owen kernel, the estimated probability of reaching the quota is 0.6221. Allowing participation to be maximally positively dependent, as under the Lovász kernel, increases this probability to 0.8427. The Owen-Lovász composition yields an even higher probability of 0.8887. Thus, holding individual participation rates fixed, the predicted reliability of the Republican bloc changes by more than twenty percentage points simply as a consequence of the assumed dependence structure.

The optimistic and pessimistic kernels make this point particularly sharply. The optimistic value is bounded below by 0.9462, whereas the pessimistic value is bounded above by 0.4014. These together can be interpreted as the possible range of bloc efficiency. The numbers show that the marginal participation probabilities alone leave substantial scope for different collective outcomes. With strong positive coordination, the same individual participation rates can support a very high probability of reaching the quota. Under an unfavorable dependence structure, the probability can be substantially below the independent benchmark.

As a descriptive benchmark, we also consider roll calls in which the Republican caucus contained exactly 238 representatives, matching the size of the Republican coalition in the Affordable Care Act vote. Among 135 such roll calls, at least 216 Republicans voted in the same direction in 116 cases, giving an empirical success frequency of 0.8593. This benchmark lies between the Lovász value of 0.8427 and the Owen-Lovász value of 0.8887. The comparison should not be interpreted as a direct validation of the calibrated participation model. The 238 representatives are not necessarily the same individuals in each of the benchmark roll calls, whereas the participation probabilities are calibrated from the members of the Affordable Care Act repeal coalition. Indeed, only three roll calls in the 115th Congress involved exactly the same Republican membership as the repeal coalition. The benchmark should therefore be interpreted as a same-size, same-quota descriptive comparison.

The application illustrates the practical role of participation kernels. Given statistically estimated individual participation probabilities, the framework translates alternative assumptions about collective voting behavior into economically interpretable probabilities of coalition success. In particular, the example shows that individual participation rates do not by themselves determine the reliability of a voting bloc, the dependence structure of participation can have quantitatively large effects on the probability of collective success.

4.4 Non-cooperative games with uncertain coalitions

A natural question is whether uncertain participation may be used in a non-cooperative framework. In this section we outline a natural candidate and propose an equilibrium concept capable of capturing limited multilateral coordination between the players as governed by a participation kernel.

The interpret a failure in participation in the non-cooperative setting we offer two options: In an *absolute* setting, there is a designated action corresponding to participation failure (for instance, staying home instead of voting in an election). While this is a direction not without independent interest, it ultimately does not alter the fundamentals of the game framework. To see this, we may take a normal-form game $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N})$ such that there is a designated action $z_i \in A_i$ denoting inaction for all players. Then, define $G_P = (N, (A_i)_{i\in N}, (u_i^P)_{i\in N})$ such that

$$u_i^P(a) = \sum_{S\subseteq N} P(S,N) u_i(a_S, z_{-S}),$$

for all $a \in \prod_{i\in N} A_i \setminus \{z\}$. That is, under uncertain participation, a set of players, drawn according to some distribution of subsets of N is inactive. In the above game, each strategy profile would retain its own unique value in u_i^P and analysis of the game follows known methods.

Alternatively, we can study participation failure in a *relative* setting. This would study a coalition's ability to subvert some status quo strategy profile. In this setting, a failure to participate amounts to keep playing the status quo action even though the player was supposed to change. The natural question to ask in this setting is whether there exist action profiles that are stable against joint deviations where coalitions might fail to coordinate given some participation kernel P.

To formalize this, consider a normal-form game $(N, (A_i)_{i\in N}, (u_i)_{i\in N})$ with set of players N, action sets A_i, and payoff functions u_i for $i \in N$. Then, given a status quo strategy profile $a \in \prod_{i\in N} A_i = A$, and an alternative $a' \in A \setminus \{a\}$ define $T(a, a') := \{i \in N : a_i \neq a'_i\}$ as the set of players who play different actions in a and a'. We simplify the notation to T whenever the status quo and the alternative are clear from the context. Consider a participation kernel P from Definition 1. Then, given a, a', and P, the *expected deviation value* for a player $i \in T$ from a to a' is

$$\Delta_i(a, a', P) = \sum_{S\subseteq T} P(S, T) u_i(a'_S, a_{-S}) - u_i(a),$$

with $a'_S = \prod_{j\in N} a'_j \in \prod_{j\in N} A_j$, $a_{-S} = \prod_{k\in N\setminus S} a_k \in \prod_{k\in N\setminus S} A_k$, and $(a'_S, a_{-S}) \in A$ denoting the action profile where members of S play the same strategy as in the alternative profile a' and members of $N \setminus S$ play the same strategy as in the status quo profile a. As we take expected values, there is a natural extension to mixed strategies.

The interpretation is as follows: if a coalition T attempts to subvert the status quo a and impose a', but coordination is imperfect, members of each subset $S \subseteq T$ may end up playing the alternative action while the remaining members $T \setminus S$ continue playing the status quo action (along with $N \setminus T$, hence the set of players with the status quo action is $N \setminus S$). If $\Delta_i(a, a', P)$ is negative, then being part of the subversive coalition T is not worth it for i in expectation and a is *stable against a' under uncertain participation.* A natural equilibrium concept is then an action profile that is stable against all others as follows:

Definition 8. In normal-form game $(N, A, (u_i)_{i\in N})$, action profile $a^* \in A$ is said to be *strongly stable under uncertain participation* for participation kernel P if for every $a' \in A \setminus \{a^*\}$ there exists an $i \in T(a^*, a')$ such that $\Delta_i(a^*, a', P) < 0$.

Strongly stable equilibria under uncertain participation accounts for plausible utilitarian welfare-improving multilateral coordination between players to act as a selection device within the set of Nash equilibria. At the same time, it is capable of defining structural barriers to such coordination. For instance, a decreasing $P(T, T)$ in the size of T captures decreasing ability to coordinate larger groups.

It is straightforward to show that the above equilibrium concept is stronger than the Nash equilibrium as long as individuals are not subject to uncertainty in their own deviations.

Proposition 6. *Fix normal-form game $(N, A, (u_i)_{i \in N})$ and participation kernel P. If $P(\{i\}, \{i\}) = 1$ for all $i \in N$, every strongly stable equilibrium under uncertain participation is a Nash equilibrium.*

The strongly stable equilibrium under uncertain participation and its original namesake, the strongly stable equilibrium (Aumann, 1959) do not contain each other. That the latter does not contain the former is trivial; whenever participation probabilities are not sufficiently high for coalitions of size larger than one, a Pareto-improving multilateral move is blocked by the high probability of discoordination. For instance, the risk dominant equilibrium of the Stag Hunt game will be stable against the Pareto dominant (and strongly stable Nash) equilibrium. With large numbers of players, epistemic models lose tractability, making statistical models of player participation with an exogenous participation kernel an attractive alternative. In such problems, uncertainty in participation becomes a key stumbling block in solving large-scale coordination problems, captured cleanly by the equilibrium concept of Definition 8.

To see that strongly stable equilibria are not necessarily strongly stable under uncertain participation, consider the 2×2 game presented in Table 2. The game

	Left	Right
Up	(2,2)	(10,1)
Down	(1,10)	(0,0)

Table 2: The strong Nash equilibrium (Up, Left) is not strongly stable against uncertain participation whenever the probability of discoordination between the players is high.

has a dominant strategy equilibrium (Up, Left), which is also its unique strict and strongly stable Nash equilibrium. Yet, a participation kernel that places most probability mass on coordination failure, for instance $P(\{i\}, \{1, 2\}) = 1/2$

for $i \in \{1, 2\}$, makes it unstable. Both players would agree to deviate to (Down, Right) as the expected utility of such an agreement under participation uncertainty is $1/2 * 1 + 1/2 * 10 = 5.5 > 2$ for both (in this paper we do not ask how such an agreement could be self-enforcing; clearly, both players have an incentive to "fail" to participate). Note that (Down, Right) is also unstable against a coalitional move to (Up, Left). The logic is reminiscent of the correlated equilibrium (Aumann, 1987) where players commit to a common signal that result in mutually beneficial coordination. Here, the coordination comes from a commonly known exogenous participation environment. We leave a complete treatment of a non-cooperative model of participation uncertainty for future work.

5 Concluding remarks

In this paper we introduced an explicit model of participation uncertainty in TU-games by the use of participation kernels. Our model has applications in a variety of economic problems from uncertainty in matching market games, insurance of two-sided markets, and in evaluating the power of voting blocks in parliamentary democracies. We identify the constant-marginal property of participation environments as a crucial feature in the behavior of the problem. This property makes use of the elegant algebraic structure, is intuitive for market behavior, and is empirically falsifiable. The idea of participation uncertainty is readily extendable to a variety of contexts; price dynamics, overselling, and general applications in non-cooperative frameworks. Specifically, in two-sided markets we identify that the constant-marginal property of the participation environment functions as a market clearing condition and as an insurability condition. Further extensions indicate that participation uncertainty may account for the stochasticity of price processes and market returns, as well as overselling.

In models of voting and calculating voting power, the assumption of independent voting through Owen (1972)'s MLE has been a key hurdle, causing difficulty in reconciling theory and empirical fact (Gelman et al., 2002, 2004). Introducing other types of constant-marginal voting broadens the set of available models with no cost in complexity. For instance, comonotonic voting through the Lovász kernel by can be used to include an ideological dimension. Compositions and mixtures can account for a combination of idiosyncratic and systemic voting behavior. Key model properties remain the same or get even stronger as the constant-marginal class, unlike the class of kernels with independent participation, are closed for operations between uncertainty environments.

Our main limitation is our exogenous treatment of uncertainty. For a clean analysis we do not unpack the incentives of economic agents that lead to uncertain participation in the first place and discuss only the resultant economic and political outcomes. Endogenous participation models exist in voting (Myerson and Weber, 1993) and auctions (Lauermann and Wolinsky, 2017, 2022), but it is hard to imagine a uniform model of endogenous participation that applies to all contexts. The toolkit we provide in this paper is thus best suited for applications that make use of statistical models of participation.

References

Algaba, E., Bilbao, J. M., Fernández, J. R., and Jimenez, A. (2004). The lovász extension of market games. *Theory and Decision*, 56(1):229–238.

Aubin, J.-P. (1981). Cooperative fuzzy games. *Mathematics of Operations Research*, 6(1):1–13.

Aumann, R. J. (1959). 16. acceptable points in general cooperative n-person games. *Contributions to the Theory of Games (AM-40), Volume IV*, pages 287–324.

Aumann, R. J. (1964). Markets with a continuum of traders. *Econometrica: Journal of the Econometric Society*, pages 39–50.

Aumann, R. J. (1987). Correlated equilibrium as an expression of bayesian rationality. *Econometrica: Journal of the Econometric Society*, pages 1–18.

Bilias, Y., Georgarakos, D., and Haliassos, M. (2010). Portfolio inertia and stock market fluctuations. *Journal of Money, Credit and Banking*, 42(4):715–742.

Borkotokey, S., Gowala, S., and Kumar, R. (2023). The expected shapley value on a class of probabilistic games. *arXiv preprint arXiv:2308.03489*.

Bossert, W., Derks, J., and Peters, H. (2005). Efficiency in uncertain cooperative games. *Mathematical Social Sciences*, 50(1):12–23.

Briggs, J., Cesarini, D., Lindqvist, E., and Östling, R. (2021). Windfall gains and stock market participation. *Journal of Financial Economics*, 139(1):57–83.

Butnariu, D. (1978). Fuzzy games: a description of the concept. *Fuzzy sets and systems*, 1(3):181–192.

Calvo, E. and Santos, J. C. (2000). A value for multichoice games. *Mathematical Social Sciences*, 40(3):341–354.

Casajus, A. (2021). Extension operators for tu games and the lovász extension. *Discrete Applied Mathematics*, 288:66–73.

Charnes, A. and Granot, D. (1976). Coalitional and chance-constrained solutions to n-person games. i: the prior satisficing nucleolus. *SIAM Journal on Applied Mathematics*, 31(2):358–367.

Charnes, A. and Granot, D. (1977). Coalitional and chance-constrained solutions to n-person games, ii: Two-stage solutions. *Operations research*, 25(6):1013–1019.

Ellison, G. and Fudenberg, D. (2003). Knife-edge or plateau: When do market models tip? *The Quarterly Journal of Economics*, 118(4):1249–1278.

Ely, J. C., Garrett, D. F., and Hinnosaar, T. (2017). Overbooking. *Journal of the European Economic Association*, 15(6):1258–1301.

Faigle, U. and Kern, W. (1992). The shapley value for cooperative games under precedence constraints. *International Journal of Game Theory*, 21(3):249–266.

Gelman, A., Katz, J. N., and Bafumi, J. (2004). Standard voting power indexes do not work: an empirical analysis. *British Journal of Political Science*, 34(4):657–674.

Gelman, A., Katz, J. N., and Tuerlinckx, F. (2002). The mathematics and statistics of voting power. *Statistical Science*, pages 420–435.

Grabisch, M. and Lange, F. (2007). Games on lattices, multichoice games and the shapley value: a new approach. *Mathematical Methods of Operations Research*, 65(1):153–167.

Grabisch, M. and Lorenzini, S. (2025). Bel coalitional games with application to market games. *International Journal of Approximate Reasoning*, 184:109466.

Grabisch, M. and Lorenzini, S. (2026). Cooperative games with uncertainty: a survey. *arXiv preprint arXiv:2603.00211*.

Granot, D. (1977). Cooperative games in stochastic characteristic function form. *Management Science*, 23(6):621–630.

Habis, H. and Herings, P. J.-J. (2011). Transferable utility games with uncertainty. *Journal of Economic Theory*, 146(5):2126–2139.

Hsiao, C.-R. and Raghavan, T. E. (1993). Shapley value for multichoice cooperative games, i. *Games and economic behavior*, 5(2):240–256.

Ieong, S. and Shoham, Y. (2008). Bayesian coalitional games. In *AAAI*, pages 95–100.

Klijn, F., Slikker, M., and Zarzuelo, J. (1999). Characterizations of a multi-choice value. *International Journal of Game Theory*, 28(4):521–532.

Lauermann, S. and Wolinsky, A. (2017). Bidder solicitation, adverse selection, and the failure of competition. *American Economic Review*, 107(6):1399–1429.

Lauermann, S. and Wolinsky, A. (2022). A common-value auction with state-dependent participation. *Theoretical Economics*, 17(2):841–881.

Lewis, J. B., Poole, K., Rosenthal, H., Boche, A., Rudkin, A., and Sonnet, L. (2026). Voteview: Congressional Roll-Call Votes Database. `https://voteview.com/`. Database accessed for the 115th United States Congress roll-call voting records.

Lovász, L. (1983). Submodular functions and convexity. In *Mathematical Programming The State of the Art: Bonn 1982*, pages 235–257. Springer.

Mallozzi, L., Scalzo, V., and Tijs, S. (2011). Fuzzy interval cooperative games. *Fuzzy Sets and Systems*, 165(1):98–105.

Mareš, M. and Vlach, M. (2001). Linear coalitional games and their fuzzy extensions. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 9(03):341–354.

Myerson, R. B. and Weber, R. J. (1993). A theory of voting equilibria. *American Political science review*, 87(1):102–114.

Owen, G. (1972). Multilinear extensions of games. *Management Science*, 18(5-part-2):64–79.

Peters, H. and Zank, H. (2005). The egalitarian solution for multichoice games. *Annals of Operations Research*, 137(1):399–409.

Rochet, J.-C. and Tirole, J. (2003). Platform competition in two-sided markets. *Journal of the european economic association*, 1(4):990–1029.

Rothstein, M. (1971). An airline overbooking model. *Transportation Science*, 5(2):180–192.

Shapley, L. S. and Shubik, M. (1971). The assignment game i: The core. *International Journal of game theory*, 1(1):111–130.

Suijs, J. and Borm, P. (1999). Stochastic cooperative games: superadditivity, convexity, and certainty equivalents. *Games and Economic Behavior*, 27(2):331–345.

Suijs, J., Borm, P., De Waegenaere, A., and Tijs, S. (1999). Cooperative games with stochastic payoffs. *European Journal of Operational Research*, 113(1):193–205.

Vissing-Jørgensen, A. and Attanasio, O. P. (2003). Stock-market participation, intertemporal substitution, and risk-aversion. *American Economic Review*, 93(2):383–391.

A Appendix: Proofs

Theorem 1

Using the fact that $D_S(R,T) = \mathbb{1}\{S \cap T = R\}$, and that P is constant-marginal if and only if

$$P(R,T) = \sum_{S:S\cap T=R} P(S,N),$$

we can write

$$P(R,T) = \sum_{S \subseteq N} P(S,N) D_S(R,T)$$

for every $R, T \subseteq N$.

Theorem 2

We show that the unanimity game u_S is an eigenvector of P for every $S \subseteq N$. Since $u_S(T) = \mathbb{1}\{S \subseteq T\}$ for any T, we have

$$(Pu_S)(T) = \sum_{R \subseteq T} P(R,T) u_S(R) = \sum_{R: S \subseteq R \subseteq T} P(R,T).$$

It is clear that for $S \not\subseteq T$ the last sum is empty, while $u_S(T) = 0$ so $(Pu_S)(T) = 0 = P(S,S)u_S(T)$ holds. If $S \subseteq T$, $u_S(T) = 1$, while the constant-marginal property means

$$\sum_{R: S \subseteq R \subseteq T} P(R,T) = P(S,S)$$

therefore $(Pu_S)(T) = P(S,S)u_S(T)$ as well.

As unanimity games form a basis, every constant-marginal P admits the stated spectral decomposition. Taking the reverse direction, any matrix that satisfies $Pu_S = P(S,S)u_S$ for all S satisfies the constant-marginal property.

Proposition 1

The statement follows from the linearity of f and Lemma 2.

Proposition 2

The left-hand side is non-empty due to (N,v) being totally balanced. The set containment then follows from Theorem 1 and the Minkowski subadditivity of the core, thus the right-hand side is also non-empty. Repeating the argument for

$$\sum_{S \subseteq T} \alpha_S \mathrm{Core}(T, v|_S) \subseteq \mathrm{Core}(T, v_P|_T)$$

gives that any subgame of v_P also has a non-empty core, hence (N, v_P) is totally balanced. For convex games, the core is Minkowski additive and convexity is preserved by restricting the game and by the probabilistic transformation for constant-marginal kernels, the argument goes through with equality.

Proposition 3

As stated in the main text, we just have to show that the submarket insurance satisfies Definition 5.

Let $(x^T)_{T\subseteq N}$ be the chosen collection of submarket core allocations and let $(\mathbf{y},\mathbf{g})$ be the corresponding insurance. By construction, every buyer who trades satisfies $y_i^T+g_i^T b_i = b_i-\pi_i = x_i$ and every seller who trades likewise has $y_i^T-g_i^T c_i = \pi_i - c_i = x_i$. Non-trading buyers and sellers simply receive $y_i^T = x_i$, hence every player receives the certainty equivalent in every state.

Since $x^T \in \mathrm{Core}(T, v|_T)$, the shadow price π^T clears the realized submarket, the number of goods given up by sellers equals the number of goods taken over by buyers in every state, so we have good-neutrality in every state.

For cash-neutrality in expectation, we calculate the expected cash in-flow to the insurer from buyers.

$$E_B = \sum_{i\in B}\sum_{T\subseteq N}\alpha_T y_i^T = \sum_{i\in B}\left(\sum_{\substack{T:i\in T\\ \text{and } b_i\geq\pi^T}}\alpha_T\pi_i - \sum_{\substack{T:i\notin T\\ \text{or } b_i<\pi^T}}\alpha_T x_i\right). \tag{12}$$

As $x_i^T = 0$ when i does not trade and $x_i^T = b_i - x_i^T$ when i trades, we have $\min\{b_i, x^T\} = b_i - x_i^T$ for all T, hence the expected price can be written as

$$\pi_i = \sum_{T\subseteq N}\alpha_T\big(\mathbb{1}_T(i)\min\{b_i,\pi^T\} + (1-\mathbb{1}_T(i))b_i\big) = b_i - \sum_{T\subseteq N}\alpha_T x_i^T = b_i - x_i.$$

where the last identity uses $x_i = \sum_{T\subseteq N}\alpha_R x_i^T$ by definition.

Hence, (12) becomes

$$\begin{aligned}
&=\sum_{i\in B}\left(\sum_{\substack{T:i\in T\\ b_i\geq\pi^T}}\alpha_T(b_i-x_i) - \sum_{\substack{T:i\notin T\\ \text{or } b_i<\pi^T}}\alpha_T x_i\right)\\
&=\sum_{i\in B}\left(\sum_{\substack{T:i\in T\\ b_i\geq\pi^T}}\alpha_T b_i - \sum_{T\subseteq N}\alpha_T x_i\right) = \sum_{T\subseteq N}\alpha_T\sum_{\substack{i\in B\cap T\\ b_i\geq\pi^T}} b_i - \sum_{i\in B} x_i.
\end{aligned}$$

By very similar arguments, the sellers', whose expected price is $\pi_i = c_i + x_i$ for $i\in S$ are paid by the insurer the following amount:

$$E_S = \sum_{j\in S}\left(\sum_{\substack{T:j\in T\\ c_j\leq\pi^T}}\alpha_T(c_j+x_j) + \sum_{\substack{T:j\notin T\\ \text{or } c_j>\pi^T}}\alpha_T x_j\right) = \sum_{T\subseteq N}\alpha_T\sum_{\substack{j\in S\cap T\\ c_j\leq\pi^T}} c_j + \sum_{j\in S} x_j$$

Taking the difference we have

$$E_B - E_S = \sum_{T \subseteq N} \alpha_T \left(\sum_{\substack{i \in B \cap T \\ b_i \geq \pi^T}} b_i - \sum_{\substack{j \in S \cap T \\ c_i \leq \pi^T}} c_j \right) - \sum_{i \in N} x_i = \sum_{T \subseteq N} \alpha_T \sum_{i \in N} x_i^T - \sum_{i \in N} x_i = 0,$$

where the second identity uses the fact that total surplus in any submarket must equal the total surpluses of the players and the third identity is by the definition of the ex ante core as the expected total surplus.

Proposition 4

P_t is convex-marginal for every $t \geq 0$ as the set of convex-marginal kernels is convex (Corollary 1). The result then follows from Proposition 1 and Lemma 2.

Proposition 6

Trivially, for every Nash profile a^* and action $a_i' \in A_i$ we have $u_i(a^*) \geq u_i(a_i', a_{-i}^*) = P(\{i\}, \{i\}) u_i(a_i', a_{-i})$ whenever $P(\{i\}, \{i\}) = 1$