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The Shapley value for airport and irrigation games

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Abstract

In this paper, we consider cost-sharing problems defined on rooted trees. We refer to these problems as cost-tree problems and to the induced transferable utility cooperative games as irrigation games. We introduce a formal notion of irrigation games and provide a characterization of this class of games. The well-known class of airport games (Littlechild and Thompson 1977) is shown to be a subclass of irrigation games. The Shapley value (Shapley 1953) is arguably the most widely used solution concept for transferable utility cooperative games. Dubey (1982) and Moulin and Shenker (1992) show, respectively, that the axiomatizations of the Shapley value due to Shapley (1953) and Young (1985) remain valid for the class of airport games. In this paper, we extend the results of Dubey (1982) and Moulin and Shenker (1992) to the broader class of irrigation games. Specifically, we provide two characterizations of the Shapley value for cost-sharing problems defined on rooted trees. In these characterization results, we relate the terminology of transferable utility games to that of cost-sharing problems, thereby establishing a bridge between the two fields.

Keywords Cost sharing · Shapley value · Rooted tree · Axiomatization of the Shapley value

JEL Classification C71

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1 Introduction

In this paper we consider cost sharing problems given by rooted trees, called *cost-tree problems*. We assign transferable utility (*TU*) cooperative games (henceforth games) to these cost sharing problems. The induced games are called *irrigation games*. For an example consider an irrigation ditch joined to the stream by a headgate and a group of users who use this ditch to irrigate their own farms. The functional and maintenance costs of the ditch are given too, and they have to be paid for by the users. One of the main questions is how to share the costs among the users. The ditch and the related users can be represented by a rooted tree.

The root of the tree is the headgate, each node represents one user, and the edges of the rooted tree represent the sections of the ditch. The users are related to the ditch by these sections. In this setting Littlechild and Owen (1973) show that the contribution vector (solution) recommended by the *sequential equal contribution*, called *sequential equal contributions rule* or *Baker-Thompson rule* (Baker 1965; Thompson 1971), where the costs of the sections are shared equally among the farmers who use them, and each farmer's total payment is the sum of his or her shares over all sections on his or her path to the root. This allocation coincides with the Shapley value (Shapley 1953).

For an empirical and axiomatic analysis of the sequential equal contribution see Aadland and Kolpin (1998) (they call it *serial cost-share rule*) who examine a real cost-sharing problem where the irrigation ditch is located in a south-central Montana community. The irrigation game by Aadland and Kolpin (1998) is defined as follows: for each nonempty set of players S the value $v(S) = -c(S)$, where $c = (c_1, \ldots, c_n) \in \mathbb{R}^N$ cost-vector gives the cost of the ditch, c_i gives the cost of section i and $c(S)$ means the minimum cost of servicing all users in S. It is easy to verify that this game is convex. A similar definition is presented in Kayi (2007), using c_i for the whole cost joining user i to the headgate.

When considering special rooted trees with no branches (i.e. chains), we arrive at the well-known class of airport games (Littlechild and Thompson 1977), therefore this class is the proper subset of the class of irrigation games. Thomson (2024) gives an overview on the results for airport games. In the literature, up to now, two axiomatizations of the Shapley value are given for airport games, those of Shapley (1953) and Young (1985). Dubey (1982) shows that Shapley's characterization is valid on the class of airport games, and Moulin and Shenker (1992) prove that Young's axiomatization is also valid on this subclass of games.

It is well known that the validity of an axiomatisation of a solution concept may vary across subclasses. For example, Shapley's axiomatization is valid on the class of monotone games but not on the class of strictly monotone games. Therefore, each subclass of games must be considered separately.

In this paper we introduce irrigation games and characterize their class. We show that the class of irrigation games is a non-convex cone which is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every irrigation game is concave. Furthermore, as a corollary we show that the class of airport games has the same characteristics as that of irrigation games.

In addition to the previously listed results, we extend the results of Dubey (1982) and Moulin and Shenker (1992) to the class of irrigation games. Furthermore, we "translate" the axioms used in the cost-sharing literature (see e.g. Thomson (2024)) to the language of transferable utility cooperative games, and show that the results of Dubey (1982) and Moulin and Shenker (1992) can be deduced directly from Shapley (1953)'s and Young (1985)'s results. Namely, we present two new variants of Shapley (1953)'s and Young (1985)'s results, and we provide Dubey (1982)'s, Moulin and Shenker (1992)'s and our characterizations as direct corollaries of the two new variants.

In our characterization results we relate the TU games terminologies to the cost sharing terminologies, therefore we bridge between the two fields.

We also note that the Shapley value (which is equivalent to the SEC rule) is stable for irrigation games, i.e. it is always in the core (Shapley 1955; Gillies 1959). This result is a simple consequence of the Ichiishi-Shapley theorem (Shapley 1971; Ichiishi 1981) and that every irrigation game is concave.

We conclude that applying the Shapley value to cost-tree problems is theoretically well-founded, therefore, since the Shapley value behaves well from the viewpoint of computational complexity theory (Megiddo 1978), the Shapley value is a desirable tool for solving cost-tree problems.

Further related literature is as follows: Granot et al. (1996) introduce the notion of standard fixed tree games. Irrigation games are equivalent to standard fixed tree games, except for that in irrigation games the tree may vary, while in the approach of Granot et al. (1996) it is fixed. Koster et al. (2001) study the core of fixed tree games. Ni and Wang (2007) characterize the rules meeting properties additivity and independence of irrelevant costs on the class of standard fixed tree games. Fragnelli and Marina (2010) characterize the SEC rule for airport games.

A widely used application of fixed tree games is the class of tree maintenance games. In the case where each non-root node is occupied by one player, these games coincide with the fixed-tree version of our irrigation games. We use the term irrigation games in order to emphasize our motivating application in water-management problems.

Ambec and Ehlers (2008) examined how to share a river efficiently among states connected to the river. They have shown that cooperation exerts positive externalities on the benefit of a coalition and explored how to distribute this benefit among the members of the coalition. By Ambec and Sprumont (2002) the location of an agent (i.e. state) along the river determines the quantity of water the agent controls, and the welfare it can thereby secure for itself. The authors call the corresponding cooperative game *consecutive game* and prove that the game is convex, therefore the Shapley value is in the core (see Shapley (1971) and Ichiishi (1981))

A further problem is presented regarding the allocation of costs and benefits from regional cooperation by Dinar and Yaron (1986), by defining *regional games*. Cooperative game-theoretic allocations are applied, like the core, the nucleolus, the Shapley value and the generalized Shapley value; and are compared with an allocation based on marginal cost pricing. Dinar et al. (1992) analyze a similar problem in the TU and the NTU settings (in the NTU case the core of the related game is non-convex, therefore the Nash-Harsányi solution is applied).

In this paper we consider only Shapley (1953)'s and Young (1985)'s axiomatizations. The validity of further axiomatizations of the Shapley value, see e.g. van den Brink (2001) and Chun (1991) among others, on the classes of airport games and irrigation games, is intended to be the topic of future research.

The setup of this paper is as follows: in the next section we introduce the concept of irrigation games and characterize the class of them. In Section 3 we present our main results: we show that Shapley (1953)'s and Young (1985)'s axiomatizations of the Shapley value work on the classes of airport games and irrigation games. In the last section we show how cost sharing axioms (see Thomson (2024)) correspond to axioms for solutions to transferable utility cooperative games, and reformulate our results in the classical cost sharing setting.

2 Airport and irrigation games

Notions, notations: $\#N$ is for the cardinality of set *N*, and 2^N denotes the class of all subsets of *N*. $A \subset B$ means $A \subseteq B$, but $A \neq B$. $A \uplus B$ stands for the union of disjoint sets *A* and *B*.

A *graph* is a pair $G = (V, E)$, where the elements of *V* are called *vertices* or nodes, and *E* stands for the ordered pairs of vertices, called *arcs* or edges. A *rooted tree* is a graph in which any two vertices are connected by exactly one simple path, and one vertex has been designated the root, in which case the edges have a natural orientation, away from the root. The tree-order is the partial ordering on the vertices of a rooted tree with $i \leq j$, if the unique path from the root to *j* passes through *i*. The *chain* is a rooted tree such that any vertices $i, j \in N$, $i \leq j$ or $j \leq i$. That is, a chain is a rooted tree with only one "branch".

For later use, we introduce the following notations:

- For any edge $e \in E$, the notation

$$e = \overline{ij}$$

means that *e* is an edge between vertices $i, j \in V$ such that $i \leq j$.

- For each vertex $i \in V$, let

$$S_i(G) = \{j \in V : j \geq i\}.$$

Thus, $S_i(G)$ is the set of successors of *i* in *G*. In particular, $i \in S_i(G)$.

- For each vertex $i \in V$, let

$$P_i(G) = \{j \in V : j \leq i\}.$$

Thus, $P_i(G)$ is the set of predecessors of *i* in *G*. In particular, $i \in P_i(G)$.

- For any subset $V' \subseteq V$, let

$$P_{V'}(G) = \bigcup_{i \in V'} P_i(G).$$

Thus, $P_{V'}(G)$ is the set of all vertices lying on paths from the root to vertices in V'.

- For any subset $V' \subseteq V$, let

$$E_{V'} = \{\overline{ij} \in E : i, j \in P_{V'}(G)\}.$$

- Then the sub-rooted tree of (V, E) induced by the vertices in $P_{V'}(G)$ is

$$(P_{V'}(G), E_{V'}).$$

Let $N \neq \emptyset$, $\#N < \infty$, and $v : 2^N \to \mathbb{R}$ be a function such that $v(\emptyset) = 0$. Then N, v are called set of players, and *transferable utility cooperative game* (henceforth game) respectively. The class of games with player set N is denoted by $\mathcal{G}^N$. It is worth noticing that $\mathcal{G}^N$ is isomorphic with $\mathbb{R}^{2^{\#N}-1}$, henceforth, we assume there is a fixed isomorphism[1] between the two spaces, and regard $\mathcal{G}^N$ and $\mathbb{R}^{2^{\#N}-1}$ as identical.

A game $v \in \mathcal{G}^N$ is *convex*, if for all $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, moreover, it is *concave*, if for all $S, T \subseteq N$, $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$.

The *dual* of game $v \in \mathcal{G}^N$ is game $\bar{v} \in \mathcal{G}^N$ such that for all $S \subseteq N$, $\bar{v}(S) = v(N) - v(N \setminus S)$. It is well known that the dual of a convex game is a concave game and vice versa.

Let $v \in \mathcal{G}^N$ and $i \in N$, and $v_i'(S) = v(S \cup \{i\}) - v(S)$, where $S \subseteq N$. v_i' is called player i's *marginal contribution function* in game v. Alternatively, $v_i'(S)$ is player i's marginal contribution to coalition S in game v.

Let $v \in \mathcal{G}^N$, players $i, j \in N$ are *equivalent*, $i \sim^v j$, if for all $S \subseteq N$ such that $i, j \notin S$, $v_i'(S) = v_j'(S)$.

Let there be given a graph $G = (V, E)$, a cost function $c : E \to \mathbb{R}_+$ and a cost tree (G, c). One possible interpretation to the presented problem, is the following. An irrigation ditch is joined to the stream by a headgate, and the users (the nodes of the graph except for the root) irrigate their farms using this ditch. The functional and maintenance costs of the ditch are given by c, and it is paid for by the users (more generally, the nodes might be departments of a company, persons, etc.). For any $e \in E$, $e = \overline{ij}$, c_e denotes the cost of connecting player j to player i.

In this paper we build on the duals of unanimity games. Given N, the *unanimity game* corresponding to coalition T for all $T \in 2^N \setminus \{\emptyset\}$ and $S \subseteq N$ is the following:

[1] The fixed isomorphism is the following: we fix a complete ordering on N, that is $N = \{1, \ldots, \#N\}$. Then for all $v \in \mathcal{G}^N$, let $v = (v(\{1\}), \ldots, v(\{\#N\}), v(\{1,2\}), \ldots, v(\{\#N-1, \#N\}), \ldots, v(N)) \in \mathbb{R}^{2^{\#N}-1}$.

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the *dual of the unanimity game* for all $T \in 2^N \setminus \{\emptyset\}$ and $S \subseteq N$ is:

$$\bar{u}_T(S) = \begin{cases} 1, & \text{if } T \cap S \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, all unanimity games are convex, and their duals are concave games (Peleg and Sudhölter 2007). Moreover, we know that the set of unanimity games $\{u_T : \emptyset \neq T \subseteq N\}$ gives a basis of $\mathcal{G}^N$ (see e.g. PelegSudhölter, 2007, pp. 153., Lemma 8.1.4).

Henceforth, we assume that there are at least two players for all cost tree games, i.e. $|V| \geq 3$ and $|N| \geq 2$. In the following we define the notion of an irrigation game. Let (G, c) be a cost tree, and N the set of players (the set of nodes, except for the root). Let us consider the nonempty coalition $S \subseteq N$, then the cost of connecting members of coalition S to the root is equal to the cost of the minimal, rooted subtree that covers members of S. This minimal spanning tree corresponding to coalition S is called the *trunk*, denoted by $\bar{S}$ (in the notation introduced above, $\bar{S} = (P_S(G), E_S)$). For all cost trees a corresponding *irrigation game* can be formally defined as follows.

Definition 1 *(Irrigation game)* For all cost-trees (G, c) and player set $N = V \setminus \{\mathrm{r}\}$, where r denotes the root, and coalition S let

$$v_{(G,c)}(S) = \sum_{e \in E_S} c_e \ ,$$

where the value of the empty sum is 0. The games given above by v are called irrigation games, their class on the set of players N is denoted by $\mathcal{G}_I^N$. Furthermore, let $\mathcal{G}_G$ denote the subclass of irrigation games that is induced by cost tree problems defined on rooted tree G.

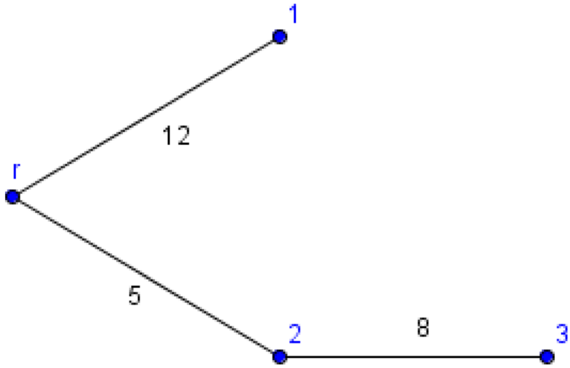
Since tree G also defines the set of players, instead of the notation $\mathcal{G}_G^N$ for set of players N, we will use the shorter form $\mathcal{G}_G$. These notations imply that

$$\mathcal{G}_I^N = \bigcup_{\substack{G(V,E) \\ N = V \setminus \{r\}}} \mathcal{G}_G .$$

Example 2 demonstrates the above definition.

Example 2 Let us consider the cost tree in Fig. 1. The rooted tree $G = (V, E)$ can be described as follows: $V = \{\mathrm{r}, 1, 2, 3\}$, $E = \{\overline{\mathrm{r}1}, \overline{\mathrm{r}2}, \overline{23}\}$. The definition of cost function c is $c(\overline{\mathrm{r}1}) = 12$, $c(\overline{\mathrm{r}2}) = 5$, and $c(\overline{23}) = 8$.

Fig. 1 Cost tree (G, c) of irrigation problem in Example 2



Therefore, the irrigation game is defined as $v_{(G,c)} = (0, 12, 5, 13, 17, 25, 13, 25)$, implying $v_{(G,c)}(\emptyset) = 0$, $v_{(G,c)}(\{1\}) = 12$, $v_{(G,c)}(\{2\}) = 5$, $v_{(G,c)}(\{3\}) = 13$, $v_{(G,c)}(\{1, 2\}) = 17$, $v_{(G,c)}(\{1, 3\}) = 25$, $v_{(G,c)}(\{2, 3\}) = 13$, and $v_{(G,c)}(N) = 25$.

The notion of airport games was introduced by Littlechild and Thompson (1977). An airport problem can be illustrated by the following example. Let there be given an airport with one runway, and k different types of planes. For each type of planes i a cost c_i is determined representing the maintenance cost of the runway required for i. For example if i stands for small planes, then the maintenance cost of a runway for them is denoted by c_i. If j is the category of big planes, then $c_i < c_j$, since big planes need a longer runway. Consequently, on player set N a partition is given: $N = N_1 \uplus \cdots \uplus N_k$, where N_i denotes the number of planes of type i, and each type i determines a maintenance cost c_i, such that $c_1 < \ldots < c_k$. Considering a coalition of players (planes) S, the maintenance cost of coalition S is the maximum of the members' maintenance costs. In other words, the cost of coalition S is the maintenance cost of the biggest plane of coalition S.

In the following we present two equivalent definitions of airport games.

Definition 3 *(Airport games I.)* For an airport problem let $N = N_1 \uplus \cdots \uplus N_k$ be the set of players, and let there be given $c \in \mathbb{R}^k_+$, such that $c_1 < \ldots < c_k \in \mathbb{R}_+$. Then the airport game $v_{(N,c)} \in \mathcal{G}^N$ can be defined as $v_{(N,c)}(\emptyset) = 0$, and for all non-empty coalitions $S \subseteq N$:

$$v_{(N,c)}(S) = \max_{i : N_i \cap S \neq \emptyset} c_i \ .$$

An alternative definition is as follows.

Definition 4 *(Airport games II.)* For an airport problem let $N = N_1 \uplus \cdots \uplus N_k$ be the set of players, and $c = c_1 < \ldots < c_k \in \mathbb{R}_+$. Let $G = (V, E)$ be a chain such that $V = N \cup \{\mathrm{r}\}$, and $E = \{\overline{\mathrm{r}1}, \overline{12}, \ldots, \overline{(|N| - 1)|N|}\}$, $N_1 = \{1, \ldots, |N_1|\}, \ldots, N_k = \{|N| - |N_k| + 1, \ldots, |N|\}$. Furthermore, for all $\overline{ij} \in E$ let $c_{\overline{ij}} = c_{N(j)} - c_{N(i)}$, where $N(i) = \{N^* \in \{N_1, \ldots, N_k\} : i \in N^*\}$.

For a cost tree (G, c), an airport game $v_{(N,c)} \in \mathcal{G}^N$ can be defined as follows. Let $N = V \setminus \{\mathrm{r}\}$ be the set of players, then for each coalition S (the empty sum is 0)

$$v_{(N,c)}(S) = \sum_{e \in \bar{S}} c_e \ .$$

Clearly, both definitions above define the same games, therefore let the class of airport games with player set N be denoted by $\mathcal{G}_A^N$. Furthermore, let $\mathcal{G}_G$ denote the subclass of airport games induced by airport problems on chain G. Note that the notation $\mathcal{G}_G$ is consistent with the notation introduced in Definition 1, since if G is a chain, then $\mathcal{G}_G \subseteq \mathcal{G}_A^N$, otherwise, if G is not a chain, then $\mathcal{G}_G \setminus \mathcal{G}_A^N \neq \emptyset$. Since not every rooted tree is a chain, $\mathcal{G}_A^N \subset \mathcal{G}_I^N$.

Example 5 Consider the airport problem (N, c'), as it is by Definition 3, in Fig. 2, that is, $N = \{\{1\} \uplus \{2,3\}\}$, $c'_{N(1)} = 5$, and $c'_{N(2)} = c'_{N(3)} = 8$ $(N(2) = N(3))$. Next, let us consider the very same airport problem as it is by Definition 4 and the cost tree in Fig. 2, where the rooted tree $G = (V, E)$ is defined as $V = \{\mathrm{r}, 1, 2, 3\}$, $E = \{\overline{\mathrm{r}1}, \overline{12}, \overline{23}\}$, and the cost function c is $c(\overline{\mathrm{r}1}) = 5$, $c(\overline{12}) = 3$ and $c(\overline{23}) = 0$.

Then the induced airport game is as follows: $v_{(G,c)} = (0, 5, 8, 8, 8, 8, 8, 8)$, i.e. $v_{(G,c)}(\emptyset) = 0$, $v_{(G,c)}(\{1\}) = 5$, $v_{(G,c)}(\{2\}) = v_{(G,c)}(\{3\}) = v_{(G,c)}(\{1,2\}) = v_{(G,c)}(\{1,3\}) = v_{(G,c)}(\{2,3\}) = v_{(G,c)}(N) = 8$.

In the following we will characterize the classes of airport and irrigation games. First we note that for every rooted tree G we have that $\mathcal{G}_G$ is a cone, therefore by definition for every $\alpha \geq 0$ it holds that $\alpha \mathcal{G}_G \subseteq \mathcal{G}_G$. Since the union of cones is also a cone, both $\mathcal{G}_A^N$ and $\mathcal{G}_I^N$ are cones. Henceforth, let $Cone\{v_i\}_{i \in N}$ denote the convex (i.e. closed to convex combination) cone spanned by the games $(v_i)_{i \in N}$.

Lemma 6 *For every rooted tree G it holds that $\mathcal{G}_G$ is a cone.*

Consequently, since G_A^N and G_I^N are unions of such fixed-tree cones, they are also cones.

In the following lemma we will show that the duals of unanimity games are airport games.

Lemma 7 *For each coalition $\emptyset \neq T \subseteq N$ there exists a chain G such that $T = S_i(G)$ for some $i \in N$, hence $\bar{u}_T \in \mathcal{G}_G$. Therefore $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}} \subset \mathcal{G}_A^N \subset \mathcal{G}_I^N$.*

Proof For every $i \in N$ take the chain G such that $T = S_i(G)$. Then let two types of players $N \setminus S_i(G)$ and $S_i(G)$, and let $c_1 = 0$ and $c_2 = 1$, where c_1 is cost of the

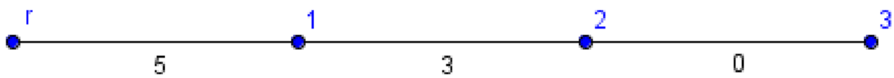


Fig. 2 Cost tree (G, c) of airport problem in Example 5

members of $N \setminus S_i(G)$ and c_2 is the cost of members of $S_i(G)$. Then the induced airport game is $v_{(G,c)} = \bar{u}_{S_i(G)}$.

On the other hand, clearly, there exists an airport game which is not the dual of any unanimity game (for example an airport game containing two non-equal positive costs). □

Let us consider Fig. 3 demonstrating the representation of dual unanimity games as airport games on player set $N = \{1, 2, 3\}$.

It is important to see the relationships between the classes of airport and irrigation games, and the convex cone spanned by the duals of unanimity games.

Lemma 8 *For every rooted tree G it holds that* $\mathcal{G}_G \subset Cone\{\bar{u}_{S_i(G)}\}_{i\in N}$. *Therefore,* $\mathcal{G}_A^N \subset \mathcal{G}_I^N \subset Cone\{\bar{u}_T\}_{T\in 2^N\setminus\{\emptyset\}}$.

Proof We divide the proof into two parts:

- First we show that $\mathcal{G}_G \subset Cone\{\bar{u}_{S_i(G)}\}_{i\in N}$. Let $v \in \mathcal{G}_G$ be an irrigation game. Since $G = (V, E)$ is a rooted tree, for all $i \in N$ we have that $|\{j \in V : \overline{ji} \in E\}| = 1$. Therefore, a notation can be introduced for the player preceding player *i*, let $i_- = \{j \in V : \overline{ji} \in E\}$. Then for all $i \in N$ let $\alpha_{S_i(G)} = c_{\overline{i_-i}}$. Now we show that $v = \sum_{i\in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}$. For the edge $\overline{i_-i}$, its cost is included in the trunk of *S* if and only if at least one player of *S* is a successor of *i*. Equivalently, $S \cap S_i(G) \neq \emptyset$. By the definition of the dual unanimity game, this condition is equivalent to $\bar{u}_{S_i(G)}(S) = 1$. Hence,

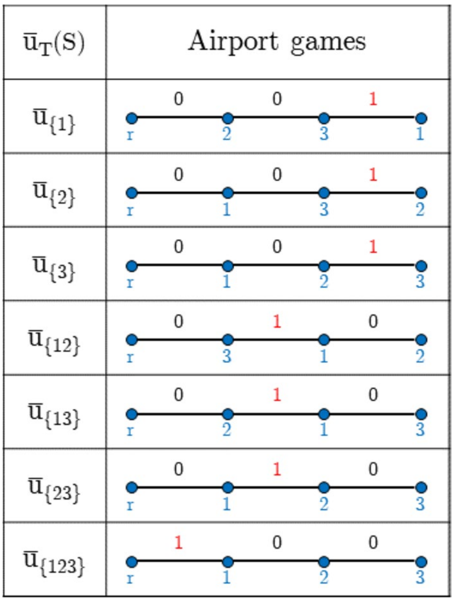
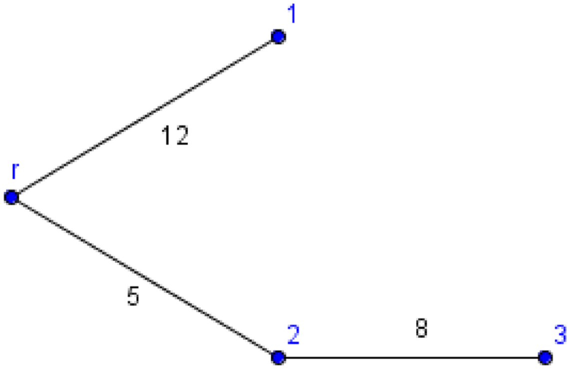


Fig. 3 Duals of unanimity games as airport games

Fig. 4 Cost tree (G, c) of irrigation problems in Examples 2 and 9



$$v_{(G,c)}(S) = \sum_{\{i \in N : S \cap S_i(G) \neq \emptyset\}} \alpha_{S_i(G)} = \sum_{i \in N} \alpha_{S_i(G)}\, \overline{u}_{S_i(G)}(S).$$

Since this equation holds for every coalition $S \subseteq N$, this proves that $v_{(G,c)} = \sum_{i \in N} \alpha_{S_i(G)}\, \overline{u}_{S_i(G)}$.

- Secondly, we show that the inclusion is proper, that is, $Cone\{\bar{u}_{S_i(G)}\}_{i \in N} \setminus \mathcal{G}_G \neq \emptyset$. Notice that is enough to consider the two-player case, the more general cases are implied by this case. Let $N = \{1, 2\}$, then $\sum_{T \in 2^N \setminus \{\emptyset\}} \bar{u}_T \notin \mathcal{G}_G$. Consequently, (2, 2, 3) is not an irrigation game. Indeed, in the case of two players the values of the singleton coalitions are $v(\{1\}) = v(\{2\}) = 2$. Two different trees can serve as the backbone of the game: (1) If the tree is a chain, where the cost of the last edge is 0, then the value of the grand coalition should be 2, while (2) if there are two edges pointing outwards from the root with cost 2 for each, then the value of the grand coalition should be 4; hence we arrive at a contradiction. □

We illustrate the above results with the Example 9.

Example 9 Let us consider the irrigation game given in Example 2 with the cost tree given in Fig. 4.

Consequently, $S_1(G) = \{1\}$, $S_2(G) = \{2, 3\}$ and $S_3(G) = \{3\}$. Moreover, $\alpha_{S_1(G)} = 12$, $\alpha_{S_2(G)} = 5$ and $\alpha_{S_3(G)} = 8$. Finally, $v_{(G,c)} = 12\bar{u}_{\{1\}} + 5\bar{u}_{\{2,3\}} + 8\bar{u}_{\{3\}} = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}$.

In the following we discuss further corollaries of Lemmata 7 and 8. First we prove that for any rooted tree G, even if the set $\mathcal{G}_G$ is convex, the set of airport and irrigation games are not convex.

Lemma 10 *Neither $\mathcal{G}_A^N$ nor $\mathcal{G}_I^N$ is convex.*

Proof Let $N = \{1, 2\}$. According to Lemma 7 we know that $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}} \subseteq \mathcal{G}_A^N$. However, $\sum_{T \in 2^N \setminus \{\emptyset\}} \frac{1}{3} \bar{u}_T \notin \mathcal{G}_I^N$, that is, (2/3, 2/3, 1) is not an irrigation game. □

The following corollary is essential for the axiomatization of the Shapley value by Young (1985) on the classes of airport and irrigation games. It is well-known that the duals of unanimity games are linearly independent vectors. By Lemma 8 for every rooted tree G and game $v \in \mathcal{G}_G$ it holds that $v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}$, where weights $\alpha_{S_i(G)}$ are well-defined. The following Lemma 11 claims that for all games $v \in \mathcal{G}_G$, if the weight of any of the basis vectors (duals of unanimity games) is decreased to 0 (erased), then we get a game corresponding to $\mathcal{G}_G$.

Lemma 11 *For every airport game* $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ *and coalition* $T^* \in 2^N \setminus \{\emptyset\}$ *it holds that* $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_A^N$. *Moreover, for every irrigation game* $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ *and* $T^* \in 2^N \setminus \{\emptyset\}$ *it holds that* $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_I^N$.

Proof Let $v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}$, G is given, and $i^* \in N$. Then the cost function c' is defined as follows: for every $e \in E$ (see proof of Lemma 8)

$$c'_e = \begin{cases} 0 & \text{if } e = \overline{i^*_- i^*}, \\ c_e & \text{otherwise.} \end{cases}$$

Then $\sum_{i \in N \setminus \{i^*\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} = v_{(G,c')}$, i.e. $\sum_{i \in N \setminus \{i^*\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$. □

The next example illustrates the above results.

Example 12 Let us consider the irrigation game in Example 2, and player 2. Then

$$c'_e = \begin{cases} 12 & \text{if } e = \overline{r1}, \\ 0 & \text{if } e = \overline{r2}, \\ 8 & \text{if } e = \overline{23}. \end{cases}$$

Moreover, $\sum_{i \in N \setminus \{2\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} = 12\bar{u}_{\{1\}} + 8\bar{u}_{\{3\}} = v_{(G,c')}$ is an irrigation game.

Finally, a simple observation.

Lemma 13 *(Koster et al. (2001)) All irrigation games are concave.*

Proof Duals of unanimity games are concave, therefore the claim follows from Lemma 8. □

Our results can be summarized as follows.

Corollary 14 *In the case of a fixed player set, the class of airport games is the union of finitely many convex cones, but it is not itself convex. Moreover, the class of airport games is a proper subset of the class of irrigation games. The class of irrigation games is likewise the union of finitely many convex cones, but it is not convex either. Furthermore, the class of irrigation games is a proper subset of the finite convex cone spanned by the duals of the unanimity games. Therefore, every irrigation game is concave, and consequently every airport game is concave as well.*

3 Solutions for irrigation games

In this section we present solutions for irrigation games. A *solution* ψ on set $A \subseteq \mathcal{G}^N$ is a set-valued mapping $\psi : A \twoheadrightarrow \mathbb{R}^N$, that is a solution assigns a set of allocations to each game. In the following we discuss two solutions, the Shapley value (Shapley 1953), and the core (Shapley 1955; Gillies 1959).

Let $v \in \mathcal{G}^N$ and

$$p^i_{Sh}(S) = \begin{cases} \dfrac{|S|!(|(N \setminus S)| - 1)!}{|N|!} & \text{if } i \notin S, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi_i(v)$, the Shapley value of player *i* in game *v* is the p^i_{Sh}-weighted expected value of all v'_i. In other words:

$$\phi_i(v) = \sum_{S \subseteq N} v'_i(S)\ p^i_{Sh}(S)\ . \tag{1}$$

Henceforth let ϕ denote the Shapley value.

It follows from the definition that the Shapley value is a single-valued solution, i.e. it is a value.

Let $v \in \mathcal{G}^N$ be a game, then the core of *v* is

$$C(v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N),\ \text{and}\ \sum_{i \in S} x_i \leq v(S),\ S \subseteq N \right\}\ .$$

The core consists of the stable allocations of the value of the grand coalition. That is, an allocation belongs to the core if it is efficient, i.e. $\sum_{i \in N} x_i = v(N)$, and no coalition has an incentive to deviate from it. In the context of irrigation games, a core allocation assigns the total cost in such a way that no group of users is required to pay more than the total cost of the ditch segments they use.

Recall the notion of a subsidy-free property (Aadland and Kolpin 1998). Adapted to our setting, an allocation rule is subsidy-free if, for every coalition $S \subseteq N$, it holds that $\sum_{i \in \bar{S}} \xi_i \leq \sum_{i \in \bar{S}} c_i$.

The following theorem describes the relationship between the subsidy-free property and the core for the case of irrigation games.

Definition 15 A value ψ on the class of games $A \subseteq \mathcal{G}^N$ is *core compatible*, if for all $v \in A$ it holds that $\psi(v) \in C(v)$.

That is, core compatibility holds, if the cost allocation rule given for an irrigation problem provides a solution that is a core allocation in the corresponding irrigation game.

Theorem 16 *A cost allocation rule ξ on the cost tree (G, c) is subsidy-free, if and only if the value generated by the cost allocation rule ξ on the irrigation game $v_{(G,c)}$ induced by the cost tree is core compatible.*

Proof For irrigation games it holds that $v_{(G,c)}(S) = v_{(G,c)}(\bar{S})$ and $\sum\limits_{i \in S} \xi_i \leq \sum\limits_{i \in \bar{S}} \xi_i$. For core allocations it holds that $\sum\limits_{i \in S} \xi_i \leq v_{(G,c)}(S)$. Since

$$\sum_{i \in S} \xi_i \leq \sum_{i \in \bar{S}} \xi_i \leq v_{(G,c)}(\bar{S}) = v_{(G,c)}(S),$$

it is sufficient to examine cases where $S = \bar{S}$. Therefore the allocations in the core are those for which

$$\sum_{i \in \bar{S}} \xi_i \leq v_{(G,c)}(\bar{S}) = \sum_{i \in \bar{S}} c_i.$$

Comparing the beginning and end of the inequality, we get the subsidy-free property. □

Among the cost allocation methods described in Aadland and Kolpin (1998) the serial and restricted average cost allocations were subsidy-free, and Aadland and Kolpin (1998) have shown that the serial allocation is equivalent to the Shapley value. However, it is more difficult to formulate which further aspects can be chosen and is worth choosing from the core. There are subjective considerations depending on the situation that may influence decision makers regarding which core allocation to choose (which are the "best" alternatives for rational decision makers). Therefore, in case of a specific distribution it is worth examining in detail, which properties a given distribution satisfies, and which properties characterize the distribution within the game class.

In the following definition we discuss properties that we will use to characterize single-valued solutions. As a reminder, the marginal contribution of player *i* to coalition *S* in game *v* is $v'_i(S) = v(S \cup \{i\}) - v(S)$ for all $S \subseteq N$. Furthermore, $i, j \in N$ are equivalent in game *v*, i.e. $i \sim^v j$ if $v'_i(S) = v'_j(S)$ for all $S \subseteq N \setminus \{i, j\}$.

Definition 17 A single-valued solution ψ on $A \subseteq \mathcal{G}^N$ is / satisfies

- Pareto optimal (*PO*), if for all $v \in A$, $\sum\limits_{i \in N} \psi_i(v) = v(N)$,
- null-player property (*NP*), if for all $v \in A$, $i \in N$, $v'_i = 0$ implies $\psi_i(v) = 0$,
- equal treatment property (*ETP*), if for all $v \in A$, $i, j \in N$, $i \sim^v j$ implies $\psi_i(v) = \psi_j(v)$,
- additive (*ADD*), if for all $v, w \in A$ such that $v + w \in A$, $\psi(v + w) = \psi(v) + \psi(w)$,
- marginality (*M*), if for all $v, w \in A$, $i \in N$, $v'_i = w'_i$ implies $\psi_i(v) = \psi_i(w)$.

Brief interpretations of the above axioms are as follows. Firstly, another commonly used name of axiom *PO* is *Efficiency*. This axiom requires that the total cost must be shared among the players. Axiom *NP* requires that if the player's marginal contribution to all coalitions is zero, i.e. the player has no influence on the costs in the given situation, then the player's share (value) must be zero.

Axiom ETP requires that if two players have the same effect on the change of costs in the given situation, then their share must be equal when distributing the total cost. Going back to our example this means that if two users are equivalent with respect to their irrigation costs, then their cost shares must be equal as well.

A solution meets axiom *ADD*, if for any two games, we get the same result by adding up the games first and then evaluating the share of players, as if we evaluated the players first and then added up their shares.

At last, axiom *M* requires that if a given player in two games has the same marginal contributions to the coalitions, then the player must be evaluated equally in the two games.

Let us consider the following observation.

Proposition 18 *Let $A, B \subseteq \mathcal{G}^N$. If a set of axioms S characterizes the value ψ on both classes of games A and B, and ψ meets the set of axioms S on the set $A \cup B$ then the set of axioms S characterizes ψ on the class $A \cup B$.*

In the following we examine two characterizations of the Shapley value on the classes of airport and irrigation games. The first one is based on the original axiomatization by Shapley (Shapley 1953).

Theorem 19 *For all rooted trees G a value ψ is PO, NP, ETP and ADD on $\mathcal{G}_G$ if and only if $\psi = \phi$, i.e. it is the Shapley value. In other words, a value ψ is PO, NP, ETP and ADD on the classes of airport and irrigation games if and only if $\psi = \phi$.*

Proof $\Rightarrow$: It is known that the Shapley value is *PO*, *NP*, *ETP* and *ADD*, see for example Peleg and Sudhölter (2007).

$\Leftarrow$: Based on Lemmata 6 and 7, ψ is defined on the cone spanned by $\{\bar{u}_{S_i(G)}\}_{i \in N}$.

Let us consider player $i^* \in N$. Then for all $\alpha \geq 0$ and players $i, j \in S_{i^*}(G)$: $i \sim^{\alpha \bar{u}_{S_{i^*}(G)}} j$, and for all players $i \notin S_{i^*}(G)$: $i \in NP(\alpha \bar{u}_{S_{i^*}(G)})$.

Then from property *NP* it follows that for all players $i \notin S_{i^*}(G)$ it holds that $\psi_i(\alpha \bar{u}_{S_{i^*}(G)}) = 0$. Moreover, according to axiom *ETP* for all players $i, j \in S_{i^*}(G)$: $\psi_i(\alpha \bar{u}_{S_{i^*}(G)}) = \psi_j(\alpha \bar{u}_{S_{i^*}(G)})$. Finally, axiom *PO* implies that $\sum_{i \in N} \psi_i(\alpha \bar{u}_{S_{i^*}(G)}) = \alpha$.

Consequently, we know that $\psi(\alpha \bar{u}_{S_{i^*}(G)})$ is well-defined (uniquely determined), and since the Shapley value is *PO*, *NP* and *ETP*, it follows that $\psi(\alpha \bar{u}_{S_{i^*}(G)}) = \phi(\alpha \bar{u}_{S_{i^*}(G)})$.

Furthermore, we know that games $\{u_T\}_{T \in 2^N \setminus \emptyset}$ give a basis of $\mathcal{G}^N$, and the same holds for games $\{\bar{u}_T\}_{T \in 2^N \setminus \emptyset}$. Let $v \in \mathcal{G}_G$ be an irrigation game. Then it follows from Lemma 8 that

$$v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \ ,$$

where for all $i \in N$: $\alpha_{S_i(G)} \geq 0$. Therefore, since the Shapley value is *ADD*, and for all $i \in N$, $\alpha_{S_i(G)} \geq 0$ it holds that $\psi(\alpha_{S_i(G)} \bar{u}_{S_i(G)}) = \phi(\alpha_{S_i(G)} \bar{u}_{S_i(G)})$, i.e. $\psi(v) = \phi(v)$.

Finally, Proposition 18 can be applied. □

In the proof of Theorem 19 we have applied a modified version of Shapley's original proof. In his proof Shapley uses the basis given by unanimity games $\mathcal{G}^N$. In the previous proof we considered the duals of the unanimity games as a basis and used Proposition 18 and Lemmata 6, 7, 8. It is worth noting that (as we will discuss it in the next section) for the class of airport games Theorem 19 was also proven by Dubey (1982), therefore in this sense our result is also an alternative proof for Dubey (1982)'s result.

In the following we consider Young's axiomatization of the Shapley value (Young 1985). This was the first axiomatization of the Shapley value not involving axiom *ADD*.

Theorem 20 *For any rooted tree G, a single-valued solution ψ on $\mathcal{G}_G$ is PO, ETP and M if and only if $\psi = \phi$, i.e. it is the Shapley value. Therefore, a single-valued solution ψ on the class of airport games is PO, ETP and M if and only if $\psi = \phi$, and a single-valued solution ψ on the class of irrigation games is PO, ETP and M if and only if $\psi = \phi$.*

Proof $\Rightarrow$: We know that the Shapley value meets axioms *PO*, *ETP* and *M*, see e.g. Peleg and Sudhölter (2007).

$\Leftarrow$: The proof is by induction, similarly to that of Young's. For any irrigation game $v \in \mathcal{G}_G$, let $\mathcal{B}(v) = |\{\alpha_{S_i(G)} > 0 : v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}\}|$. Clearly, $\mathcal{B}(\cdot)$ is well-defined.

If $\mathcal{B}(v) = 0$ (i.e. $v \equiv 0$), then axioms *PO* and *ETP* imply that $\psi(v) = \phi(v)$.

Let us assume that for any game $v \in \mathcal{G}_G$ for which $\mathcal{B}(v) \leq n$ it holds that $\psi(v) = \phi(v)$. Furthermore, let $v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$ such that $\mathcal{B}(v) = n + 1$.

Then one of the following two points holds.

1. There exists player $i^* \in N$ such that there exists $i \in N$ that it holds that $\alpha_{S_i(G)} \neq 0$ and $i^* \notin S_i(G)$. Then Lemmata 8 and 11 imply that $\sum_{j \in N \setminus \{i\}} \alpha_{S_j(G)} \bar{u}_{S_j(G)} \in \mathcal{G}_G$

   , and for the marginal contributions it holds that

$$\left(\sum_{j \in N \setminus \{i\}} \alpha_{S_j(G)} \bar{u}_{S_j(G)} \right)'_{i^*} = v'_{i^*} \ ,$$

   therefore, from axiom *M*

$$\psi_{i^*}(v) = \psi_{i^*}\left(\sum_{j \in N \setminus \{i\}} \alpha_{S_j(G)} \bar{u}_{S_j(G)}\right),$$

i.e. $\psi_{i^*}(v)$ is well-defined (uniquely determined) and due to the induction hypothesis, $\psi_{i^*}(v) = \phi_{i^*}(v)$.

2. There exist players $i^*, j^* \in N$ such that for all $i \in N$: $\alpha_{S_i(G)} \neq 0$ it holds that $i^*, j^* \in S_i(G)$. Then $i^* \sim^v j^*$, therefore, axiom *ETP* implies that $\psi_{i^*}(v) = \psi_{j^*}(v)$. Axiom *PO* implies that $\sum_{i \in N} \psi_i(v) = v(N)$. Therefore, $\psi(v)$ is well-defined (uniquely determined), consequently, since the Shapley value meets *PO*, *ETP* and *M*, due to the induction hypothesis, $\psi(v) = \phi(v)$.

Finally, we can apply Proposition 18. □

In the above proof we apply the idea of Young's proof, therefore we do not need the alternative approaches provided by Moulin (1988) and Pintér (2015). The applied approach works, because Lemma 11 ensures that when the induction step is applied, we do not leave the considered class of games. It is also worth noting that (as we will discuss it in the following) for the class of airport games Theorem 20 was also proven by Moulin and Shenker (1992), therefore in this sense our result is also an alternative proof for Moulin and Shenker (1992)'s result.

Finally, Lemma 13 and the well-known results of Shapley (1971) and Ichiishi (1981) imply the following corollary.

Corollary 21 *For any irrigation game v, $\phi(v) \in C(v)$, i.e. the Shapley value is in the core. Moreover, since every airport game is an irrigation game, for any airport game v we have that $\phi(v) \in C(v)$.*

The above corollary shows that the Shapley value belongs to the core for both classes of games considered. In addition to satisfying the stability requirements imposed by the core, the Shapley value provides a "fair" allocation by rewarding each player according to his or her contribution.

4 Cost sharing results

In this section we reformulate our results using the classic cost sharing terminology. In order to unify the different terminologies used in the literature, we exclusively use Thomson (2024)'s notions. First we introduce the notion of a *rule*, which is in practice analogous to the notion of a cost-sharing rule introduced by Aadland and Kolpin (1998). Let us consider the class of allocation problems defined on cost trees, i.e. the set of cost trees. A *rule* is a mapping which assigns a cost allocation to a cost tree allocation problem, providing a method by which the cost is allocated among the players. Note that the rule is a single-valued mapping. The analogy between the rule and the solution is clear, the only significant difference is that while the solution can be a set-valued mapping, the rule is single-valued.

Let us introduce the rule known in the literature as *sequential equal contributions rule*.

Definition 22 *(SEC rule)* For all cost trees (G, c) and for all players i the distribution according to the *SEC rule* is given as follows:

$$\xi_i^{SEC}(G,c) = \sum_{j \in P_i(G) \setminus \{\mathrm{r}\}} \frac{c_{\overline{j\_j}}}{|S_j(G)|} \; .$$

In the case of airport games, where graph G is a chain, the SEC rule can be given as follows, for any player i:

$$\xi_i^{SEC}(G,c) = \frac{c_1}{n} + \cdots + \frac{c_{\overline{i\_i}}}{n-i+1} \; ,$$

where the players are ordered according to their positions in the chain, i.e. player i is at the ith position of the chain.

Littlechild and Owen (1973) have shown that the SEC rule and the Shapley value are equivalent on the class of airport games, however this equivalence holds on the class of irrigation games too.

Proposition 23 *For every cost-tree (G, c) we have that $\xi(G,c) = \phi(v_{(G,c)})$, where $v_{(G,c)}$ is the irrigation game generated by the airport problem (G, c). In other words, for cost-tree problems the SEC rule and the Shapley value are equivalent.*

Proof Consider a cost-tree (G, c) and the related irrigation game $v_{(G,c)} = \sum_{i \in N} \alpha_{S_i(G)} \overline{u}_{S_i(G)}$, where $\alpha_{S_i(G)} = c_{\overline{i\_i}}$, $i \in N$.

Then for every $i \in N$

$$\xi_i^{SEC}(G,c) = \sum_{j \in P_i(G) \setminus \{\mathrm{r}\}} \frac{c_{\overline{j\_j}}}{|S_j(G)|} = \sum_{i \in N} \frac{\alpha_{S_i(G)}}{|S_i(G)|} = \sum_{i \in N} \alpha_{S_i(G)} \phi_i(\overline{u}_{S_i(G)}) = \phi_i(v_{(G,c)}) \, ,$$

where ϕ_i denotes the Shapley value of player i.

$\square$

In the following we consider certain properties of *rules* (see Thomson, 2024).

Definition 24 Let $G = (V, E)$ be a rooted tree. A rule χ defined on the set of cost trees denoted by G satisfies

- non-negativity, if for each cost function c, $\chi(G,c) \geq 0$,
- cost boundedness, if for each cost function c, $\chi(G,c) \leq \left(\sum_{e \in E_{P_i(G)}} c_e \right)_{i \in N}$,

- efficiency, if for each cost function c, $\sum\limits_{i \in N} \chi_i(G,c) = \sum\limits_{e \in E} c_e$,
- equal treatment of equals, if for each cost function c and pair of players $i, j \in N$, $\sum\limits_{e \in E_{P_i(G)}} c_e = \sum\limits_{e \in E_{P_j(G)}} c_e$ implies $\chi_i(G,c) = \chi_j(G,c)$,
- conditional cost additivity, if for any pair of cost functions c, c', $\chi(G, c + c') = \chi(G,c) + \chi(G,c')$,
- independence of at-least-as-large costs, if for any pair of cost functions c, c' and player $i \in N$ such that for each $j \in P_i(G)$, $\sum\limits_{e \in E_{P_j(G)}} c_e = \sum\limits_{e \in E_{P_j(G)}} c'_e$,

  $\chi_i(G,c) = \chi_i(G,c')$.

The interpretations of the properties of rules defined above are as follows. Non-negativity requires that, for every problem, the rule assign a non-negative cost allocation vector. Cost boundedness requires that the individual costs constitute upper bounds for the components of the cost allocation vector. Efficiency states that the components of the cost allocation vector must sum to the maximal cost. Equal treatment of equals requires that players with equal individual costs be assigned equal costs. Conditional cost additivity requires that, when two cost trees with the same underlying tree are added, the resulting cost allocation vector be equal to the sum of the cost allocation vectors corresponding to the individual problems. Finally, independence of at-least-as-large costs means that the amount paid by a player is independent of the costs of the segments that the player does not use.

These properties are similar to the axioms we defined in Definition 17. The following proposition formalizes the similarity.

Proposition 25 *Let G be a rooted tree, χ be a rule defined on cost trees (G, c), a solution ψ be defined on $\mathcal{G}_G$ such that $\chi(G,c) = \psi(v_{(G,c)})$ for any cost function c. Then, if χ satisfies*

- non-negativity and cost boundedness, then ψ is *NP*,
- efficiency, then ψ is *PO*,
- equal treatment of equals, then ψ is *ETP*,
- conditional cost additivity, then ψ is *ADD*,
- independence of at-least-as-large costs, then ψ is *M*.

Proof *NN* and *CB* $\Rightarrow$ *NP*: Clearly, player *i* is *NP*, if and only if $\sum_{e \in E_{P_i(G)}} c_e = 0$. Then *NN* implies that $\chi_i(G,c) \geq 0$, and from *CB*: $\chi_i(G,c) \leq 0$. Consequently, $\chi_i(G,c) = 0$, therefore $\psi(v_{(G,c)}) = 0$.

E $\Rightarrow$ *PO*: From the definition of irrigation games (Definition 1): $\sum_{e \in A} c_e = v_{(G,c)}(N)$, therefore $\sum_{i \in N} \psi_i(v_{(G,c)}) = \sum_{i \in N} \chi_i(G,c) = \sum_{e \in E} c_e = v_{(G,c)}(N)$.

ETE ⇒ *ETP*: Clearly, if $i \sim^{v(G,c)} j$ for $i, j \in N$, then $\sum_{e \in E_{P_i(G)}} c_e = \sum_{e \in E_{P_j(G)}} c_e$, so $\chi_i(G,c) = \chi_j(G,c)$. Therefore, if $i \sim^{v(G,c)} j$ for $i, j \in N$, then $\psi_i(v_{(G,c)}) = \psi_j(v_{(G,c)})$.

CCA ⇒ *ADD*: $\psi(v_{(G,c)} + v_{(G,c')}) = \psi(v_{(G,c+c')}) = \chi(G, c+c') = \chi((G,c)+(G,c')) = \chi(G,c) + \chi(G,c') = \psi(v_{(G,c)}) + \psi(v_{(G,c')})$.

IALC ⇒ *M*: It is easy to check that if for cost trees (*G*, *c*), (G, c') and player $i \in N$ it holds that $(v_{(G,c)})'_i = (v_{(G,c')})'_i$, then for each $j \in P_i(G)$: $\sum_{e \in E_{P_j(G)}} c_e = \sum_{e \in E_{P_j(G)}} c'_e$, therefore $\chi_i(G,c) = \chi_i(G,c')$. Consequently, $(v_{(G,c)})'_i = (v_{(G,c')})'_i$ implies $\psi_i(v_{(G,c)}) = \psi_i(v_{(G,c')})$. □

It is worth noting that all but the efficiency point are tight, i.e. efficiency and the Pareto-optimal property are equivalent, in the other cases the implication holds in only one way. Therefore the cost-sharing axioms are stronger than the game theoretical axioms.

The above results and Theorem 19 imply Dubey (1982)'s result as a direct corollary.

Theorem 26 *(Dubey, 1982) A rule χ on the class of airport games satisfies non-negativity, cost boundedness, efficiency, equal treatment of equals and conditional cost additivity, if and only if $\chi = \xi$, i.e. χ is the SEC rule.*

Proof ⇒: With a simple calculation it can be shown that the SEC rule satisfies properties *NN*, *CB*, *E*, *ETE*, and *CCA* (see e.g. Thomson (2024)).

⇐: Proposition 25 implies that we can apply Theorem 19 and thereby get the Shapley value. Then Proposition 23 implies that the Shapley value and the SEC rule coincide. □

Moulin and Shenker (1992)'s result can be deduced from the results above and Theorem 20, similarly to Dubey (1982)'s result.

Theorem 27 *(MoulinShenker, 1992) A rule χ on the class of airport problems satisfies efficiency, equal treatment of equals and independence of at-least-as-large costs, if and only if $\chi = \xi$, i.e. χ is the SEC rule.*

Proof ⇒: It can be shown with a straightforward calculation that the SEC rule satisfies the *E*, *ETE* and *IALC* properties (see e.g. Thomson (2024)).

⇐: Based on Proposition 25, Theorem 20 can be applied and thereby get the Shapley value. Then Proposition 23 implies that the Shapley value and the SEC rule coincide. □

Finally, in the following two theorems we extend the results of Dubey (1982) and Moulin and Shenker (1992) to any cost tree allocation problem. The proofs of these results follow the same arguments as those used in the previous two theorems.

Theorem 28 *A rule χ on the class of cost-tree problems satisfies non-negativity, cost boundedness, efficiency, equal treatment of equals and conditional cost additivity, if and only if $\chi = \xi$, i.e. χ is the SEC rule.*

Theorem 29 *A rule χ on the class of cost-tree problems satisfies efficiency, equal treatment of equals and independence of at-least-as-large costs, if and only if $\chi = \xi$, i.e. χ is the SEC rule.*

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