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Cabau, Noemie and Tenev, Anastas

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Corvinus University of Budapest

Bridging Network Gaps Under Limited Perception

Noémie Cabau[a] and Anastas P. Tenev[b,c]

Abstract

We consider a model of network formation in which agents benefit from intermediation insofar as they are the *only bridge* between agents who would otherwise be unconnected. Their payoffs depend proportionally on their frequency as intermediaries. We analyze two types of payoffs: global, based on the intermediation across the whole network; and local, based on an agent's *perceived* centrality in a subnetwork, measured within a fixed radius around the agent. These capture two measurements of network importance: objective and perceived (subjective). Agents have an incentive to form links so that they become key intermediaries in their (perceived) network and to bypass other key intermediaries in it. However, because payoffs are determined relative to other agents' contributions, an agent might refrain from forming links if this would expose them to relatively more essential players. In case agents' payoffs are global, the pairwise stable networks are connected and have at most one (central) agent with a nonzero payoff. In the case where agents' payoffs are local, we describe the pairwise stable networks and some characteristics of the individual payoffs within them. A class of networks we call *augmented stars* (stars in which spokes may be connected but no cycle encompasses all spokes) is pairwise stable for both global and all possible local payoffs.

Keywords: Social Networks; Structural Holes; Network Formation; Intermediation
JEL classification: C72, D85

1 Introduction

An agent fills a *structural hole* in a network if their links connect otherwise disconnected groups, that is, they bridge gaps in the network (cf. the seminal work by Burt (1992)). Some reasons individuals filling structural holes can outperform their peers include having access to different and novel ideas, controlling the information flow between groups, and being perceived as more "important" by others. The extent to which an agent is central, in the structural-hole sense, within their "ego-network" of direct neighbors can predict performance and has been linked, among other outcomes, to managers' career achievements (Burt, 1992, 2004, 2015). Higher structural hole scores are achieved when one has many contacts that do not have a link between them (e.g., see Burt's notion of *network constraint*, reformulated by Everett and Borgatti (2020) for the case of unweighted and

[a]Institute of Finance, Corvinus University of Budapest, Fővám tér 8, 1093 Budapest, Hungary. E-mail: noemie.cabau@uni-corvinus.hu

[b]Institute of Economics, Corvinus University of Budapest, Fővám tér 8, 1093 Budapest, Hungary. E-mail: ap.tenev@uni-corvinus.hu

[c]Corresponding author.

undirected networks).[1] We stay faithful to this notion and propose a centrality measure that depends on the marginal relative contribution one makes in one's network.

We analyze a model of network formation in which agents form links to maximize their relative centrality within the network they perceive. Agents can *only* benefit from their position if they are the only bridge between otherwise disconnected agents and their payoff increases proportionally the more often they act as an essential intermediary between pairs of agents. While the number of agent pairs for which a person is an intermediary matters, the distance between the agents in the pair does not. That is, being the only mediator between agents who are two steps apart is as valuable as being the only mediator between agents who are three or more steps apart. We consider two benchmarks for our setup. The first benchmark considers agents with complete knowledge of the network and, hence, their objective position within it. The second benchmark has agents with very limited information about their network and who only see their immediate neighbors. Finally, we analyze the intermediate cases in which agents know the structure of the network a fixed number of steps away ("radius" of interest) from themselves but do not have full information about it. The main motivation for this paper comes from the empirical evidence outlined above and is, in this sense, behavioral as it measures agents' own perceived importance in a network.

Our results focus on the architecture of, and the payoff distribution in pairwise stable networks. When agents only consider their immediate network of neighbors, they always perceive their neighbors as inessential – since extremities of networks cannot lie in between other agents, which is assumed to be the sole source of benefits in this game. Thus, the incentive to form links reduces to *expanding one's network,* until finding someone to whom no one else (among one's current neighbors) is linked. As soon as such an agent is found, the player earns the highest payoff of 1. A nonempty pairwise stable network either has all agents having a payoff of 1 or they do not, in which case the network is dense and precludes agents with a null payoff from finding an agent to whom no one around them is linked (all other individuals are thus within two links of the agent).

The incentive to form links is markedly different when agents consider the entire network. Because link formation never expands the network seen by individuals – as they already see it all – the incentive to form links is purely to reduce competition from other essential intermediaries. This competitive pressure gives rise to nonempty pairwise stable networks in which either no agent is essential for any connection (and all players get a 0 payoff) or one agent is uniquely essential and faces no competition, i.e., all have zero except for this one agent, who earns the maximal payoff of 1.

When agents see beyond the network of their immediate neighbors without nonetheless accounting for the whole network, both incentives to form links – expansion and bypassing competitors – are present. However, links that expand one's network now come at the cost of increasing both the number and the essentiality of some other intermediaries in the network. Whether the benefits and costs of links balance each other out is highly network-specific. By bounding payoffs in pairwise stable networks, we show that nonempty ones have all agents maintaining links, and that those with a payoff of 0 are bundled together in the same component.

When trying to unify our results, two main findings emerge. First, there is not, generally, an inclusion relationship between the sets of pairwise stable networks for different

[1] This can also be tied to the notion of betweenness centrality. For a model which balances maximizing weighted betweenness centrality and linking costs to obtain results for small-world networks, see Samoylenko et al. (2023).

radii of interest. However, these sets are the same provided that the radius of interest exceeds the number of players. Finally, stars and some close variants of stars are shown to be pairwise stable for any radius of interest.

1.1 Related literature

Our model contributes to the economic modeling literature related to structural holes. Several papers have explicitly modeled this phenomenon. Goyal and Vega-Redondo (2007) consider a model in which agents form links in order to create surplus, gain rents from intermediation, and to bypass others who share the intermediation rent. We consider a more extreme case in which rent from a transaction exists only when a person is unavoidable as a middleman between two agents, which means that links are formed to bypass others. Further, we consider situations in which agents receive payoffs only from the part of the network they observe, thereby adding behavioral and informational aspects to our model. However, in this case, the incentive to expand one's network is not always there. An agent may prefer to stay blissfully ignorant, thinking of themselves as a big fish in a small pond rather than a small fish in a big pond. So, in these cases, our setup can create stable, disconnected networks. Kleinberg et al. (2008) focus on the structure of equilibria in a game of network formation in which intermediation benefits accrue to agents if they lie on a path of length 2 between others. However, this path need not be unique. Additional differences from our setup include their choice of equilibrium concept (pure Nash vs. pairwise stability), their use of explicit linking costs, and their focus on the computational aspects of equilibria. Buskens and Van de Rijt (2008) employ Burt's constraint measure and simulations to find that, in stable networks, no one can maintain a structural advantage in the long run, and that dominant stable networks distribute benefits evenly. In our model, this need not be the case. In fact, in our model, benefits are distributed equally only when no agents are essential, and everyone gets a zero payoff. Instead, agents can obtain a positive payoff only if they are asymmetrically essential in the network.

The way we model individual contributions to one's network builds upon that of Cabau (2026). In her model, as in ours, an agent is said to contribute to connecting a pair of individuals if they are essential for the connection. However, we do not assume that one's own connections, i.e., the agents whom one can reach, are a source of benefits as we focus on pure intermediation benefits. Furthermore, we define an agent's network benefit as their *relative* rather than their absolute contribution. The change from absolute to relative network contributions has important implications for agents' incentives as follows. A link to an agent who offers access to parts of the network otherwise unreachable mechanically increases one's absolute contribution, since the agent now mediates more connections. However, network expansion may reduce one's relative contribution to it, especially when newly added parts of the network are highly centralized around a few individuals. This explains why the set of nonempty disconnected network architectures is relatively larger in this game than in Cabau's.

Our setup is also reminiscent, to a lesser extent, of the model by Bochet et al. (2021) in which agents differ in their perceived network. However, they focus on payoffs that differ from ours in that the perceived network is not an expression of the part of the network the agents care about; rather, it is an agent-specific misspecification of the actual network that distorts the payoffs. Finally, motivated by applications such as Tullock contests and Cournot games, having agents with symmetric network positions corresponds to an equal split of positive payoffs in their setup; in our case, this necessarily leads to a null payoff.

The current setup also connects to models of connectivity under node failure, e.g. Billand et al. (2016). However, we focus on the potential benefits that agents can gain from their crucial dominant position in the network, rather than designing networks that minimize the likelihood of network failure by reducing overreliance on specific agents.

2 Setup

We introduce the model, then examples that illustrate the payoffs and their properties.

2.1 Model

There is a finite set $N = \{1, \ldots, n\}, n > 3$ of agents. Let i be any typical member of N. A strategy s_i for agent i is a set of other agents with whom i proposes to form links. We restrict attention to pure strategy profiles for all agents. The set of agent i's pure strategies is $S_i = \mathcal{P}(N \setminus \{i\})$, the powerset of $N \setminus \{i\}$. We denote by $S = \Pi_{i=1}^n S_i$ the space of pure strategy profiles. A link between i and j is formed if, and only if, both $i \in s_j$ and $j \in s_i$; we denote the formed link by ij and it is undirected. Any given strategy profile $s = (s_1, \ldots, s_n)$ then induces a *global network* $g(s)$ whose set of nodes is the set of agents N and whose set of links is $L(g(s))$. Throughout, we say that agents i and j are connected if there is a path between them. There is a *path* linking agents i and j in some network g if either the link ij exists in g, or there is a sequence of links, $ii_1, \ldots, i_k i_{k+1}$ where $j \equiv i_{k+1}$ and all agents along the sequence are distinct. We denote by $d(i, j; g)$ the geodesic *distance* between i and j in network g. Note that it is finite if i and j are connected; if they are not, we set $d(i, j; g) = +\infty$.

Graph-theoretic concepts. We introduce a few additional graph theoretic concepts. A *component* A is a set of agents that are connected to each other, and there is no strict superset A' of A for which this is true. A network g is said to be *connected* if it has a unique component, and *disconnected* otherwise. The *empty* network is the network of no links. A *star* network has one central agent with links to all other agents; each other agent has a single link, to the central agent. A *cycle* has agents, $i_1, \ldots, i_k$ with $k \geq 3$, maintaining the following sequence of links $i_1 i_2, \ldots, i_{k-1} i_k, i_k i_1$.

Payoffs. Agents form links to increase some measure of their *perceived* centrality. We suppose that each agent has some local network of interest, denoted g_i^x, where $x \in \mathbb{N}_{>0}$ refers to the radius of agent i's network. Formally, given a global network g, the set of agents and links in g_i^x are, respectively,

$$N(g_i^x) = \{j \in N : d(i, j; g) \leq x\}, \quad L(g_i^x) = \{ab \in L(g) : a, b \in N(g_i^x)\}.$$

An agent perceives himself as central to the extent of his relative contribution to his network's (g_i^x) connections.

For any given network g, consider the network g_{-i} that results if i leaves the network g. Formally, g_{-i} is defined on the set of nodes $N(g) \setminus \{i\}$ and on the set of links $L(g_{-i}) = \{ab : a, b \in N(g) \setminus \{i\}\}$. This way, agent i's absolute contribution to g is:

$$c_i(g) = \left| \left\{ \{a, b\} \in (N(g) \setminus \{i\})^2 : d(a, b; g) < +\infty, \quad d(a, b; g_{-i}) = +\infty \right\} \right| \tag{1}$$

and his relative contribution is:[2]

$$r_i(g) = \begin{cases} \frac{c_i(g)}{\sum_{j \in N(g)} c_j(g)} & \text{if } c_j(g) > 0 \text{ for some } j \in N(g), \\ 0 & \text{otherwise.} \end{cases}$$

That is, agent i's absolute contribution to g is the number of unordered pairs of agents that cease to be connected if i and all links adjacent to them are removed from g. The relative contribution $r_i(g)$ measures the share of essential links in the specific network which are attributable to the agent, when accounting for the essential links of all individuals in g.

It is worth noting that, unlike in other games of network formation, reaching others is *not* a source of benefits in this game. An agent benefits from their links to the extent they connect others; indeed, one's absolute contribution counts the number of times the agent acts as an essential "bridge" in others' connections and hence can thwart a potential communication/ cooperation between others.

For $u : S \to \mathbb{R}_+$, agent i's *payoff* is given by his relative contribution to his local network g_i^x,

$$u_i(s) = r_i(g_i^x(s)) \tag{2}$$

An agent, therefore, has a non-zero payoff only if he is *essential* for connecting some pair of individuals. If the agent disappears from the network, those individuals would have no path to one another. This payoff function also implies that if a network has no key intermediaries everyone will have a zero payoff.[3,4] In effect, this means that if everyone has a symmetric absolute contribution, the rents from intermediation are nonexistent (since everyone cannot be essential at the same time). Only asymmetric intermediation contributions produce positive payoffs.

Stability notion. Our equilibrium concept is *pairwise stability* (Jackson and Wolinsky, 1996). That is, in a pairwise equilibrium, no agent is willing to cut a link, and no pair of agents can gain from forming a link (at least one of them strictly).

Definition 1 (Pairwise equilibrium)**.** *A strategy profile s^* is a pairwise equilibrium if the following conditions hold:*

- *For any agent $i \in N$ and for each $s_i \in \{s_i' \in S_i : s_i' \subset s_i^*, |s_i^* \setminus s_i'| = 1\}$,*

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*).$$

- *For any pair of agents $i, j \in N$ and the strategy pair $(s_i, s_j) = (s_i^* \cup \{j\}, s_j^* \cup \{i\})$, if*

$$u_i(s_i, s_j, s_{-i-j}^*) > u_i(s^*) \quad \Rightarrow \quad u_j(s_i, s_j, s_{-i-j}^*) > u_j(s^*).$$

[2]This type of proportional payoff can result from an α-asymmetric Nash bargaining, c.f. Dagan and Volij (1993). For further non-cooperative support for the Asymmetric Nash bargaining solution, see Miyakawa (2006); Laruelle and Valenciano (2008); Britz et al. (2010). The papers imply that in a bargaining game where the proposer in each period is drawn from an invariant probability distribution given by the relative contributions, the stationary equilibrium payoffs would converge to the Asymmetric Nash bargaining solution with that probability distribution as the weight vector.

[3]For example, the potential rents which can be extracted from connecting others disappear in a fierce Bertrand-type competition once there are multiple intermediaries between different agents. If they can substitute for each other, their competition for rents brings them to a payoff of 0.

[4]In the special case where the agents play an equilibrium that gives them all zero payoff, one can set the agents' bargaining powers to $1/n$, resulting in the symmetric Nash bargaining solution.

At times, we use a refinement of pairwise stability due to Goyal and Vega-Redondo (2007). A strategy profile is a bilateral equilibrium if no agent or pair of agents gains from cutting links or from proposing a link to each other, and at least one of them strictly benefits. This concept refines the seminal pairwise stability concept of Jackson and Wolinsky (1996) by allowing pairs of players to cut and add links simultaneously.

Definition 2 (Bilateral equilibrium)**.** *A strategy profile s^* is a bilateral equilibrium if the following conditions hold:*

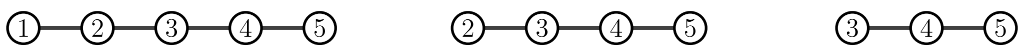
- *For any agent $i \in N$ and every $s_i \in S_i$, $u_i(s^*) \geq u_i(s_i, s^*_{-i})$.*
- *For any pair of agents $i, j \in N$ and every strategy pair (s_i, s_j),*

$$u_i(s_i, s_j, s^*_{-i-j}) > u_i(s^*) \quad \Rightarrow \quad u_j(s_i, s_j, s_{-i-j}) < u_j(s^*).$$

2.2 Illustrative examples

Before moving to the analysis, we demonstrate the mechanics of the model (Example 1) and then we show that increasing the radius of interest has an ambiguous effect on an agent's payoffs (Example 2). That is, as agents see a bigger part of the network, they might realize that they, but also others, are not as central as they initially thought. In the most extreme cases, they might realize they are not essential at all and are encompassed in a big cycle.

Example 1 (Payoffs)**.** *Consider the global network g with $n = 5$ shown in Figure 1a. If $x = +\infty$ for all agents, agent 1 has a zero payoff, since if the agent is removed from the network, the others would still be connected. Therefore $c_1(g) = 0$. The same reasoning applies to agent 5. Agent 2 is essential for connecting agent 1 to agents 2, 3 and 5. Thus, $c_2(g) = 3$. By symmetry, $c_4(g) = 3$. Finally, agent 3 is essential for connecting agent 1 with agents 4 and 5, agent 2 with agents 4 and 5. Thus, $c_3(g) = 4$. Overall, $\sum_{i \in N(g)} c_i(g) = 10$, hence payoffs are $u_1(g) = u_5(g) = 0, u_2(g) = u_4(g) = \frac{3}{10}$ and $u_3(g) = \frac{4}{10}$.*



(a) Network $g = g_3^2$. (b) Network g_4^2. (c) Network g_4^1.

Figure 1: Network g form different perspectives.

Suppose now that $x = 2$ for agent 4, and consider his local network g_4^2 in Figure 1b. Given the global network g shown in Figure 1a, agent 1 does not belong to g_4^2 since the latter is three links away from agent 4. Agents 2 and 5 are end–points of g_4^2; as such, agent 4 perceives them as inessential, and $c_2(g_4^2) = c_5(g_4^2) = 0$. Agent 4 perceives agent 3 as equally central. Contributions as perceived by agent 4 are then $c_2(g_4^2) = c_5(g_4^2) = 0$ and $c_3(g_4^2) = c_4(g_4^2) = 2$, which yields the payoff of $u_4(g_4^2) = \frac{1}{2}$ to agent 4.

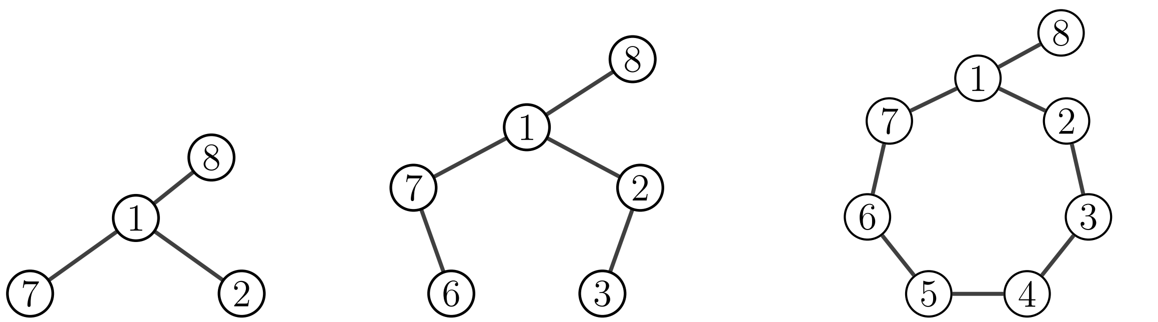
Agent 3's local network is the global network in Figure 1a. It can be verified that $c_1(g_3^2) = c_5(g_3^2) = 0$, $c_2(g_3^2) = c_4(g_3^2) = 3$, $c_3(g_3^2) = 4$, hence agent 3's payoff is $u_3(g_3^2) = \frac{4}{10}$.

Finally, when $x = 1$, the central agents 2, 3 and 4 perceives themselves as the center of a star as shown in Figure 1c. This yields the maximum payoff of 1 to each of them. Table 2 below summarizes the observations.

Agent	1	2	3	4	5
$x = +\infty$	0	0.3	0.4	0.3	0
$x = 1$	0	1	1	1	0
$x = 2$	0	0.5	0.4	0.5	0

Figure 2: Payoffs of all agents under different radii of interest x. $\triangle$

Example 2 (Payoffs are not monotonic in the radius of interest)**.** *Fixing the global network, an agent's payoff in his subnetwork g^x is generally not monotonic in x. That is, if we consider the same network, if agent 1 only sees within a range $x = 1$ (Figure 3a), they might have the maximum payoffs. Expanding the range to $x = 2$ reveals that other agents are also essential intermediaries, and this yields lower than maximum payoffs (Figure 3b). Finally, seeing the whole network might once again reveal that the ones who were perceived as essential intermediaries, i.e., 7 and 2, are actually interchangeable and therefore have a 0 contribution, leaving 1 to be the only indispensable player (Figure 3c).*



(a) Network g_1^1 (b) Network g_1^2 (c) Global network $g_1^\infty = g_1^3$

Agent	1
$x = 1$	1
$x = 2$	0.5
$x = +\infty$	1

(d) Payoffs of agent 1 depending on the radius of interest

Figure 3: Networks from agent 1's perspective, g_1^x, for different radii of interest x and the corresponding payoffs. $\triangle$

The nature of the payoff function makes the dynamics of this model fundamentally different from those of many other models (e.g., Goyal and Vega-Redondo (2007); Cabau (2026)) in an important way, because new connections do not create surplus in and of themselves and are not necessarily beneficial. For example, the centers of the two stars in Figure 4 below will not want to form a link for any radius of interest $x \neq \infty$. This happens because, by seeing only their component, they feel satisfied that they are the only important intermediary; they maintain a monopoly over their sphere of influence and avoid competition with players in other spheres.

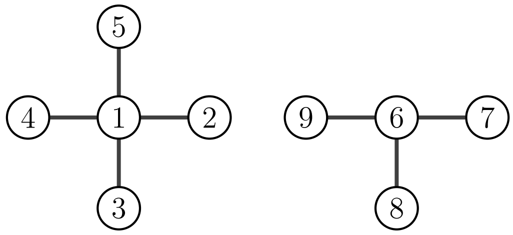


Figure 4: For any $x \neq \infty$ the centers of the two stars, 1 and 6, will not have an incentive to connect. They both have a payoff of 1 and hence no further incentive to create new links.

We further explore the differences between this model and other setups in Appendix B.

3 Analysis

This section fully characterizes the set of equilibrium payoff vectors in the case where agents have a radius of interest either $x = 1$ or $x = +\infty$. When $x = 1$, each agent considers their "ego-network" of direct neighbors; when instead $x = +\infty$, they consider their objective structural position in the global network. Afterwards we provide partial characterizations for the intermediate cases, i.e. $1 < x < +\infty$. The last subsection analyses to what extent the pairwise equilibria for different x overlap.

For each possible radius x, we first consider pairwise stability, then the bilateral equilibrium concept as a refinement of the former. Doing so allows us to "advantageously relocate" agents with a null payoff in a pairwise stable network by having them cut all their links, and form one with an agent elsewhere in the network that may benefit from it. Generally, applying this equilibrium notion removes nonempty networks with several 0-payoff agents, but preserves the pattern of links among those with positive payoffs.

3.1 Equilibria when agents care about their global network

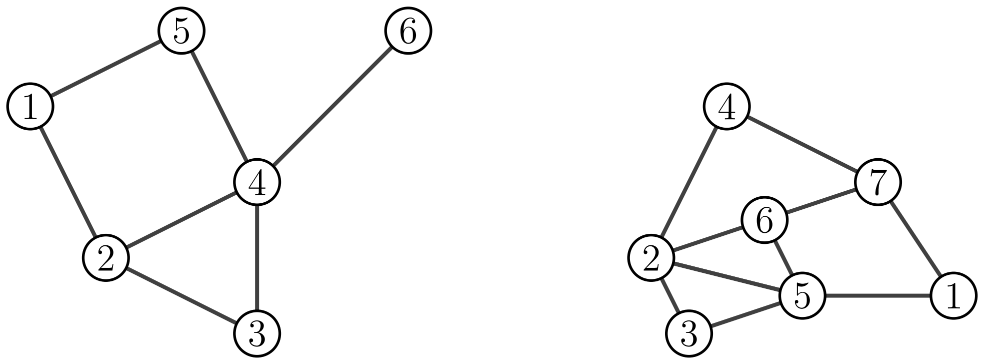
Suppose that $x = +\infty$, so that each agent cares about their own, and others' contributions to the connections across the global network. Our first result concerns pairwise-stable networks: in any such network, *at most* one agent has a strictly positive payoff. Therefore, either equilibrium networks are egalitarian with all agents having a null payoff, or they are as unequal as possible, with all agents having a null payoff except for one who earns the highest possible payoff of 1.

Proposition 1. *Suppose $x = +\infty$. A strategy profile s^* is a pairwise equilibrium if, and only if, $u_i(s^*) > 0$ for at most one agent. If such an agent exists, then $u_i(s^*) = 1$. Furthermore, a pairwise stable network is either connected or empty.*

The proofs of all statements are relegated to Appendix A. The logic behind Proposition 1 is as follows. When $x = +\infty$, any agent who earns a positive payoff imposes a competitive pressure on all other agents in the network – i.e., this agent contributes to the denominators of all players' payoffs. Also, one makes a positive contribution only if one is essential for connecting at least one pair of individuals – graphically, one must lie on the unique path connecting two agents. Observe the following: along that path, any pair of individuals such that one is located upstream, the other downstream of the positive payoff agent would, by forming a link between themselves, strictly decrease the contribution made by the positive payoff agent without any further consequences on anyone else's contributions. Whether this deviation is profitable depends on the deviating agents' payoffs. If one of them has a positive payoff originally, the deviation is profitable, and the network is not pairwise stable. This implies that in pairwise stable networks, at most one agent per component earns a positive payoff.

The second part of the statement is about the connectedness of the network. Clearly, the empty network is pairwise stable because no two agents forming a link can benefit from forming a link. If the network is not empty and there is more than one component, it must be the case that each component has at least one agent with a null payoff, as argued earlier. However, such agents would strictly benefit from forming a link; therefore, a network is a pairwise equilibrium if it is either connected or empty.

Figure 5 below presents two examples of pairwise stable networks based on global payoffs. On the left panel, agent 4 acts as the main intermediary and has the highest payoff of 1 (illustrating that agents can have positive payoffs in a global network). In the right panel, no agent is essential, and they all earn a payoff of 0.



(a) Pairwise stable network with $n = 6$. Agent 4 has a payoff of 1, the rest have 0.

(b) Pairwise stable network with $n = 7$. All agents have a null payoff.

Figure 5: Only global payoffs matter.

The next proposition states that the empty network is the only network that can be sustained in a bilateral equilibrium.

Proposition 2. *Suppose $x = +\infty$. There exists an essentially unique bilateral equilibrium, and it maps to the empty network.*

A formal proof is omitted; the result rests on the following intuitive ideas. A bilateral equilibrium is also a pairwise equilibrium, since the second concept refines the first. As such, let us restrict attention to pairwise equilibria. We first discuss the case of the empty network. The network is pairwise stable, since adding a unique link between agents never improves their payoff (since payoffs depend on pure intermediation in others' connections). The empty network is also a bilateral equilibrium because the allowed deviations effectively lead to the formation of at most one new link. As just mentioned, adding a link to the empty network does not strictly improve the payoff of the agents adjacent to it.

The remaining pairwise equilibria form connected networks, with at least $n-1$ agents having a null payoff. Consider any such network, and an agent i with a null payoff. Let agent i deviate by cutting all his links and having a link with another agent, j, who also has a null payoff. After the deviation, agent i is an endpoint of the network and earns a payoff of 0. The deviation is then payoff–equivalent to i. However, the deviation renders agent j essential in all connections with i. Therefore, if i and j were not linked in the original network, j is now willing to form a link with i after the latter has cut all his links. If i and j were linked in the original network, the link persists, and agent j is strictly better off after that i has cut all his other links. The original network is, therefore, not a bilateral equilibrium.

3.2 Equilibria when agents only care about their immediate local neighborhood

Suppose instead that $x = 1$, so that they only consider their ego-network of direct neighbors. It is worthwhile noting that any agent perceives the others in his local network of interest as inessential, i.e., $c_j(g_i^1) = 0$ for any $i \in N$, for all $j \in N(g_i^1) \setminus \{i\}$.[5] Thus, each agent's payoff is either 1 or 0 in any network. In particular, an agent earns a null payoff if, and only if, there is a cycle that encompasses all agents in i's local network.

Our first result provides a characterization of the pairwise equilibria. The proposition emphasizes how equilibrium payoffs depend on the connectedness of the global network.

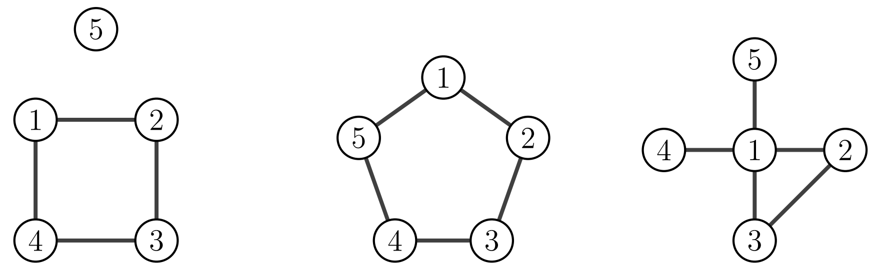
Proposition 3. *Suppose $x = 1$. A strategy profile s^* is a pairwise equilibrium if, and only if, either*

1. *$g(s^*)$ is the empty network, and $u_i(s^*) = 0$ for all agents $i \in N$.*
2. *s^* is such that $g(s^*)$ is disconnected non-empty. Further, $u_i(s^*) = 1$ for each $i \in N$ who have links adjacent to them in $g(s^*)$, and otherwise $u_i(s^*) = 0$ and i is an isolated agent.*
3. *s^* is such that $g(s^*)$ is connected, and $u_i(s^*) = 1$ for all $i \in N$.*
4. *s^* is such that $g(s^*)$ is connected, and if $u_i(s^*) = 0$ for some $i \in N$, then $d(i, j; g(s^*)) \leq 2$ for any $j \in N$ – hence, the diameter of $g(s^*)$ is at most 4.*

Here we provide the main logic behind Proposition 3. Part 1 follows immediately from the observation that adding a link in an empty network does not strictly improve the payoff of either agent and there are no links to remove. In case of a disconnected network (statement 2), adding a link between two components does not strictly benefit either agent if their initial payoffs are 1, or if one of them has a payoff of 0 and has no links (if the agent has links, then the additional link would make them essential). By a similar logic, if all agents have the maximum payoff of 1, then no possible improvement can be made (statement 3). Finally, for statement 4, if a network has an agent with a null payoff, the agent cannot be more than two steps away from any other agents because the zero-payoff agent can strictly benefit from forming a link (the other agent being at least indifferent). This implies that the diameter of the network is at most 4, since any two agents would be at a distance of at most 2 from the zero-payoff agent.

Figure 6 presents network architectures that can serve as an example for the non-trivial cases outlined in Proposition 3. The next result characterizes the bilateral equilibria.

[5]This is a general feature of our model. All peripheral agents at a distance x are perceived as inessential.



(a) statement 2 (b) statement 3 (c) statement 4

Figure 6: A selection of pairwise stable networks related to Proposition 3.

Proposition 4. *Suppose $x = 1$. A strategy profile s^* is a bilateral equilibrium if, and only if, either*

1. *$g(s^*)$ is the empty network, and $u_i(s^*) = 0$ for all agents $i \in N$.*

2. *s^* is such that $g(s^*)$ is disconnected non-empty. Further, $u_i(s^*) = 1$ if $i \in N$ has links adjacent to them in $g(s^*)$, and otherwise, $u_i(s^*) = 0$ and i is an isolated agent.*

3. *s^* is such that $g(s^*)$ is connected, and $u_i(s^*) = 1$ for all in $\in N$.*

The intuition is as follows. Recall that the set of bilateral equilibria is a subset of the set of pairwise equilibria. Comparing Propositions 3 and 4, one notices that the bilateral concept eliminates the connected networks in which some agents (but not all) have a null payoff. Consider any such network and agent. Since $x = 1$ for this agent, it means that in g_i^1, there is a cycle that encompasses all agents in $N(g_i^1(s^*))$. For agent i to earn a payoff of 1, it is sufficient that any one of his neighbors cuts their link in the aforementioned cycle. For this neighbor, whom we refer to as j, cutting the said link never decreases their payoff. The reason is as follows. First, after the deviation, j still perceives his neighbors as inessential. Thus, a change in j's payoff can only come from a change in j's own contribution to his network's connections. If j's original contribution was strictly positive (hence $u_j(s^*) = 1$), then it could not owe to the deleted link since that link was with an agent to whom i is also connected, and $i \in N(g_j^1(s^*))$. Thus, j's deviation leaves him indifferent, and strictly improves i's payoff. Therefore, s^* is not an equilibrium.

3.3 Intermediate cases

In this section, we suppose that the agents' radius of interest, x, is such that $1 < x < \infty$. The next proposition and its corollary provide necessary conditions on the architectures of pairwise stable networks and on the distribution of payoffs across the network. We show, among other things, that agents with a null payoff have links; if several such agents exist, they belong to the same component.

Proposition 5. *Suppose that $1 < x < \infty$. If the strategy profile s^* is a pairwise equilibrium, then one of the following statements is true.*

1. *$g(s^*)$ is the empty network.*

2. *$g(s^*)$ is nonempty, all agents $i \in N$ for whom $u_i(s^*) = 0$ have links, and they belong to the same component.*

3. *$g(s^*)$ has at most one acyclic component, and it is a star.*

Corollary 1. *Suppose that $1 < x < \infty$. For any given component A in a pairwise stable network $g(s^*)$, either one of the two following statements applies:*

- *$\frac{1}{x} > u_i(s^*) > 0$ for all $i \in A$: all agents in A have a strictly positive payoff, and it is strictly lower than $1/x$.*

- *$u_i(s^*) = 0$ for some $i \in A$.*

We give some intuition for the results in Proposition 5 and the corollary. First, the empty network is pairwise stable for any radius of interest, since adding a link between two individuals can never benefit either. We turn to Statement 2. Consider first nonempty, disconnected networks in which several agents have a null payoff, and at least one of them has links. If there is another zero-payoff agent in another component, a profitable deviation is for them to form a link. Indeed, the agent(s) with links strictly benefit(s): the formed link now merges two components into one, and they act as essential bridges in inter-component connections. If one of the two agents had no links initially, the latter is indifferent between accepting and rejecting. To see why, note that accepting the link changes the agent's structural location from being isolated to being an endpoint of some component, which, in either case, gives them a null payoff. This leaves us with nonempty networks in which agents with a null payoff are "isolated", having no links. Therefore, all agents with links in these networks have a strictly positive payoff. In Appendix A, we derive an upper bound on the payoffs of such agents: they cannot exceed $1/x$. However, for such low payoffs, we show that each agent would strictly benefit from forming a link with an isolated player. As argued earlier, the isolated agent does not oppose the link formation. Therefore, these networks allow profitable bilateral deviations and are consequently not pairwise stable. This finally leaves us with networks that satisfy Statement 3: agents with a null payoff are not isolated—i.e., they have links — and if several of them have zero payoff, then they belong to the same component. If the component is acyclic, it is a tree, which means there is at least one agent i who is a leaf in this tree and hence has a payoff of zero. Since the radius of interest is greater than 1, there will be two neighboring agents with positive payoffs who observe the zero-payoff agent, one of them strictly closer to i. The agent farther away will want to bypass the closer one and gain greater relative influence by connecting to i. This would strictly benefit the agent and leave i indifferent.

To illustrate Proposition 5, Figure 7 below provides an example of a pairwise stable network in which all agents have a positive payoff. Further, Figure 8 in the next section specifically illustrates statement 2.

Proposition 6. *Suppose that $1 < x < \infty$. If s^* is a bilateral equilibrium, then one of the following statements is true:*

1. *$g(s^*)$ is the empty network, and $u_i(g^*) = 0$ for all $i \in N$.*

2. *$g(s^*)$ is non-empty, and $u_i(s^*) > 0$ for all $i \in N$.*

3. *$g(s^*)$ is connected, and $u_i(s^*) = 0$ for a unique agent $i \in N$. Furthermore, $g_i^x = g(s^*)$.*

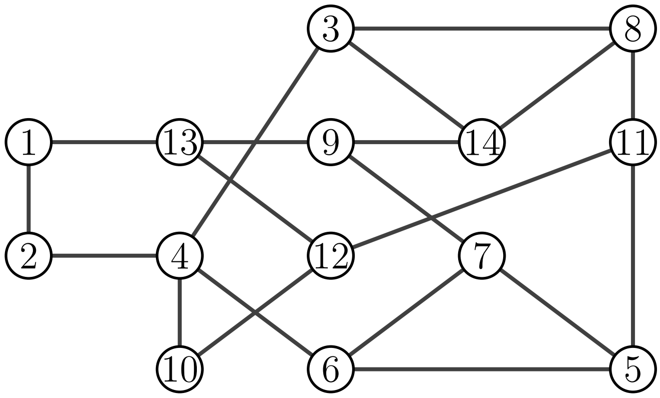


Figure 7: A simulated pairwise stable network with $n = 14$ and $x = 2$. All nodes have payoffs in the range $\{0.32, 0.39\}$.

The bilateral equilibrium refines pairwise stability by retaining networks in which at most one agent has a null payoff. We restrict attention to networks in which all agents with a null payoff belong to the same component (by Statement 3 in Proposition 5), and there are at least two. Let them deviate as follows: either of them cuts all their links, and they mutually offer to form a link. In effect, the agent who did not cut all their links is essential to all agents in the component for reaching the other; the other, as argued previously, is an endpoint of the component and thus earns a payoff of zero. Hence, one agent strictly benefits, the other is indifferent, and the deviation is consequently strictly profitable.

Statement 3 in Proposition 6 further claims that the agent with zero payoff "sees" the whole network $g(s^*)$ – hence, $g(s^*)$ is connected. To see why, assume that this is not the case. The same deviation mentioned above can be applied to our zero-payoff agent and to anyone the latter does not see. The deviation strictly benefits the other agent because their payoff cannot exceed the bound of $1/x$ that we derived in the proof of Proposition 5, rendering the deviation strictly beneficial to them. Our zero-payoff agent does not oppose the formation of the link, as this changes their structural location to an endpoint of the component, yielding the same payoff of zero.

3.4 Discussion

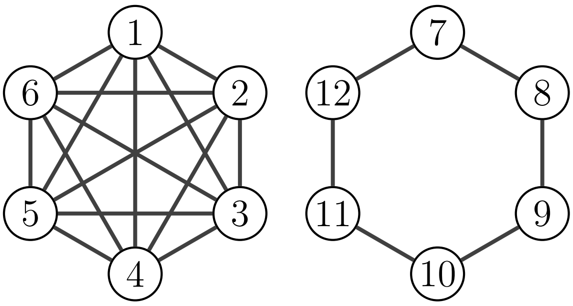
The reader may have noticed that the set of pairwise stable networks and payoffs when agents care about their contribution to the whole network (i.e., $x = +\infty$) is a strict subset of those when $x = 1$. One may wonder whether this inclusion relationship holds in general. In particular, is the set of pairwise stable networks and payoffs shrinking as a player's local network radius expands? This section shows that this is not the case in general.

Proposition 7. *Generally, there is no inclusion relation between the sets of pairwise stable networks for different radii. In particular, there exist two networks g, g' such that g is pairwise stable when $x = 1$, and not pairwise stable when $x = 2$; by contrast, g' is not pairwise stable when $x = 1$, and it is pairwise stable when $x = 2$.*

Example 3 below shows two networks such as g and g' in Proposition 7. The detailed computations of it are in Appendix B.

This particular network structure in Figure 8 proves useful to show that: (1) pairwise stable networks for the case $x = 2$ can feature diverse payoffs, with some agents earning neither 0 nor 1; (2) the set of pairwise stable networks when $x = 1$ is neither included, nor does it include, the set of pairwise stable networks when $x = 2$.

Example 3 (Counterexample)**.** *$|N| \geq 12$, and the network partitions the set of players into two components: one component includes 6 agents who form a cycle, and* ***the rest of the individuals form a complete network*** *(see Figure 8).*



(a) The network

Node	1 to 6	7 to 12
$x = 1$	0	1
$x = 2$	0	$\frac{4}{10}$

(b) The local payoffs of every node.

Figure 8: A network with $n = 12$ which is pairwise stable for $x = 2$, but not for $x = 1$. In contrast, the circle consisting of $n = 12$ nodes is stable for $x = 1$ but not for $x = 2$. △

We can now also provide some contrast between the pairwise equilibria we describe for different radii of interest. We will call an *augmented star* a network g with the following properties: (i) there is exactly one agent $i \in N$ such that for every $j \neq i, j \in N$ there is a link ij, and (ii) no cycle encompasses all $j \neq i, j \in N$. Note that per our definition, a star network is an augmented star.

Proposition 8. *Augmented stars and the empty network can be supported in a pairwise equilibrium for all $1 \leq x \leq \infty$. The empty network is the unique network that can be sustained as a bilateral equilibrium for all $1 \leq x \leq \infty$.*

The statement follows from combining the observations in Proposition 1, Proposition 3 statement 4, and Proposition 5 statements 2 and 3.[6]

Proposition 7 states that the sets of pairwise equilibria for different radii of interest are generally not included in each other. However, they overlap for a specific class of networks (Proposition 8). The next proposition addresses the question of whether there is a radius of interest after which the set of pairwise equilibria stops changing.

Proposition 9. *For all radii of interest $x \geq N$, the set of networks to which pairwise equilibria map is the same: it consists of the empty network, and all connected networks such that either $u_i(s^*) = 0$ for all $i \in N$, or else $u_i(s^*) = 1$ for one $i \in N$, and $u_i(s^*) = 0$ for all $j \in N \setminus \{i\}$. Furthermore, for each $x \geq N$, there exists an essentially unique bilateral equilibrium, and it maps to the empty network.*

[6]Proposition 5 statements 2 and 3 say that augmented stars are candidates for pairwise stability. Here, we show that they are pairwise stable. First, the center of the augmented star does not have a strictly profitable deviation, since it already earns the maximum payoff of 1. For the other peripheral agents, it can be directly verified that all deviations allowed by pairwise stability are payoff-equivalent. Therefore, these networks are pairwise stable.

Note that when $x \geq N$, the local network of an agent is the entire component to which they belong. As such, the logic is the same as the one behind Proposition 1 (where $x = +\infty$) applies. Consequently, its conclusion applies, and pairwise stable networks have the same properties as those listed in Proposition 1, and bilateral equilibria have the properties in Proposition 2.

4 Conclusion

This paper analyzes a model of network formation in which agents seek to maximize their *relative* contribution to the set of connections in their network of interest. An agent is said to contribute to a network connection if he is essential to it, and an agent's network of interest is defined as the subnetwork that covers all individuals within an x-links radius of the agent.

Our main equilibrium notion is pairwise stability, and pairwise-stable networks arise from the following incentives and trade-offs. First, links are always beneficial when they circumvent others' attempts at being essential, keeping one's own network of interest constant. Second, when forming a link that expands one's own network of interest, an agent trades off the benefits from mediating more connections with the cost of facing more competitors for essentiality. Typically, the cost exceeds the benefit when an agent links to someone who contributes relatively more than the current average network participant. For this reason, some pairwise stable networks are nonempty and disconnected, as agents may lose relative centrality when trying to bridge two components.

We provide a full characterization of pairwise networks in the two extreme cases: when agents consider their relative contributions to the overall network (i.e., $x = +\infty$) and in their ego-network of direct neighbors (i.e., $x = 1$). We show that in either case, equilibrium payoffs are either 0 (the agent is then inessential in all connections in their relevant network) or 1 (they are the unique essential agent in their relevant network). However, the reasons these payoffs result in equilibrium differ markedly. When agents care only about their immediate neighborhood, they earn the maximal payoff of 1 whenever they are essential for connecting at least one pair of neighbors. Consequently, the incentive to form links is limited to expanding one's network until finding an agent to whom one's current neighbors are not directly linked. The incentive to form links that circumvent one's direct neighbors' contributions is absent simply because such contributions cannot exist when $x = 1$. By contrast, this incentive is key when agents care about their contribution to the global network. There, forming a link does not expand one's own network of interest, since the said network is the entire network. Therefore, agents with positive contributions gain from links that create alternative paths between the same pair of individuals, merely for the sake of canceling the contribution of some others to the said connected pair. This implies that, in a component, at most one agent can earn a positive payoff. This, in turn, drives the conclusion that nonempty pairwise stable networks are connected. Indeed, two agents with a null payoff and belonging to two different components always gain from linking, as they now become essential to the connections between the two components.

In intermediate cases, both incentives are present and render a full characterization difficult. We nonetheless present necessary conditions that reduce the set of candidate (nonempty) networks. We find, among other things, that agents with a null payoff must all have links and belong to the same component. In general, when we apply the more demanding notion of bilateral equilibrium, nonempty networks that have many zero-payoff agents are no longer stable. However, the empty network is the only architecture supported by bilateral equilibrium across all possible radii of interest. In contrast, despite the absence of inclusion between the sets of pairwise stable networks for different radii of interest, their intersection is nonempty. Star networks and some close variants are pairwise stable for any value of x.

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A Appendix: Proofs

Proof of Proposition 1

The proof rest on the following two lemmas.

Lemma A.1. *If s^* is a pairwise equilibrium, $u_i(s^*) > 0$ for some $i \in N$, then $u_j(s^*) = 0$ for all $j \in N \setminus \{i\}$ such that $d(i, j; g(s^*)) < \infty$.*

Proof. Given $s^* \in S$, let $g^* := g(s^*)$. Note, s^* is a pairwise equilibrium if and only if g^* is pairwise stable. Suppose $u_i(s^*) > 0$ for some $i \in N$. Therefore, $\exists a, b \in N \setminus \{i\}$ such that $a \neq b$, $d(i, a; g^*) < +\infty$, $d(i, b; g^*) < +\infty$ and $d(a, b; g^*_{-i}) = +\infty$ (i.e., agent i is essential for connecting a and b). If two agents can form a link and benefit from it, the starting setup is not an equilibrium. Let $g' \equiv g^* + ab$. Then $c_a(g') = c_a(g^*)$ and $c_b(g') = c_b(g^*)$, that is, the node contribution of a and b is the same in both g^* and g'. However, i's contribution decreases, $c_i(g') < c_i(g^*)$. Finally, for any $j \in N, j \neq a, b$ it is true that $c_j(g') \leq c_j(g^*)$. This implies that:

1. if $r_a(g^*) = 0$ then $r_a(g') = 0$ (and if $r_b(g^*) = 0$ then $r_b(g') = 0$).

2. if $r_a(g^*) > 0$ then $r_a(g') > r_a(g^*)$ (and if $r_b(g^*) > 0$ then $r_b(g') > r_b(g^*)$).

Hence g is pairwise stable only if $r_a(g^*) = r_b(g^*) = 0$, since then reducing the contribution of i does not strictly benefit either a or b. □

Lemma A.1 implies that in a pairwise stable network, each component has *at most* one agent with nonzero payoff.

Lemma A.2. *If s^* is a pairwise equilibrium, then $g(s^*)$ is either connected or empty.*

Proof. In the empty network, adding a link is payoff-equivalent to having none, for all pairs of agents. Also, there are no links to be removed. Thus, the empty network is pairwise stable, and $s^*_i = \{\emptyset\}$ for all $i \in N$ is a pairwise equilibrium.
The rest of the proof is by contradiction. Suppose g is a non-empty, disconnected pairwise stable network. Thus, there is a component A in g s.t. $|A| > 0$. By Lemma A.1, there is some agent $i \in A$ for whom $r_i(g) = 0$. Consider any $j \in N \setminus A$ such that $r_j(g) = 0$ as well. Such an agent exists; to see this, note that either j is a singleton and then $r_j(g) = 0$, or j belongs to some component B with $|B| > 1$, and one knows from Lemma A.1 that there is an agent in B who has a zero payoff (and j is such an agent). Let $g' = g + ij$; note that $r_i(g') > 0 = r_i(g)$ (since $i \in A$ and $|A| > 1$), and $r_j(g') \geq 0 = r_j(g)$. It follows that g is not pairwise stable, a contradiction. □

Lemma A.3 characterizes the set of pairwise stable networks.

Lemma A.3. *A pairwise stable network g^* is either:*

1. *empty and $r_i(g) = 0$ for all $i \in N$ or*

2. *connected and:*

   - *either $r_i(g) = 0$ for all $i \in N$,*
   - *or $r_i(g) = 1$ for some $i \in N$ and $r_j(g) = 0$ for all $j \in N \setminus \{i\}$.*

Proof. Lemmas A.1 and A.2 prove everything except the statement $r_i(g) = 1$ for i the agent with a nonzero payoff. Notice that if agent i has $c_i(g) > 0$, and $r_j(g) = c_j(g) = 0$ for all $j \in N \setminus \{i\}$, then:

$$r_i(g) = \frac{c_i(g)}{\sum_{j \in N} c_j(g)} = \frac{c_i(g)}{c_i(g)} = 1.$$

□

Proof of Proposition 3

In the empty network, adding a link between two agents is payoff-equivalent for them, and there are no links to be removed. Therefore, the empty network is pairwise stable.

Consider a nonempty disconnected network g.

Only if. Since g is nonempty, there is an agent i with links. Let A be the component to which i belongs. Let $g' = g + ij$ for some $j \notin A$. Then, $c_i((g')_i^1) > c_i(g_i^1)$, so i's node contribution increases in $(g')_i^1$. This implies that:

1. if $r_i(g_i^1) = 0$ then $r_i((g')_i^1) = 1$.
2. if $r_i(g_i^1) = 1$ then $r_i((g')_i^1) = 1$ and if $r_j(g_j^1) = 1$ then $r_j((g')_j^1) = 1$.
3. if $r_j(g_j^1) = 0$ and j has no links in g, then $r_j((g')_j^1) = 0$. However, if j has some links in g, then $r_j((g')_j^1) = 1$.

Therefore, forming the link ij weakly benefits both agents. For the network to be pairwise stable, it must that the link formation does not strictly benefit either agent. This is the case when:

1. $r_i(g_i^1) = r_j(g_j^1) = 1$ or
2. $r_i(g_i^1) = 1$ and j has no links.

If. Consider a disconnected non-empty global network g where $r_i(g_i^1) = 1$ for all agents with links. If there are some isolated agents without links, $r_i(g_i^1) = 0$. Clearly, removing or adding a link cannot improve the payoff of i if $r_i(g_i^1) = 1$. Adding a link with an isolated agent cannot improve his payoff, as the agent's local network is a single link, which gives him a null payoff.

Consider a connected network g.

Only if. Suppose $r_i(g_i^1) = 0$ for some $i \in N$. Therefore, in g_i^1, there is a cycle that encompasses all agents in $N(g_i^1)$. Suppose there is some j s.t. $d(i, j; g) > 2$. Thus, no agent in $N(g_i^1)$ has a link with j and symmetrically, no agent in $N(g_j^1)$ has a link with i. This immediately implies that, for $g' = g + ij$, $r_i((g')_i^1) = 1$ and $r_j((g')_j^1) = 1$. Therefore,

1. if $r_j(g_j^1) = 1$, accepting the link is payoff-equivalent for j.
2. if $r_j(g_j^1) = 0$, j strictly prefers to accept the link.

In both cases, g would not be pairwise stable, a contradiction. Hence, when $u_i(g) = 0$, then $d(i, j; g) \leq 2$ for all $j \in N$.

If. Consider a global network g as described in Proposition 3 Statements 3 and 4. If $r_i(g_i^1) = 1$ for some $i \in N$, adding or removing links cannot further improve i's payoff. If $r_i(g_i^1) = 0$, then there is a cycle in g_i^1 that encompasses all agents in $N(g_i^1)$. Because of this cycle, i cannot strictly gain from cutting a link. Consider the instance where a link is added between i and some agent $j \notin N(g_i^1)$. Since $d(i, j; g) = 2$, then an agent in $N(g_i^1) \setminus \{i\}$, say k, has a link with j in the original global network g. Since any $\ell \in N(g_i^1) \setminus \{i\}$ and j can access k without using i's links, they can all access j without using the new link ij. Thus, i's payoff remains the same, 0.

Proof of Proposition 4

If. For the empty network, see the previous proof (since, in effect, at most one new link can be formed).

If $u_i(s) = 1$, then no deviation can strictly improve the agent's payoff. Therefore, any strategy $s \in S$ that satisfies the condition in statement (3) (i.e., $u_i(s) = 1$ for all $i \in N$) is a bilateral equilibrium. Consider the strategy profiles s that satisfy condition 2: $u_i(s) = 1$ for all agents with links, $u_i(s) = 0$ for the agents without links in $g(s)$. No deviaition can improve the payoff of i if $u_i(s) = 1$; neither is it possible for i such that $u_i(s) = 0$, since i has no links and adding one link with i (recall, the deviations allowed by the bilateral criterion add, in effect, at most one new link to the network) makes i an endpoint of some component. Being an endpoint always confers the payoff of 0, regardless of the agent's local network. Thus, s is a bilateral equilibrium.

Only if. Let $s \in S$ be a strategy profile such that $u_i(s) = 0$ for some $i \in N$ with links in $g(s)$. Therefore, in $g_i^1(s)$, there exists a cycle that encompasses all agents in $N(g_i^1)$. Furthermore, if any $j \in N(g_i^1) \setminus \{i\}$ cuts their link in the cycle (let this link be referred to as jk), then $u_i(s') = 1$, for $s' = (s'_j, s_{-j})$ and $s'_j = s_j \setminus \{k\}$. We now show that $u_j(s) = u_j(s')$. The result is immediate if $u_j(s) = 0$, as cutting one's own links when $x = 1$ cannot improve one's payoff. Thus, consider the case where $u_j(s) = 1$. Hence, $c_j(g_j^1) > 0$. Note that $i, k \in N(g_j^1)$; thus the cycle ij, jk, ki exists in $N(g_j^1)$. It immediately follows that removing the link jk does not decrease j's payoff. In conclusion, $u_i(s') > u_i(s)$ and $u_j(s') = u_j(s)$. Therefore, s is not a bilateral equilibrium.

Proof of Proposition 5

The first part of the proposition readily follows.

We prove Statement 3 of the proposition by contradiction. Suppose that a pairwise stable network is acyclic and is not a star. Then its diameter is strictly greater than 2. Consider a terminal node ℓ and $k \in N(g)$, such that $1 < d(\ell, k) \leq x$ and observe that such a k would always exist. The terminal node ℓ has payoff $u_\ell = 0$, while node k has a strictly positive payoff, $u_k > 0$. Notice that there exists a node j, which is on the path $k - \ell$ such that $r(g_k^x) > 0$. Node k would strictly benefit from connecting to the leaf ℓ (this, because k's local network remains the same after adding $k\ell$, and the deviation merely eliminates the contribution of j), while the leaf would be indifferent. This contradicts the initial assumption that the network is pairwise stable.

We prove Statement 2 of the proposition with several consecutive Lemmas. First, we introduce a definition.

Definition A.3. *For $i, \ell \in N$ and $x \geq d^1 \geq 1$, $x \geq d^2 \geq 1$:*

$$C^{d^1 d^2}_{\ell,i}(g) = \left| \left\{ \{k, k'\} | k, k' \in N(g) \backslash \{\ell\}, d(i, k; g) = d^1, d(i, k'; g) = d^2, d(k, k'; g_{-\ell}) = +\infty \right\} \right|$$

In words, the above counts the number of unordered pairs of agents for whom ℓ is essential, controlling for the distance of each pair member to agent i. Importantly, it does not imply that agent i is also essential for the unordered pairs. Throughout, we abuse notation and drop the subscript i, as g will be taken as g^x_i.

When $\ell = i$ these numbers can be used to express i's contribution $c_i(g^x_i)$ as

$$c_i(g^x_i) = \sum_{d^1=1}^{x} \sum_{d^2=d^1}^{x} C^{d^1 d^2}_i(g^x_i).$$

Lemma A.4. *For every $i \in N(g)$ and $1 \leq x < +\infty$ and $j \in N(g^{x-1}_i)$, if $c_j(g^x_i) = 0$, then $c_j(g^x_j) = 0$.*

Proof. Notice that when $j \in N(g^{x-1}_i)$ it is true that every $j' \in N(g^1_j)$ also belongs to $N(g^{x-1}_i)$. Thus, if $c_j(g^x_i) = 0$ that means that i sees a cycle encompassing all neighbors of j. Then it must be the case that $c_j(g^x_j) = 0$. □

Put simply, Lemma A.4 states that if an agent j in some other agent i's local network g^x_i is perceived by i as not contributing (i.e., $c_j(g^x_i) = 0$), and $1 < d(i, j; g^x_i) < x$, then agent j's contribution to his own network of interest must be null (i.e., $c_j(g^x_j)$) – hence, j's payoff is null. Lemma A.5 and Lemma A.6 introduce two technical observations which help us bind the payoffs in Lemma A.7.

Lemma A.5. *If $u_i(g) > 0$ for every $i \in N(g)$ and $1 \leq x < +\infty$ it holds that*

$$\sum_{d^1=1}^{x} \sum_{d^2=d^1}^{x} (d^1 + d^2 - 2) C^{d^1 d^2}_i(g^x_i) < \sum_{\ell \in N(g^x_i) \backslash \{i\}} c_\ell(g^x_i).$$

Proof. The above inequality holds trivially when $x = 1$. Therefore, we set $x > 1$ in the remainder. The first term in the sum, which corresponds to the case where $d^1 = d^2 = 1$, is null. Therefore, we restrict attention to focus to either $d^1 > 1$ or $d^2 > 1$. If agent i is essential for connecting a distant pair of agents $k, k' \in N$ with fixed distances $d(i, k; g^x_i) = \overline{d^1} > 1$ or $d(i, k'; g^x_i) = \overline{d^2} > 1$, there are two possible options:

- there is a unique path $k - i - k'$. This means that all agents who lie along this unique path joining k and k' are also essential for that connection. That is, there are $(\overline{d^1} + \overline{d^2} - 2)$ connections counted to $\sum_{\ell \in N(g^x_i) \backslash \{i\}} C^{\overline{d^1 d^2}}_\ell(g^x_i)$.

- the path $k - i - k'$ is not unique. That is, there exist at least two $m, m' \in N(g), m \neq i, k, k'$ and $m' \neq i, k, k'$ such that m and m' each lies on a path from i to k' and neither m nor m' is essential for the connecting i and k' in g^x_i. Therefore, because $x > 1$ it must be the case that $\exists j \in N$ such that $d(i, j; g^x_i) < \overline{d^2} = d(i, k'; g^x_i)$ and which lies on the path $i - k'$, i.e. there is a path $k - i - j - k'$. Observe that there will be $\overline{d^2}-1$ agents with this property (because that is the number of agents other than i, k' along the shortest path joining i to k'). Moreover, every $j' \in N(g_j^1)$ also belongs to $N(g_i^x)$ or in other words, it is visible in the local network of i. By the initial assumption that $u_i(g) > 0$ for all $i \in N(g)$, it follows from Lemma A.4 that $c_j(g_i^x) > 0$ and hence the neighbors of j are not encompassed in a cycle in the network g_i^x. This implies that $\exists j' \in N(g_j^1)$ such that j is essential for the connection $j' - i$ in network g_i^x. Overall, this means that for a selected path $k - i - k'$ in which i is essential there must be nodes j, j' such that j is essential in the connection $k - j'$ (and $k - j'$ is counted in $c_j(g_i^x)$).

Notice that the argument above can be applied in the symmetric case for $k - \tilde{m} - i - k'$ and $k - \tilde{m}' - i - k'$ in case $\exists \tilde{m}, \tilde{m}' \in N(g)$. If this is not the case, then there is a unique path $k - i$ and the argument in the bullet point above applies.

Therefore, we can conclude that every connection counted to the contribution of i in network g_i^x, $c_i(g_i^x)$, implies that there are at least $(\overline{d^1} + \overline{d^2} - 2)$ connections counted to $\sum_{\ell \in N(g_i^x)\setminus\{i\}} C_\ell^{\overline{d1}\overline{d2}}(g_i^x)$, i.e. the contributions of all other agents in the network as perceived by i.

Overall, we can conclude that:

$$(d^1 + d^2 - 2) C_i^{d^1 d^2}(g_i^x) \leq \sum_{\ell \in N(g_i^{x-1})\setminus\{i\}} C_\ell^{d^1 d^2}(g_i^x)$$

$$\begin{aligned}\sum_{d^1=1}^{x} \sum_{d^2=d^1}^{x} (d^1 + d^2 - 2) C_i^{d^1 d^2}(g_i^x) &\leq \sum_{\ell \in N(g_i^{x-1})\setminus\{i\}} \sum_{d^1=1}^{x} \sum_{d^2=d^1}^{x} C_\ell^{d^1 d^2}(g_i^x) \\ &= \sum_{\ell \in N(g_i^{x-1})\setminus\{i\}} c_\ell(g_i^x) = \sum_{\ell \in N(g_i^x)\setminus\{i\}} c_\ell(g_i^x)\end{aligned}$$

Further, agent k such that $d(i, k; g_i^x) = x - 1$ will be essential in the connection $i - \ell$, where ℓ is such that $d(i, \ell; g_i^x) = x$ and $\ell \in N(g_k^1)$. This will make the last inequality strict, i.e.:

$$\sum_{d^1=1}^{x} \sum_{d^2=d^1}^{x} (d^1 + d^2 - 2) C_i^{d^1 d^2}(g_i^x) < \sum_{\ell \in N(g_i^x)\setminus\{i\}} c_\ell(g_i^x).$$

□

Lemma A.6. *If $u_i(g) > 0$ for every $i \in N(g)$ and $1 \leq x < +\infty$ with $1 \leq d^1 \leq d^2 \leq x$ it holds that*

$$C_i^{d^1-1\ d^2-1}(g_i^x) \leq C_i^{d^1 d^2}(g_i^x).$$

Proof. Fix an agent $i \in N$ and a local neighborhood for i, g_i^x. Consider the set $\mathcal{C}_i^{d^1-1, d^2-1}(g_i^x)$ of unordered pairs (k, ℓ) such that $d(i, k; g_i^x) = d^1 - 1$, $d(i, \ell; g_i^x) = d^2 - 1$, $d^1 + d^2 \leq x$, and i is essential for connecting k to ℓ in g_i^x. The cardinality of this set is, by construction, $C_i^{d^1-1, d^2-1}(g_i^x)$. We adopt the convention that the first element of the unordered pair is distant from i by the first distance mentioned in the power of $\mathcal{C}$; i.e., $(k, \ell) \in \mathcal{C}_i^{d, d'}(g_i^x)$ indicates that $d(i, k; g_i^x) = d$ and $d(i, \ell; g_i^x) = d'$.

We first claim that our current assumptions imply that for each $(k, \ell) \in \mathcal{C}_i^{d^1-1, d^2-1}(g_i^x)$, there is a pair $(k', \ell') \in \mathcal{C}_i^{d^1, d^2}(g_i^x)$ where $k' \in N(g_k^1)$ and $\ell' \in N(g_\ell^1)$. Suppose this is not the case. Then, one of the following must be true.

- Either $N(g_k^1) \subseteq N(g_i^{d^1-1})$, or $N(g_\ell^1) \subseteq N(g_i^{d^2-1})$. Wlog, suppose the inclusion holds for agent k. In words, either agent k has no links, or the agents with whom k has links are all reachable from i via a path (weakly shorter than d^1) that does not include any of k's links. In such instances, $c_k(g_i^x) = 0$; by Lemma A.4, therefore, $u_k(g) = 0$. But this contradicts our assumption that $u_j(g) > 0$ for all $j \in N$.

- Neither $N(g_k^1) \subseteq N(g_i^{d^1-1})$ nor $N(g_\ell^1) \subseteq N(g_i^{d^2-1})$. Therefore, $\exists k' \in N(g_k^1)$, $\ell' \in N(g_\ell^1)$, such that $d(i, k'; g_i^x) = d^1$ and $d(i, \ell'; g_i^x) = d^2$. If $(k', \ell') \notin \mathcal{C}_i^{d^1,d^2}(g_i^x)$, then agent i is inessential for the connection between k' and ℓ' in g_i^x. Meaning, there is path connecting k' to ℓ' in g_i^x that does not pass through i. Since $kk' \in g_i^x$ and $\ell\ell' \in g_i^x$, k and ℓ could use this path connecting k' to ℓ' to access one another. But then, $(k, \ell) \notin \mathcal{C}_i^{d^1-1,d^2-1}(g_i^x)$.

Our next claim refines the first. We show that two distinct pairs $(k, \ell), (a, b) \in \mathcal{C}^{d^1-1,d^2-1}(g_i^x)$ can be mapped to two *distinct* pairs $(k', \ell'), (a', b') \in \mathcal{C}_i^{d^1,d^2}(g_i^x)$. Notice that proving this is sufficient for concluding that $C_i^{d^1,d^2}(g_i^x) \geq C^{d^1-1,d^2-1}(g_i^x)$.

Notice first that our previous observation implies that for each pair $(k, \ell) \in \mathcal{C}_i^{d^1-1,d^2-1}(g_i^x)$, there exists $(k', \ell') \in \mathcal{C}_i^{d^1,d^2}(g_i^x)$ with $k' \in N(g_k^1), \ell' \in N(g_\ell^1)$, such that k is essential for connecting i to k', and ℓ is essential for connecting i to ℓ'. Below, we prove additionally that if there is an agent $m \in N(g_i^x) \setminus \{k, k'\}$ who is also essential for connecting i to k', then $d(i, m; g_i^x) < d(i, k; g_i^x)$. (The same observation will hold about agent ℓ and his connecting agents i and ℓ'.)

To see this, suppose there is an agent $m \in N(g_i^x) \setminus \{k, k'\}$ who lies on a path connecting i to k', and $d(i, m; g_i^x) \geq d(i, k; g_i^x)$. Because agent k is assumed to be essential for the connection between i and k', the path that connects i to k' and that passes through m first passes through k. As such, m cannot lie on the shortest path from i to k'. It follows that m is inessential for connecting i and k'.

Together, the two previous observations we have made establish the following. For two distinct pairs $(k, \ell), (a, b) \in \mathcal{C}_i^{d^1-1,d^2-1}(g_i^x)$, there exists two pairs $(k', \ell'), (a', b')$ such that:

1. $k' \in N(g_k^1)$, $a' \in N(g_a^1)$, $k' \neq a'$, $d(i, k'; g_i^x) = d(i, a'; g_i^x) = d^1$. Also, k is essential for connecting i to k', and a is essential for connecting i to a'.

2. $\ell' \in N(g_\ell^1)$, $b' \in N(g_b^1)$, $\ell' \neq b'$, $d(i, \ell'; g_i^x) = d(i, b'; g_i^x) = d^2$. Also, ℓ is essential for connecting i to ℓ', and b is essential for connecting i to b'.

3. Hence, $(k', \ell') \neq (a', b')$.

4. Agent i is essential for connecting k' to ℓ', and i is essential for connecting a' to b'.

This finishes the proof that $C_i^{d^1-1,d^2-1}(g_i^x) \leq C_i^{d^1,d^2}(g_i^x)$. □

Lemma A.7. *Consider a connected network g in which $u_i(g) > 0$ for all $i \in N$ and suppose that $1 < x < +\infty$. For each $i \in N$, the following is true*

$$r_i\left(g_i^x\right) < \frac{1}{x},$$

or, equivalently to the above inequality,

$$\frac{c_i(g_i^x)}{\sum_{j \in N(g_i^x) \setminus \{i\}} c_j(g_i^x)} < \frac{1}{x-1}.$$

Proof. Consider a connected network g, and consider any agent i. If $u_i(g) > 0$ for all $i \in N(g)$, then each agent in $N\left(g_i^{x-1}\right)$ has at least two links adjacent to them. Otherwise, the agent would have a null payoff.

It can be directly verified that

$$r_i(g_i^x) < \frac{1}{x} \Longleftrightarrow c_i(g_i^x) < \frac{1}{x-1} \sum_{\ell \in N(g_i^x) \setminus \{i\}} c_\ell(g_i^x).$$

Next, we use Lemma A.5 and A.6 to establish the second inequality.

$$\begin{aligned}
&\frac{1}{x-1} \sum_{\ell \in N(g_i^x) \setminus \{i\}} c_\ell(g_i^x) - c_i(g_i^x) = \frac{1}{x-1} \sum_{\ell \in N(g_i^x) \setminus \{i\}} c_\ell(g_i^x) - \sum_{d^1=1}^{x} \sum_{d^2=d^1}^{x} C_i^{d^1 d^2}(g_i^x) \\
&> \frac{1}{x-1} \sum_{d^1=1}^{x} \sum_{d^2=d^1}^{x} (d^1 + d^2 - 2) C_i^{d^1 d^2}(g_i^x) - \sum_{d^1=1}^{x} \sum_{d^2=d^1}^{x} C_i^{d^1 d^2}(g_i^x) \\
&= \sum_{d^1=1}^{x} \sum_{d^2=d^1}^{x} \left(\frac{(d^1 + d^2 - 2)}{(x-1)} - 1 \right) C_i^{d^1 d^2}(g_i^x) \\
&= \sum_{d^1=1}^{x} \sum_{\Delta=0}^{x-a} \left(\frac{d^1 + (d^1 + \Delta) - 2}{x-1} - 1 \right) C_i^{d^1, d^1+\Delta} \\
&= \sum_{d^1=1}^{x} \left(\frac{2(d^1-1)}{x-1} - 1 \right) C^{d^1 d^1}(g_i^x) + \sum_{d^1=1}^{x-1} \left(\frac{2(d^1-1)+1}{x-1} - 1 \right) C^{d^1, d^1+1}(g_i^x) + \dots \\
&+ \left(\frac{x-1}{x-1} - 1 \right) C^{1,x}(g_i^x) \\
&= \sum_{d^1=1}^{\lfloor \frac{x}{2} \rfloor} \left(\frac{2(d^1-1)}{x-1} - 1 \right) (C^{d^1, d^1} - C^{x-d^1+1, x-d^1+1}) \\
&+ \dots + \sum_{d^1=1}^{\lfloor \frac{x-k}{2} \rfloor} \left(\frac{2(d^1-1)+k}{x-1} - 1 \right) (C^{d^1, d^1+k} - C^{x-d^1+1-k, x-d^1+1}) \\
&+ \dots + \left(\frac{x-2}{x-1} - 1 \right) (C^{1,x-1} - C^{2,x}) \\
&\geq 0
\end{aligned}$$

The first strict inequality owes to Lemma A.5; the last inequality to zero owes to Lemma A.6. □

Lemma A.8. *Suppose that $1 < x < +\infty$. In non-empty pairwise stable networks, all agents have links, and all those with a null payoff belong to the same component.*

Proof. In some network, if there are two agents with a null payoff belonging to two different components, and one of them has links, then there is a strictly profitable deviation whereby they form a link between them. The agent with links strictly benefits; if the second does not have any link in the initial network, the deviation is payoff-equivalent as the said agent keeps their null payoff.

Consider a network in which all agents with a null payoff have no links. Suppose further that the network is not empty. We will study the profitability of a deviation whereby an agent j with links (thus, with a strictly positive payoff) and agent i form a link. It is clear that i does not oppose the link formation since, regardless of whether it forms, i's payoff is 0. The next paragraph gives a sufficient condition on j's initial payoff for the deviation to strictly improve j's payoff.

Consider the local network of agent j after the deviation, $g_j^x + ij$. Observe first that $N(g_j^x + ij) = N(g_j^x) \cup \{i\}$, and that $c_i(g_j^x + ij) = 0$, since agent i has no links before the deviation. Therefore,

$$\sum_{\ell \in N(g_j^x + ij)} c_\ell(g_j^x + ij) = \sum_{\ell \in N(g_j^x)} c_\ell(g_j^x + ij). \tag{3}$$

We want to characterize a lower bound on j's payoff after the deviation. To do so, we determine the increase in j's absolute contribution c_j, and an upper bound on the increase in the aggregate contribution of the other agents in j's local neighborhood. First, for agent j,

$$c_j(g_j^x + ij) = c_j(g_j^x) + (|N(g_j^x)| - 1). \tag{4}$$

Meaning, j's contribution increases by the number of new connections from each agent in $N(g_j^x) \setminus \{j\}$ toward agent i. Indeed, the fact that agent i has initially no links (thus, no links with any agent in $N(g_j^x)$) implies that the set of connections in j's local network expands by the connections between each agent in $N(g_j^x) \setminus \{j\}$ and i, and that j is essential for each of them.

Consider the agents in $N(g_j^x) \setminus \{j\}$, and first consider those who are 1 link away from agent j, i.e., $N(g_j^1) \setminus \{j\}$. At most, the aggregate contribution made by these agents increases by $|N(g_j^x)| - |N(g_j^1)|$; mathematically,

$$\sum_{\ell \in N(g_1^x) \setminus \{j\}} c_\ell(g_j^x + ij) \leq (|N(g_j^x)| - |N(g_j^1)|) + \sum_{\ell \in N(g_1^x) \setminus \{j\}} c_\ell(g^x).$$

Intuitively, the inequality binds when they, collectively, are essential for connecting the agents at distances $2, 3 \dots, x$ from agent j to agent i. It is important to remark that we are not claiming that the contribution of *each* agent in $N(g_j^1) \setminus \{j\}$ increases by $(|N(g_j^x)| - |N(g_j^1)|)$, as this is impossible. In fact, generally, two agents a and b are each essential for connecting some agents c and d only if a and b are located on the shortest path connecting c and d. As such, they cannot be both equidistant from an agent located on the same path (in our case, this third agent is j, and c or d is i). A similar reasoning holds for the agents in $N(g_j^z)$, for any $z \in \{2, \dots, x\}$: at most, the aggregate contribution made by these agents increases by $|N(g_j^x)| - |N(g_j^z)|$, which is the number of connections between agent i and the agents located at distances $z + 1, \dots, x$ from agent j in $N(g_j^x)$. Therefore,

$$\sum_{\ell \in N(g_j^x) \setminus \{j\}} c_\ell(g_j^x + ij) \leq \sum_{\ell \in N(g_j^x) \setminus \{j\}} c_\ell(g_j^x) + \sum_{z=1}^{x} (|N(g_j^x)| - |N(g_j^z)|)$$

Combining this with (3) and our earlier result that $c_j(g_j^x + ij) = c_j(g_j^x) + (|N(g_j^x)| - 1)$ yields the following conclusion:

$$u_j(g_j^x + ij) = \frac{c_j(g_j^x + ij)}{c_j(g_j^x + ij) + \sum_{\ell \in N(g_j^x + ij)} c_\ell(g_j^x + ij)}$$

$$\geq \frac{c_j(g_j^x) + (|N(g_j^x)| - 1)}{\sum_{\ell \in N(g_j^x)} c_\ell(g_j^x) + (|N(g_j^x)| - 1) + \sum_{z=1}^{x}(|N(g_j^x)| - |N(g_j^z)|)}.$$

Therefore, the deviation is profitable if:

$$\frac{c_j(g_j^x) + (|N(g_j^x)| - 1)}{\sum_{\ell \in N(g_j^x)} c_\ell(g_j^x) + (|N(g_j^x)| - 1) + \sum_{z=1}^{x}(|N(g_j^x)| - |N(g_j^z)|)} > u_i(g_j^x)$$

Replacing $u_j(g_j^x)$ by $\frac{c_j(g_j^x)}{\sum_{\ell \in N(g_j^x)} c_\ell(g_j^x)}$ and re-arranging yields:

$$\frac{c_j(g_j^x)}{\sum_{\ell \in N(g_j^x)\setminus\{j\}} c_\ell(g_j^x)} < \frac{|N(g_j^x)| - 1}{\sum_{z=1}^{x}(|N(g_j^x)| - |N(g_j^z)|)} \tag{5}$$

Recall that we are currently assuming that j, and all agents in j's component have a positive payoff. As such, by Lemma A.7, it holds that:

$$\frac{c_j(g_j^x)}{\sum_{\ell \in N(g_j^x)\setminus\{j\}} c_\ell(g_j^x)} < \frac{1}{x-1}. \tag{6}$$

We show below that:

$$\frac{1}{x-1} \leq \frac{|N(g_j^x)| - 1}{\sum_{z=1}^{x}(|N(g_j^x)| - |N(g_j^z)|)}. \tag{7}$$

But this is immediate, since it is equivalent to:

$$(x-1)(|N(g_j^x)| - 1) \geq \sum_{z=1}^{x}(|N(g_j^x)| - |N(g_j^z)|) = \sum_{z=1}^{x-1}(|N(g_j^x)| - |N(g_j^z)|)$$

Combining the relations in (6) and (7) yields exactly the inequality in (5); therefore, the deviation is profitable. Consequently, the original network is not be pairwise stable.

Therefore, all agents with a null payoffs must have links, and they must belong to the same component. This finishes the proof. □

Proof of Proposition 6

Let $g(s^*)$ be the network to which some bilateral equilibrium s^* maps. Consider the case where $g(s^*)$ is nonempty. Recall that a bilateral equilibrium maps to a pairwise stable network. Therefore, by Lemma A.7, each agent in N has links in $g(s^*)$. Consider these networks with several agents with null payoff. Then, any two of them could deviate as follows: one removes first all his links, then, the two agents form a link between them. The agent who initially removed all links still has a null payoff. The other agent now earns a strictly positive payoff, as he is essential for connecting all agent sin the component to the individual with whom he has formed the link.

We now show that in a bilateral equilibrium, if an agent has a null payoff (let us call him i), then $g_i^x = g(s^*)$. By contradiction, assume $g_i^x \neq g(s^*)$. Thus, there is an agent j such that $j \notin N(g_i^x)$. Equivalently, $d(i, j; g(s^*)) > x$. Therefore, $i \notin g_j^x$ either. Moreover, $u_j(s^*) > 0$, and by Lemma A.7,

$$u_j(s^*) < \frac{1}{x}.$$

Consider the following deviation by agents i and j: i cuts all his links, then i and j mutually offer each other to form a link.

It is immediate that the deviation leaves agent i's payoff unchanged, and the latter still earns 0. By the proof of Lemma A.8, agent j's payoff strictly increases because $u_j(s^*) < 1/x$ and i has no other link than ij after the deviation. Therefore, s^* is not a bilateral equilibrium, implying a contradiction.

B Examples

Example 3

Consider network in Figure 8 and take $x = 2$. Each agent in the complete component of g earns a zero payoff, and each agent i in the cycle earns $\frac{4}{10}$ (see Example 1 above) – since their local network g_i^2 is a line on 5 agents and agent i holds the central – third – position. The rest of the proof shows that the network is pairwise stable for $x = 2$. Consider first any agent in the complete component. Cutting a link yields the same payoff of 0. For any agent in the cycle, cutting a link yields a strictly lower payoff (namely, 0). Thus, no agent in N can strictly gain from cutting a link. We now turn to the deviations, which consist of adding a link. First, consider the case where two agents in the cycle component add a link between them. After the deviation, their local neighborhoods coincide with the entire cycle; as a result, their respective payoffs strictly decrease and reach 0. Last, consider adding a link between the complete component and the cycle. The deviation strictly improves the payoff of the agent in the complete component. However, we show below that the agent in the cycle sees his payoff strictly decrease. Let x be the number of agents in the complete component. Notice that, given the restrictions on the total number of individuals and the number in the cycle, $y \geq 6$. The payoff to the member of the cycle after the deviation is:

$$\frac{4+4y}{4+4y+2\cdot 0+2(3+y)+5(y-1)},$$

which is strictly lower than the agent's original payoff of $\frac{4}{10}$ when $y \geq 6$. This completes the proof that the network in Figure 8 is pairwise stable for $x = 2$.

By Proposition 3 Statement (b), the network is not pairwise stable when $x = 1$ – as adding a link between the two components yields the same payoff to the agent in the cycle (namely, 1), and strictly improves the payoff of the other agent in the complete component. So, the network in Figure 8 is pairwise stable for $x = 2$ and not stable for $x = 1$.

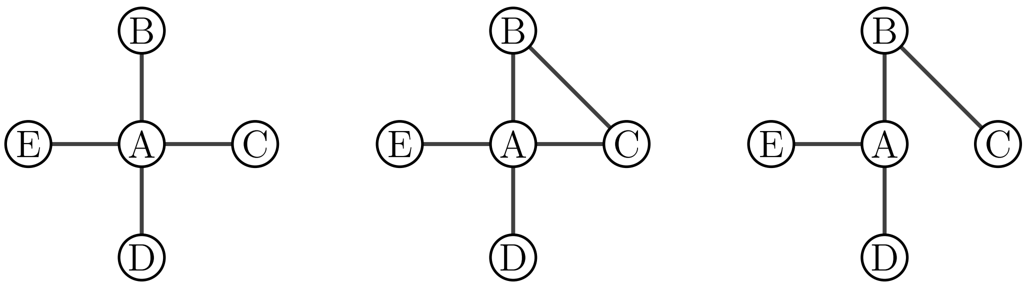
Consider a cycle encompassing all $|N| \geq 12$ agents. By Proposition 3 Statement (c), the network is pairwise stable when $x = 1$ since each agent then earns a payoff of 1. Below, we show that the network is not pairwise stable when $x = 2$. Without loss of generality, suppose that the network is entirely defined by the following sequence of links: $1-2\ldots-|N|-1$. Currently, g_i^2 is a line on 5 individuals along which i holds the central (third) position, and $u_i(g) = \frac{4}{10}$ for each $i \in N$. Let agents indexed 1 and 3 add a link between them. We refer to g' as $g + 13$. Note that after the deviation, the local networks of agents 1 and 3 have the *same architecture*, and both include the following cycle: $1-2-3-1$. Each of them earns the following payoff after the deviation:

$$\frac{4+2}{(4+2)+3\times 0+4+4} = \frac{6}{14} > \frac{4}{10}$$

We briefly comment on the payoff after the deviation. Since agents 1 and 3 are symmetric, we will reason with agent 1. Before the deviation, agents 2 and $|N|$ were the agents essential in some of the connections within 1's local network. After the deviation, these agents are agents 3 and $|N|$. Agents $2 \in N(g_1^2)$ and $3 \in N(g_1'^2)$ are essential for 4 connections; agent $|N|$'s contribution increases by 1 (as he further connects $|N|-1$ and 4 in $g_1'^2$) and the contribution of agent 1 increases by 2. Therefore, agent 1's relative contribution increases – as the aggregate increase in contributions from essential agents other than agent 1 is lower than our agent 1's increase in contribution. So, the cycle with $|N| \geq 12$ is pairwise stable for $x = 1$ but not for $x = 2$. △

Comparison to other measures

We borrow the example used in Buskens and Van de Rijt (2008) to show that the payoff structure in this paper produces results which are different Burt's constraint measure (cf. Everett and Borgatti (2020)), Burt's size measure (cf. Borgatti (1997)), betweenness and eigenvalue centrality.



(a) Network g_A (b) Network g'_A (c) Network g''_A

Figure B.1: Example networks

Measure for A	Constraint	Size	Betweenness	Eigenvalue	λ_A
g_A	$\frac{1}{4}$		6	$\frac{1}{3}$	
g'_A	$\frac{13}{32}$		5	≈ 0.3	
g''_A	$\frac{1}{3}$		5	≈ 0.31	
Order	$g \succ g'' \succ g'$	$g \succ g' \sim g''$	$g \succ g' \sim g''$	$g \succ g'' \succ g'$	$g \succ g' \succ g''$