

Wasserstein rigidity over $\mathbb{R}^n$ with smooth norms

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Abstract

We study p – Wasserstein spaces $\mathcal{W}_p(\mathbb{R}^n, d_N)$ over $\mathbb{R}^n$ equipped with a norm metric d_N . We show that, if the norm is smooth enough, then the Wasserstein space is isometrically rigid whenever $p \neq 2$. We also show that, even when $p = 2$, we can recover the isometric rigidity of the Wasserstein space $\mathcal{W}_2(\mathbb{R}^n, d_N)$ when N is an l_q – norm and $q > 2$.

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1 | INTRODUCTION AND MAIN RESULTS

For a fixed $p \geq 1$ and a complete separable metric space (X, d_X) , the p -Wasserstein space $\mathcal{W}_p(X, d_X)$ is the set of Borel probability measures with finite p -moment, equipped with the so-called Wasserstein distance $d_{\mathcal{W}_p}$. More precisely, we say that a Borel probability measure μ is in the p -Wasserstein space if

$$\int_X d_X^p(x, x_0) d\mu(x) < \infty$$

for some (and hence any) $x_0 \in X$, and the Wasserstein distance is given by

$$d_{\mathcal{W}_p}(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \left(\iint_{X \times X} d_X^p(x, y) d\pi(x, y) \right)^{\frac{1}{p}} \right\},$$

where $\Pi(\mu, \nu)$ is the set of couplings between μ and ν , that is, the set of probability measures $\pi \in \mathcal{P}(X \times X)$ with $p_{1\#}\pi = \mu$ and $p_{2\#}\pi = \nu$, for the projection operators $p_1(x, y) = x$ and $p_2(x, y) = y$; here $p_{\#}$ denotes the push-forward operation, which we define below.

The Wasserstein space $\mathcal{W}_p(X, d_X)$ inherits a number of properties from its base space (X, d_X) , see [1, 8, 16, 17] for an overview. For example, if (X, d_X) is complete and separable, then so is $\mathcal{W}_p(X, d_X)$.

Another property of any Wasserstein space is that it contains an isometric copy of its base space. Indeed, the map $x \rightarrow \delta_x$ is an isometric embedding from the space (X, d_X) to $\mathcal{W}_p(X, d_X)$, where δ_x is the Dirac mass located at $x \in X$. Furthermore, the convex hull of Dirac masses (i.e., the set of measures with finite support) is dense in $\mathcal{W}_p(X, d_X)$ (see, e.g., [17, Theorem 6.18]). If ϕ is an isometry (i.e., a distance-preserving and bijective map) on (X, d_X) , it induces an isometry of $\mathcal{W}_p(X, d_X)$ by the push-forward operation. More precisely, $\phi_{\#} : \mathcal{W}_p(X, d_X) \rightarrow \mathcal{W}_p(X, d_X)$ defined by

$$\phi_{\#}(\mu)(A) = \mu(\phi^{-1}(A)) \quad \text{for } A \subset X$$

is an isometry of $\mathcal{W}_p(X, d_X)$. An isometry Φ of the Wasserstein space that can be written as $\Phi = \phi_{\#}$ is called a trivial isometry. If the isometry group of $\mathcal{W}_p(X, d_X)$ contains only trivial isometries, we say that the Wasserstein space is isometrically rigid.

In [12], Kloeckner showed that for $p = 2$, the space $\mathcal{W}_2(\mathbb{R}^n, d_E)$ (where d_E is the usual Euclidean distance) is not rigid. That is, its isometry group contains nontrivial isometries, called shape preserving isometries when $n \geq 2$. Another example of a nonrigid space was given in [10], where Gehér et al. showed that $\mathcal{W}_1([0, 1], |\cdot|)$ admits nontrivial, so-called mass splitting isometries.

In contrast, Bertrand and Kloeckner showed in [5, 6] that if (X, d_X) is Hadamard, that is, a Riemannian manifold with negative sectional curvature, then $\mathcal{W}_2(X, d_X)$ is isometrically rigid. Furthermore Santos-Rodriguez proved in [14] that the same is true when (X, d_X) is a manifold with strictly positive sectional curvature. Rigidity in the case of the sub-Riemannian Heisenberg group was shown by Balogh et al. in [4]; in [3], the same authors together with Kiss showed that the space $\mathcal{W}_p(\mathbb{R}^2, d_{\max})$ is isometrically rigid for any $p \geq 1$, where $d_{\max}$ is the distance induced by

the maximum norm on the plane. This shows that there is an abundance of metric spaces where the associated p -Wasserstein space is rigid.

On the other hand, there is also an abundance of nonrigid Wasserstein spaces, as shown in the recent result by Che et al., who proved in [7] that for any Hilbert space (H, d_H) and any proper metric space (Y, d_Y) , $\mathcal{W}_2(H \times Y, d_{H \oplus_2 Y})$ (where $\oplus_2$ denotes the l_2 -product metric) is not rigid.

In this paper, we study the isometric rigidity of the Wasserstein spaces $\mathcal{W}_p(X, d_X)$ for the class of general normed spaces of $(\mathbb{R}^n, d_N)$. We recall that in [11], Gehér et al. showed that, if $(\mathbb{R}^n, d_H)$ is a Hilbert space with d_H the distance induced by a scalar product, then for any $p \geq 1$ with $p \neq 2$, the space $\mathcal{W}_p(\mathbb{R}^n, d_H)$ is isometrically rigid. We generalize their result in the following way: instead of requiring that the norm comes from an inner product, we show that only the smoothness of the norm is enough to ensure the isometric rigidity of the Wasserstein space. Our first theorem goes as follows:

Theorem 1.1. *If $N : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a strictly convex norm that is C^2 -smooth, then the Wasserstein space $\mathcal{W}_p(\mathbb{R}^n, d_N)$ is isometrically rigid for all $p \in [1, \infty)$, $p \neq 2$.*

Here we say that the norm N is C^2 -smooth if at any point $x \in \mathbb{R}^n \setminus \{0\}$, N is (at least) twice continuously differentiable at x . Since the one-dimensional case has already been proven in [10], we will assume in this paper that $n \geq 2$.

Our proof will be done in three steps. We first prove a geometric characterization of Dirac masses, which will in particular imply that for any $x \in \mathbb{R}^n$ and any isometry of the Wasserstein space Φ , there exists $y \in \mathbb{R}^n$ such that $\Phi(\delta_x) = \delta_y$. In our second step, we show a dimension-upgrading result, which says that if an isometry Φ acts trivially on measures supported on certain subspaces of $\mathbb{R}^n$, then Φ has to act trivially on measures supported on the whole $\mathbb{R}^n$. In the third step, we show that, using the C^2 -smoothness of the norm, we can find a proper subspace L such that, if a measure is supported on L , then so is its image by any isometry Φ . Finally, we combine these steps and, with an induction argument on the dimension of $\mathbb{R}^n$, we can prove the isometric rigidity of the Wasserstein space $\mathcal{W}_p(\mathbb{R}^n, d_N)$.

We cannot expect this result to hold in general for $p = 2$. Indeed, as mentioned above, Kloeckner [12] and Che et al. [7] gave examples of Wasserstein spaces over certain special types of normed spaces in $\mathbb{R}^n$ which allowed shape-preserving isometries.

In our proof of Theorem 1.1, we only need the condition $p \neq 2$ for certain steps; we can thus adapt the proof when the norm N is an l_q -norm to show isometric rigidity in the case $p = 2$, where the l_q -norms are given by $N_q(x) = \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}}$. Specifically, we prove the following theorem:

Theorem 1.2. *If $q > 2$ and d_q is the distance function induced by the l_q -norm, then the Wasserstein space $\mathcal{W}_2(\mathbb{R}^n, d_q)$ is isometrically rigid.*

This theorem only considers the case $q > 2$; we will show that $\mathcal{W}_2(\mathbb{R}^n, d_q)$ is also isometrically rigid when $1 \leq q < 2$ in our upcoming paper [15].

Our paper is structured in the following way: in the next section, we show the metric characterization of Dirac masses. In Section 3, we prove the dimension-upgrading proposition. In Section 4, we will show Theorem 1.1, first when $p < 2$, then for $p > 2$; we finish the section by adapting the $p > 2$ -argument to show Theorem 1.2.

2 | METRIC CHARACTERIZATION OF DIRAC MASSES

When investigating the isometric rigidity of Wasserstein spaces, a metric characterization of Dirac masses is very often a key tool. Indeed, since isometries preserve distances, such a characterization shows that the image under an isometry of a Dirac mass is another Dirac mass. If that is the case, then for an isometry $\Phi : \mathcal{W}_p(\mathbb{R}^n, d_N) \rightarrow \mathcal{W}_p(\mathbb{R}^n, d_N)$, the map $\psi : (\mathbb{R}^n, d_N) \rightarrow (\mathbb{R}^n, d_N)$ defined by $\Phi(\delta_x) = \delta_{\psi(x)}$ is an isometry of $\mathbb{R}^n$, and $(\Phi \circ \psi_{\#}^{-1})(\delta_x) = \delta_x$ for all $x \in \mathbb{R}^n$. If we can show that $\Phi \circ \psi_{\#}^{-1}$ is the identity on $\mathcal{W}_p(\mathbb{R}^n, d_N)$, we can conclude that $\Phi = \psi_{\#}$ is a trivial isometry. Thus we will in the following sections sometimes assume that $\Phi(\delta_x) = \delta_x$, since this can be obtained by composition with the trivial isometry $\psi_{\#}^{-1}$, if we already know that Dirac masses are invariant.

In the special case where N is a Hilbert norm, a metric characterization of Dirac masses has been found in [11, Lemma 3.5]. As we will see, the same characterization holds for a much more general class of metric spaces (X, d) . In what follows, we say that a triple (μ, ν, η) of measures in $\mathcal{W}_p(X, d)$ is $\mathcal{W}_p$ -aligned if μ, ν and η are distinct and

$$d_{\mathcal{W}_p}(\mu, \nu) + d_{\mathcal{W}_p}(\nu, \eta) = d_{\mathcal{W}_p}(\mu, \eta). \quad (2.1)$$

As a reminder, a curve $\gamma : [a, b] \rightarrow X$ is called a (unit speed) geodesic if $d(\gamma(s), \gamma(t)) = |s - t|$ for any $s, t \in [a, b]$. In this section, we prove a characterization of Dirac masses in the general case where (X, d) is a uniquely geodesic and geodesically complete Polish space; when $p = 1$, we also assume that (X, d) is nonbranching. We recall that a general metric space (X, d) is geodesically complete if any two points $x, y \in X$ can be connected by a geodesic which can be arbitrarily extended (i.e., for any geodesic $\gamma : [a, b] \rightarrow X$, there exists a geodesic $\gamma' : (-\infty, \infty) \rightarrow X$ with $\gamma'([a, b]) = \gamma$). If for every two points $x, y \in X$ the geodesic between x and y is unique, we say that (X, d) is uniquely geodesic; furthermore, (X, d) is called nonbranching if any two geodesics which coincide on an open interval (i.e., $\gamma_1((a, b)) = \gamma_2((a, b))$) must necessarily coincide on their entire length.

We point out that the normed spaces $(\mathbb{R}^n, d_N)$ considered in the assumptions of Theorems 1.1 and 1.2 fulfill the assumptions of Proposition 2.1. Indeed, a norm N is said to be strictly convex if for all distinct points $x_1, x_2 \in \mathbb{R}^n$ with $N(x_1) = N(x_2) = 1$ we have

$$N((1-t)x_1 + tx_2) < 1$$

for any $t \in (0, 1)$. By Proposition 7.2.1 of [13] any normed space with a strictly convex norm is uniquely geodesic; since in $(\mathbb{R}^n, d_N)$ straight lines are always geodesics, the nonbranching property follows from the uniqueness of geodesics.

Thus, using Corollary 2.4 and the argument above, we will in the next section assume that isometries $\Phi : \mathcal{W}_p(\mathbb{R}^n, d_N) \rightarrow \mathcal{W}_p(\mathbb{R}^n, d_N)$ preserve Dirac masses, that is, $\Phi(\delta_x) = \delta_x$ for any $x \in \mathbb{R}^n$.

Proposition 2.1. *Let $p \geq 1$, and consider a uniquely geodesic and geodesically complete Polish space (X, d) ; if $p = 1$, assume additionally that (X, d) is nonbranching. Then, for a measure $\mu \in \mathcal{W}_p(X, d)$, the following are equivalent:*

- (1) μ is a Dirac mass, that is, there exists an $x \in X$ such that $\mu = \delta_x$.
- (2) For all $\nu \in \mathcal{W}_p(X, d)$, $\mu \neq \nu$, there exists an $\eta \in \mathcal{W}_p(X, d)$ such that the triple (μ, ν, η) is $\mathcal{W}_p$ -aligned.

Proof. We start by showing that (1) $\Rightarrow$ (2). Take $x \in X$, set $\mu = \delta_x$ and let $\nu \in \mathcal{W}_p(X, d)$ with $\mu \neq \nu$. Consider the dilation map $D_x : X \rightarrow X$ given by $D_x(y) = \gamma_{x,y}(2a)$ for $a = d(x, y)$, where γ is the unique geodesic such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(a) = y$ (and $D_x(x) = x$). Then we have in particular that $d(x, D_x(y)) = 2d(x, y)$ and $d(y, D_x(y)) = d(x, y)$. We claim that, with the measure $\eta := (D_x)_\# \nu$, (μ, ν, η) is $\mathcal{W}_p$ -aligned. Since μ is a Dirac mass, the only transport plan between μ and η is given by $\pi_0 = \delta_x \times \eta$. Thus we have

$$\begin{aligned} d_{\mathcal{W}_p}(\mu, \eta) &= \inf_{\pi \in \Pi(\mu, \eta)} \left(\iint_{X \times X} d^p(\tilde{x}, z) d\pi(\tilde{x}, z) \right)^{1/p} \\ &= \left(\int_X d^p(x, z) d\eta(z) \right)^{1/p} = \left(\int_X d^p(x, D_x(y)) d\nu(y) \right)^{1/p} \\ &= \left(\int_X 2^p d^p(x, y) d\nu(y) \right)^{1/p} = 2d_{\mathcal{W}_p}(\mu, \nu). \end{aligned}$$

So we get that $d_{\mathcal{W}_p}(\mu, \nu) = \frac{1}{2}d_{\mathcal{W}_p}(\mu, \eta)$.

By the triangle inequality, we have $d_{\mathcal{W}_p}(\mu, \eta) \leq d_{\mathcal{W}_p}(\mu, \nu) + d_{\mathcal{W}_p}(\nu, \eta)$, giving the bound $\frac{1}{2}d_{\mathcal{W}_p}(\mu, \eta) \leq d_{\mathcal{W}_p}(\nu, \eta)$. On the other hand, since D_x is a transport map from ν to η ,

$$d_{\mathcal{W}_p}(\nu, \eta) \leq \left(\int_X d^p(y, D_x(y)) d\nu(y) \right)^{1/p} = \left(\int_X d^p(x, y) d\nu(y) \right)^{1/p} = d_{\mathcal{W}_p}(\mu, \nu) = \frac{1}{2}d_{\mathcal{W}_p}(\mu, \eta)$$

showing that $\frac{1}{2}d_{\mathcal{W}_p}(\mu, \eta) \geq d_{\mathcal{W}_p}(\nu, \eta)$. This proves that (2.1) holds, and thus (1) $\Rightarrow$ (2).

We now show that (2) $\Rightarrow$ (1). Assume for contradiction that the measure μ is not a Dirac mass, and thus has two distinct points in its support, x_1 and x_2 . Consider the geodesic γ between x_1 and x_2 (with $\gamma(0) = x_1$), set $y = \gamma(\frac{1}{3}d(x_1, x_2))$ and let $\nu = \delta_y$ be the Dirac mass located at y . By the assumption, there exists a measure $\eta (= \eta_y)$, as the measure depends on the choice of y) such that (2.1) holds. Then

$$\begin{aligned} d_{\mathcal{W}_p}(\mu, \eta) &\leq \left(\iint_{X \times X} d^p(x, z) d(\mu \times \eta)(x, z) \right)^{1/p} \\ &\leq \left(\iint_{X \times X} (d(x, y) + d(y, z))^p d(\mu \times \eta)(x, z) \right)^{1/p} \\ &\leq \left(\iint_{X \times X} d^p(x, y) d(\mu \times \eta)(x, z) \right)^{1/p} + \left(\iint_{X \times X} d^p(y, z) d(\mu \times \eta)(x, z) \right)^{1/p} \\ &= \left(\int_X d^p(x, y) d\mu(x) \right)^{1/p} + \left(\int_X d^p(y, z) d\eta(z) \right)^{1/p} \\ &= d_{\mathcal{W}_p}(\mu, \nu) + d_{\mathcal{W}_p}(\nu, \eta), \end{aligned} \tag{2.2}$$

where in the third inequality, we used the Minkowski inequality.

At this point, we split the proof into two cases depending on the value of p , and start the proof for $p > 1$.

Since (μ, ν, η) are $\mathcal{W}_p$ -aligned, the inequalities in (2.2) are saturated. As $p > 1$, the Minkowski inequality becomes an equality only when there is a constant λ such that

$$d(x, y) = \lambda d(y, z) \text{ for } (\mu \times \eta)\text{-a.e.}(x, z). \quad (2.3)$$

Then, by the definition of the support of a measure, for all $\varepsilon > 0$, the balls $B(x_1, \varepsilon)$ and $B(x_2, \varepsilon)$ have positive μ -measure. Thus we can find $z' \in \text{supp}(\eta)$, $x'_1 \in B(x_1, \varepsilon)$ and $x'_2 \in B(x_2, \varepsilon)$ such that $d(x'_1, y) = \lambda d(y, z')$ and $d(x'_2, y) = \lambda d(y, z')$. By our choice of y , if ε is small enough we can guarantee that $d(x'_2, y) > d(x'_1, y) > 0$. But then $d(x'_1, y) = \lambda d(y, z') = d(x'_2, y)$, which is the desired contradiction. Therefore μ is indeed a Dirac mass, finishing the proof when $p > 1$.

When $p = 1$, the saturation of the inequalities in (2.2) implies

$$d(x, z) = d(x, y) + d(y, z) \text{ for } (\mu \times \eta)\text{-a.e.}(x, z). \quad (2.4)$$

For $\varepsilon > 0$, we can find $x'_1 \in B(x_1, \varepsilon)$, $x'_2 \in B(x_2, \varepsilon)$, and $z \in \text{supp}(\eta)$ such that

$$d(x'_i, z) = d(x'_i, y) + d(y, z) \quad (2.5)$$

holds for $i = 1, 2$.

Consider L_1 (resp. L_2) the unique geodesic between x'_1 (resp. x'_2) and z . Then $y \in L_1$ (resp. $y \in L_2$). Indeed, consider the curve $\tilde{L}_1$ obtained by concatenating the geodesics from x'_1 to y and from y to z . We show that $\tilde{L}_1$ is a geodesic. Let t_0 be defined by the equation $\tilde{L}_1(t_0) = y$. Then for $s, t < t_0$ or $s, t > t_0$, the equation $d(\tilde{L}_1(s), \tilde{L}_1(t)) = |s - t|$ holds by construction. If $s < t_0 < t$, then

$$d(\tilde{L}_1(s), \tilde{L}_1(t)) \leq d(\tilde{L}_1(s), y) + d(y, \tilde{L}_1(t)) = (t_0 - s) + (t - t_0) = (t - s).$$

Note that, since $t > t_0$, $\tilde{L}_1$ is a geodesic between $\tilde{L}_1(t)$ and z , and thus

$$d(\tilde{L}_1(t), z) = |(t - t_0) - d(y, z)| = |(t - d(x'_1, y)) - d(y, z)| = d(x'_1, z) - t;$$

similarly, since $s < t_0$, we have $d(0, \tilde{L}_1(s)) = (s - 0) = s$. Thus, by the triangle inequality we have that

$$d(x'_1, z) \leq d(\tilde{L}_1(0), \tilde{L}_1(s)) + d(\tilde{L}_1(s), \tilde{L}_1(t)) + d(\tilde{L}_1(t), z) \leq s + (t - s) + (d(x'_1, z) - t) = d(x'_1, z),$$

which directly implies $d(\tilde{L}_1(s), \tilde{L}_1(t)) = (s - t)$, and $\tilde{L}_1$ is a geodesic. In particular, if $y \notin L_1$, then L_1 and $\tilde{L}_1$ are two distinct geodesics between x'_1 and z , contradicting the assumption of the uniqueness of geodesics in (X, d) . Thus $y \in L_1$ (and $y \in L_2$). Assume that $y \neq z$. Then, since $L_1 = L_2$ between z and y , we have by the nonbranching condition that $L_1 = L_2$. In particular z, y, x'_1 , and x'_2 are on the same geodesic; if ε is small enough, then y lies between x'_1 and x'_2 . As by equation (2.5) y also lies between x'_1 and z , respectively, between x'_2 and z , we necessarily get that $y = z$, giving the contradiction. In particular this implies that $\nu = \eta$, and (μ, ν, η) is not $\mathcal{W}_1$ -aligned, finishing the proof. $\square$

Remark 2.2. For $p > 1$, the proof uses the ideas of Lemmata 2.1 and 2.2 of [5], where Bertrand and Kloeckner considered the case $p = 2$. We can recover their original assumption (where the Polish

space (Y, d_Y) is only locally compact and geodesically complete) using a selection argument and Arzelà–Ascoli, we refer to the proof of Proposition 2.1 of [5] for more details.

Remark 2.3. In contrast, the conditions of unique geodesics and of nonbranching are necessary in the case $p = 1$. Indeed, in Proposition 7 of [3], Balogh et al. showed that, for the plane equipped with the maximum norm (which is a branching space), the second condition of Proposition 2.1 is equivalent to the measure μ being supported on an extremal line, instead of a single point.

As the above proposition offers a metric characterization of Dirac masses, we have the following important corollary.

Corollary 2.4. *Let $p \geq 1$, and consider a metric space (X, d) with the same assumptions as in Proposition 2.1. Consider $\Phi : \mathcal{W}_p(X, d) \rightarrow \mathcal{W}_p(X, d)$ an isometry. Then there exists an isometry $\psi : (X, d) \rightarrow (X, d)$ such that $(\Phi \circ \psi_{\#}^{-1})(\delta_x) = \delta_x$ for all $x \in X$.*

Proof. For $x \in X$, set $\mu = \delta_x$, the Dirac mass supported on x . We first show that $\Phi(\mu)$ is also a Dirac mass. For this, consider a measure $\nu' \in \mathcal{W}_p(X, d)$, $\nu' \neq \Phi(\mu)$. Then, since μ is a Dirac mass, there exists a measure $\eta \in \mathcal{W}_p(X, d)$ with η different from μ and $\Phi^{-1}(\nu')$ such that $(\mu, \Phi^{-1}(\nu'), \eta)$ is $\mathcal{W}_p$ -aligned. Since distances are preserved under isometry, we have that $(\Phi(\mu), \nu', \Phi(\eta))$ is also $\mathcal{W}_p$ -aligned. Since this is true for any $\nu' \neq \Phi(\mu)$, by Proposition 2.1, $\Phi(\mu)$ is a Dirac mass, that is, there exists $y \in X$ such that $\Phi(\delta_x) = \delta_y$.

To finish the proof, we define the map $\psi : X \rightarrow X$ by the relation $\delta_{\psi(x)} = \Phi(\delta_x)$. Since Φ is an isometry of $\mathcal{W}_p(X, d)$, it is easy to see that ψ is an isometry of (X, d) , and that $\Phi \circ \psi_{\#}^{-1}(\delta_x) = \delta_x$. $\square$

3 | UPGRADING RIGIDITY

As mentioned in the introduction, it is shown in [10] that, for $p \neq 2$, the Wasserstein space $\mathcal{W}_p(\mathbb{R}, |\cdot|)$ is rigid. As a consequence, if an isometry Φ of $\mathcal{W}_p(\mathbb{R}^n, d_N)$ globally preserves measures that are supported on a line, then it acts as a trivial isometry on those measures. One of the main points of our proof will be to show that such a line exists; this will be the result of the next section.

In this section, we show how we can utilize rigidity on a proper linear subspace to prove rigidity on the entire space. In particular, we show that, if an isometry Φ fixes both measures supported on such a subspace and measures supported on an appropriately chosen “complementary” linear subspace, then Φ is the identity on the whole Wasserstein space.

We recall that a linear subspace is a space $L \subset \mathbb{R}^n$ such that, if $v_1, v_2 \in L$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, then $\lambda_1 v_1 + \lambda_2 v_2 \in L$. If L is a linear subspace and $v_0 \in \mathbb{R}^n$ is a vector, then the space $v_0 + L$ is called an affine subspace.

Consider a proper linear subspace $L \subset \mathbb{R}^n$. We say that a norm N projects uniquely onto the linear subspace L if, for any $x \in \mathbb{R}^n$, there exists a unique $\hat{x} \in L$ such that $d_N(x, \hat{x}) \leq d_N(x, y)$ for all $y \in L$. Then we denote by $P_L(x) = \hat{x}$ the projection map of $x \in \mathbb{R}^n$ onto L . Since for any $v_0 \in \mathbb{R}^n$, we have $d_N(x, y - v_0) = d_N(x + v_0, y)$, it is clear that if N projects uniquely onto a linear subspace L , then it also projects uniquely onto the affine subspace $v_0 + L$, and we define $P_{v_0+L}(x)$ analogously to the linear subspace case.

In this section, we will assume that the norm N projects uniquely onto the proper subspace L . It is easy to see that, if the norm is strictly convex as in Theorems 1.1 and 1.2, this assumption always holds. Indeed, consider $x \in \mathbb{R}^n$ and L a proper subspace such that there exist $\hat{x}_1, \hat{x}_2 \in L$ minimizing the distance $d_N(x, y)$ over all $y \in L$. Without loss of generality, we can assume $d_N(x, \hat{x}_1) = d_N(x, \hat{x}_2) = 1$. If the norm N is strictly convex we have that $N(\frac{1}{2}(\hat{x}_1 + \hat{x}_2) - x) < 1$ if $\hat{x}_1 \neq \hat{x}_2$. But since L is a proper linear subspace, we have $\hat{x}_3 := \frac{1}{2}(\hat{x}_1 + \hat{x}_2) \in L$ and $d_N(\hat{x}_3, x) < 1$, contradicting the minimality assumption of $\hat{x}_1$ and $\hat{x}_2$. Thus, $\hat{x}_1 = \hat{x}_2$ and the projection is unique.

The main result of this section will be the following proposition.

Proposition 3.1. *For $p \geq 1$, assume that there exists a linear subspace $L \subset \mathbb{R}^n$ such that N projects uniquely onto L and let $\Phi : W_p(\mathbb{R}^n, d_N) \rightarrow W_p(\mathbb{R}^n, d_N)$ be an isometry such that $\Phi(\mu) = \mu$ for every measure μ supported on any translate of L . Assume also that for the linear subspace $H \subset P_L^{-1}(0)$ given by Lemma 3.8, N projects uniquely onto H and, for any $\nu \in \mathcal{W}_p(H, d_{N,H})$, we have $\Phi(\nu) = \nu$. Then $\Phi(\mu) = \mu$ for all $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$.*

Thus, it will be enough in the next section to prove that there exists a linear subspace L and an associated subspace H such that we have rigidity for measures supported on these subspaces; Proposition 3.1 then gives the rigidity of the whole Wasserstein space.

Notice that this result holds for all $1 \leq p < \infty$, including the special case $p = 2$.

We present a few properties of the projection operators. The following lemma is due to Fletcher and Moors [9]:

Lemma 3.2. *If L is a linear subspace of $(\mathbb{R}^n, d_N)$, then for any $x \in \mathbb{R}^n$, $k \in L$ and $\lambda \in \mathbb{R}$, we have*

$$P_L(\lambda x + k) = \lambda P_L(x) + k.$$

In particular, this lemma implies that the set $S = P_L^{-1}(0)$ of points that project onto 0 is homogeneous: if $x \in S$, then the entire line supporting the segment between 0 and x is contained in S . Also, for any $k \in L$, the preimage of k , $S_k = P_L^{-1}(k)$ is simply the preimage of 0 translated by the vector k , that is, $S_k = S + k$.

If we know the projection of a vector onto a subspace, we can say the following about its projection onto the translation of the subspace:

Lemma 3.3. *If L is a linear subspace and $v_0, v_1 \in \mathbb{R}^n$, $L' = v_0 + L$, we have*

$$P_{L'-v_1}(x) = P_{L'}(x + v_1) - v_1$$

for any $x \in \mathbb{R}^n$.

Proof. Set $\hat{x} = P_{L'}(x)$. Then, for any $y \in L'$, $d_N(x, y) = d_N(x - v_1, y - v_1)$, and the unique minimum of the right expression over all $y - v_1 \in L' - v_1$ is attained at $y = \hat{x}$. Thus we have $P_{L'-v_1}(x - v_1) = \hat{x} - v_1 = P_{L'}(x) - v_1$. $\square$

We can now rewrite Lemma 3.2 in the case of an affine subspace.

Lemma 3.4. *If L is a linear subspace of $(\mathbb{R}^n, d_N)$ and $v_0 \in \mathbb{R}^n$, then for any $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $k \in L$, if $P_{L+v_0}(x) = \hat{x}$, we have*

$$P_{L+v_0}(\hat{x} + \lambda(x - \hat{x}) + k) = \hat{x} + k.$$

Proof. Since $\hat{x} \in L + v_0$, we have that $v_0 + L - \hat{x} = L$, and using Lemmata 3.2 and 3.3 we get

$$\begin{aligned} P_{L+v_0}(\hat{x} + \lambda(x - \hat{x}) + k) &= P_{L+v_0-\hat{x}}(\lambda(x - \hat{x}) + k) + \hat{x} \\ &= \lambda P_L(x - \hat{x}) + \hat{x} + k = \lambda(P_{L+\hat{x}}(x) - \hat{x}) + \hat{x} + k = \hat{x} + k. \end{aligned}$$

□

Thus, like in the linear case, the preimage of a point $S_k = P_{L'}^{-1}(k)$ is a collection of lines passing through k , and is a translation of the preimage of v_0 , $P_{L'}^{-1}(v_0)$.

We write the set of measures supported on the affine subspace L' as $\mathcal{W}_p(L', d_{N,L'})$. The next lemma shows that the push-forward of the projection operator onto L' defines a projection operator from $\mathcal{W}_p(\mathbb{R}^n, d_N)$ to $\mathcal{W}_p(L', d_{N,L'})$.

Lemma 3.5. *Consider the affine subspace $L' \subset \mathbb{R}^n$ and let $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$. Then $\hat{\mu} = P_{L'\#}\mu$ is the unique measure in $\mathcal{W}_p(L', d_{N,L'})$ such that*

$$d_{W_p}(\mu, \hat{\mu}) \leq d_{W_p}(\mu, \nu)$$

for all $\nu \in \mathcal{W}_p(L', d_{N,L'})$.

Proof. We recall that a set $\Gamma \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is called c -cyclically monotone for a cost function $c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ if for any set of points $(x_i, y_i)_{i=1}^M \subset \Gamma$ with $M \geq 1$, we have

$$\sum_{i=1}^M c(x_i, y_{i+1}) \geq \sum_{i=1}^M c(x_i, y_i),$$

where we define $x_{M+1} = x_1$. We first look at the set

$$S = \{(x, P_{L'}(x)) : x \in \mathbb{R}^n\}$$

and show that it is c -cyclically monotone with respect to the cost $c(x, y) = d_N^p(x, y)$.

To do that, take a set of points $(x_i, P_{L'}(x_i))_{i=1}^M \subset S$. Then, by the property of the projection operator, we have that

$$d_N^p(x_i, P_{L'}(x_i)) \leq d_N^p(x_i, P_{L'}(x_{i+1}))$$

for any $1 \leq i \leq M$ (where we set $x_{M+1} = x_1$). By summing over i , we get that

$$\sum_{i=1}^M d_N^p(x_i, P_{L'}(x_i)) \leq \sum_{i=1}^M d_N^p(x_i, P_{L'}(x_{i+1})),$$

showing that S is c -cyclically monotone. Applying [8, Corollary 2.6.8] shows that $P_{L'}$ is an optimal transport map between μ and $P_{L'\#}\mu = \hat{\mu}$, and therefore we have that

$$d_{\mathcal{W}_p}^p(\mu, \hat{\mu}) = \int_{\mathbb{R}^n} d_N^p(x, P_{L'}(x)) d\mu(x).$$

Now, consider a measure $\nu \in \mathcal{W}_p(L', d_{N,L'})$ and π an optimal coupling between μ and ν . Since by the definition of the projection operator, we have that

$$d_N(x, P_{L'}(x)) \leq d_N(x, y)$$

for any $y \in L'$, we get that

$$d_{\mathcal{W}_p}^p(\mu, \nu) = \int_{\mathbb{R}^n \times \mathbb{R}^n} d_N^p(x, y) d\pi(x, y) \geq \int_{\mathbb{R}^n \times \mathbb{R}^n} d_N^p(x, P_{L'}(x)) d\pi(x, y) = d_{\mathcal{W}_p}^p(\mu, \hat{\mu}), \quad (3.1)$$

and we have the inequality from the lemma.

To show the uniqueness of $\hat{\mu}$, we notice that the inequality (3.1) becomes an equality only if $d_N^p(x, y) = d_N^p(x, P_{L'}(x))$ for π -almost every pair $(x, y) \in \mathbb{R}^n \times L'$. Since the projection onto L' is unique, this implies that $y = P_{L'}(x)$ for π -almost any pair $(x, y) \in \mathbb{R}^n \times L'$; thus $\pi = (\text{Id} \times P_{L'})_{\#}\mu$, and $\nu = P_{L'\#}\mu = \hat{\mu}$, showing the uniqueness of $\hat{\mu}$. $\square$

The next lemma shows that, if a Wasserstein isometry Φ leaves all measures supported on an affine subspace H' invariant, then the projection operator $P_{H'}$ and the isometry Φ commute.

Lemma 3.6. For $p \geq 1$, L a linear subspace, $v_0 \in \mathbb{R}^n$ and $L' = v_0 + L$, let

$$\Phi : \mathcal{W}_p(\mathbb{R}^n, d_N) \rightarrow \mathcal{W}_p(\mathbb{R}^n, d_N)$$

be an isometry such that $\Phi(\nu) = \nu$ for all $\nu \in \mathcal{W}_p(L', d_{N,L'})$. Then we have the relation

$$\Phi(P_{L'\#}\mu) = P_{L'\#}\Phi(\mu)$$

for all $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$.

Proof. Let $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$, and set $\hat{\mu} = P_{L'\#}\mu$. By the assumptions, we have

$$\Phi(\hat{\mu}) = \hat{\mu} \in \mathcal{W}_p(L', d_{N,L'}).$$

Then $d_{\mathcal{W}_p}(\mu, \hat{\mu}) = d_{\mathcal{W}_p}(\Phi(\mu), \Phi(\hat{\mu})) = d_{\mathcal{W}_p}(\Phi(\mu), \hat{\mu})$. Using Lemma 3.5, we have that for $\nu \in \mathcal{W}_p(L', d_{N,L'})$,

$$d_{\mathcal{W}_p}(\Phi(\mu), \nu) = d_{\mathcal{W}_p}(\mu, \Phi^{-1}(\nu)) = d_{\mathcal{W}_p}(\mu, \nu) \geq d_{\mathcal{W}_p}(\mu, \hat{\mu}) = d_{\mathcal{W}_p}(\Phi(\mu), \Phi(\hat{\mu})).$$

Thus $\Phi(\hat{\mu})$ minimizes the expression $d_{\mathcal{W}_p}(\Phi(\mu), \nu)$ among measures $\nu \in \mathcal{W}_p(L', d_{N,L'})$. By Lemma 3.5, it is the unique minimizer, and we have that $\Phi(\hat{\mu}) = P_{L'\#}\Phi(\mu)$, proving the lemma. $\square$

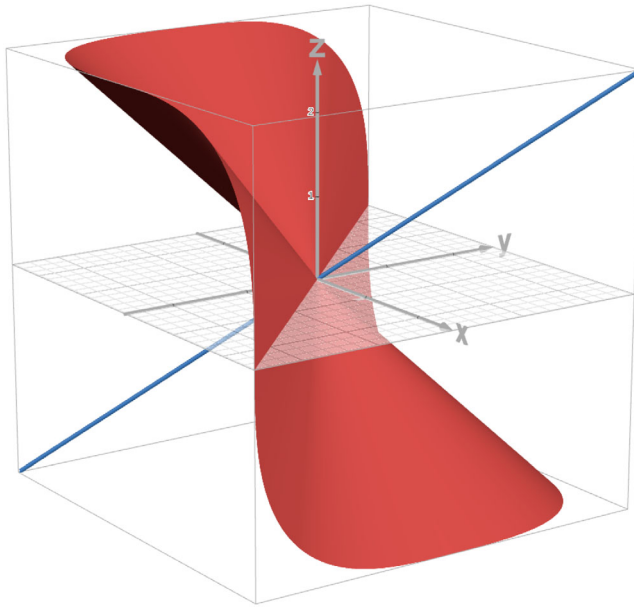


FIGURE 1 In blue the subspace L , in red the surface $S = P_L^{-1}(\{0\})$. Created using Desmos.

This lemma has a very useful consequence. Indeed, consider a linear subspace L such that $\Phi(\mu) = \mu$ for any measure μ supported on L . Then we have

$$P_{L\#}(\Phi(\mu)) = \Phi(P_{L\#}\mu) = P_{L\#}\mu.$$

Thus, since μ and $\Phi(\mu)$ have the same projection onto L , we have that $\Phi(\mu)$ is supported on the set $P_L^{-1}(\{0\}) + \text{supp}(\mu)$. In Proposition 3.1, the main result of this section, we want to use this property to conclude that, if the isometry Φ acts as the identity on measures supported on L and on $P_L^{-1}(\{0\})$, then Φ has to be the identity on the whole Wasserstein space over $\mathbb{R}^n$. This approach works very well if the set $P_L^{-1}(\{0\})$ is a linear subspace; this happens for example if the norm N is an l_q -norm, and we choose $L = \{te_i | t \in \mathbb{R}\}$, if e_i is a canonical base vector. It is then easy to see that $P_L^{-1}(\{0\})$ is the hyperplane H_i defined by $x_i = 0$. Unfortunately, as the following example shows, the preimage of the projection operator is not always a linear subspace.

Example 3.7. We consider the l_4 norm on $\mathbb{R}^3$ given by $N_4(x, y, z) = \sqrt[4]{x^4 + y^4 + z^4}$, and the linear subspace $L = \{(t, t, t) | t \in \mathbb{R}\}$. Then $S := P_L^{-1}(\{0\})$ is not a linear subspace (Figure 1).

Indeed, we have that

$$(x, y, z) \in S \Leftrightarrow x^4 + y^4 + z^4 \leq (x - t)^4 + (y - t)^4 + (z - t)^4 \text{ for all } t \in \mathbb{R}.$$

This is equivalent to saying that the global minimum of the convex function $f(t) := N^4(x - t, y - t, z - t)$ is attained at $t = 0$; which is exactly equivalent to the equality

$$x^3 + y^3 + z^3 = 0, \tag{3.2}$$

and S is the set of points in $\mathbb{R}^3$ satisfying (3.2).

This surface is obviously not a linear subspace; for example, while both $P_1 = (1, -1, 0)$ and $P_2 = (0, -1, 1)$ are in S , $P_1 + P_2$ is not in S .

If $P_L^{-1}(\{0\})$ is not a linear subspace, the idea of Proposition 3.1 does not work anymore, since we need the intersection between $P_L^{-1}(\{0\})$ and L to be a single point to show the finite support of $\Phi(\mu)$. Further, the proof of Theorem 1.1 (which will be presented at the end of Section 4) uses an induction argument on the dimension n . If $P_L^{-1}(\{0\})$ is not a linear subspace, we cannot use the induction assumption to assume that Φ acts as the identity on $P_L^{-1}(\{0\})$.

To counter this problem, we show in the next lemma that the geometry of the preimage of the projection operator restricts the possible support points of the image of a measure under an isometry. Indeed, we show that while $P_L^{-1}(\{0\})$ might not be a linear subspace, there exists a linear subspace $H \subset P_L^{-1}(\{0\})$ (which might be trivial, $H = \{0\}$) such that for any measure $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$ and any isometry Φ of the Wasserstein space fixing pointwise all measures supported on a translate of L , the image $\Phi(\mu)$ will be supported on the set $H + \text{supp}(\mu)$.

Lemma 3.8. *Consider a normed space $(\mathbb{R}^n, d_N)$ and a linear subspace L such that N projects uniquely onto L . Then there exists a linear subspace H such that $P_L(H) = \{0\}$ and that has the following additional property: if Φ is an isometry of the Wasserstein space such that for all $v_0 \in \mathbb{R}^n$, measures ν supported on $L + v_0$ are fixed, that is, $\Phi(\nu) = \nu$, then $\text{supp}(\Phi(\mu)) \subset H + \text{supp}(\mu)$ for all $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$.*

In later applications of this lemma, we will always work in cases where the set $H + \text{supp}(\mu)$ is closed and hence the closure may be omitted.

Proof. Before we start the general proof, we look at the two-dimensional case. Indeed, for $n = 2$, the possible linear subspaces are $\{0\}$, $\mathbb{R}^2$ and lines passing through 0. In the first two cases, we set $H = \mathbb{R}^2$ or $H = \{0\}$, and the lemma is trivially true. If the linear subspace L is a line, then by Lemma 3.2 the set $P_L^{-1}(0)$ is also a line passing through 0. Setting $H := P_L^{-1}(0)$, we immediately have that H is a linear subspace and that $P_L(H) = \{0\}$. The condition $\text{supp}(\Phi(\mu)) \subset H + \text{supp}(\mu)$ is a simple application of Lemma 3.6. Thus, we will now assume that $n \geq 3$. Furthermore, if $L = \{0\}$ or $L = \mathbb{R}^n$, then once again the lemma trivially holds, and we thus assume that $0 < \dim(L) < n$.

We also assume in this proof that any isometry Φ of the Wasserstein space $\mathcal{W}_p(\mathbb{R}^n, d_N)$ that we consider fixes measures supported on a translated subspace L , that is, Φ fulfills the assumption of the lemma.

We consider the set

$$\mathcal{G} = \left\{ \sum_{k=1}^M a_k \delta_{x_k} : M \geq 1, \sum_{k=1}^M a_k = 1, x_k \in \mathbb{R}^n \text{ for all } 1 \leq k \leq M \right. \\ \left. \text{and if } k \neq k', \text{ then } P_{L+x_i}(x_k) \neq P_{L+x_i}(x_{k'}) \text{ for all } i \leq M \right\}.$$

We first show the following claim.

Claim 1. The set $\mathcal{G}$ is dense in $\mathcal{W}_p(\mathbb{R}^n, d_N)$.

It is well known (see the proof of Proposition 7.1.5 in [2]) that the set of finitely supported measures is dense in $\mathcal{W}_p(\mathbb{R}^n, d_N)$. Thus we only need to show that $\mathcal{G}$ is dense with respect to finitely supported measures.

Consider a measure $\mu = \sum_{i=1}^M a_k \delta_{x_k}$ and set $\varepsilon > 0$. Let m be the minimum of all strictly positive distances between the projections of the support points of μ onto $L + x_i$,

$$m := \min_{i,k,k' \leq M} \{d_N(P_{L+x_i}(x_k), P_{L+x_i}(x'_k)) | d_N(P_{L+x_i}(x_k), P_{L+x_i}(x'_k)) > 0\}. \quad (3.3)$$

If the set over which we take the minimum is empty, we simply set $m = 1$. Since the set of points $P_{L+x_i}(x_k)$ is finite, we have that $m > 0$.

Let $v \in L$ be a unit vector. We now slightly perturb the measure μ to ensure that it is in $\mathcal{G}$. For this consider the factor $m' = \min\{m/2, \varepsilon/2\}$. Then construct the measure

$$\mu' = \sum_{k=1}^M a_k \delta_{x_k + 2^{-k} m' v}.$$

Now consider any $i, k \leq M$ and $k' \leq M, k \neq k'$. Then we have that

$$\begin{aligned} & d_N(P_{L+x_i+2^{-i} m' v}(x_k + 2^{-k} m' v), P_{L+x_i+2^{-i} m' v}(x'_k + 2^{-k'} m' v)) \\ &= d_N(P_{L+x_i}(x_k + 2^{-k} m' v), P_{L+x_i}(x'_k + 2^{-k'} m' v)) \\ &= d_N(P_{L+x_i}(x_k) + 2^{-k} m' v, P_{L+x_i}(x'_k) + 2^{-k'} m' v) \\ &= N(P_{L+x_i}(x_k) - P_{L+x_i}(x'_k) - (2^{-k'} - 2^{-k}) m' v) \\ &\geq |N(P_{L+x_i}(x_k) - P_{L+x_i}(x'_k)) - (2^{-k'} - 2^{-k}) m'| > 0 \end{aligned}$$

where we have used that $L + \lambda v = L$ and Lemma 3.4 to give the equality steps. The final strict inequality is obtained by considering the two different scenarios. If $P_{L+x_i}(x_k) = P_{L+x_i}(x'_k)$, then since $k \neq k'$, it is immediate to see that the inequality holds. When $P_{L+x_i}(x_k) \neq P_{L+x_i}(x'_k)$, then by the definition of m' we have that $d(P_{L+x_i}(x_k), P_{L+x_i}(x'_k)) > m'$, and the strict inequality still holds. In particular, we thus have that $\mu' \in \mathcal{G}$.

It now remains to show that μ' is an approximation of μ . To see this, consider the transport plan π between μ and μ' which sends every support point to its translated support point, that is

$$\pi = \sum_{k=1}^M a_k \delta_{x_k, x_k + 2^{-k} m' v}.$$

Then we have that

$$d_{\mathcal{W}_p}^p(\mu, \mu') \leq \sum_{k=1}^M a_k d^p(x_k, x_k + 2^{-k} m' v) = \sum_{k=1}^M a_k (2^{-k} m')^p \leq (m')^p \sum_{k=1}^M 2^{-k} \leq (m')^p.$$

Since we have $m' \leq \varepsilon$, this shows that $d_{\mathcal{W}_p}(\mu, \mu') \leq \varepsilon$, and thus μ' approximates μ , proving the claim.

Now take $\mu = \sum_{k=1}^M a_k \delta_{x_k} \in \mathcal{G}$. By Lemma 3.6, for any $v \in \mathbb{R}^n$,

$$P_{L+v\#}(\mu) = \Phi(P_{L+v\#}(\mu)) = P_{L+v\#}(\Phi(\mu)).$$

We can then write

$$P_{L+v\#}(\Phi(\mu)) = P_{L+v\#}(\mu) = \sum_{k=1}^M a_k \delta_{P_{L+v}(x_k)}. \quad (3.4)$$

Thus, for every $v \in \mathbb{R}^n$ and every point $y \in \text{supp}(\Phi(\mu))$, there exists $k \leq M$ such that $P_{L+v}(y) = P_{L+v}(x_k)$.

Before we continue, we give a brief overview of the steps of the proof. As a first step, we will, for every $i \leq M$, look at the space $S_i = P_{L+x_i}^{-1}(x_i)$, that is, the set of points such that their projection onto the affine subspace $L + x_i$ is exactly x_i . Then, for any point $y \in \text{supp}(\Phi(\mu))$ such that $y \in S_i$, we show that locally, S_i looks the same around x_i as it does around y . This means that, for any $v \in \mathbb{R}^n$ small enough, $x_i + v \in S_i$ if and only if $y + v \in S_i$. This is exactly the statement of Equation (3.5).

Since the set S_i is homogeneous with center x_i , we can extend this local behavior into a global statement. Thus, as our second step, we show Equation (3.7), which tells us that (unless $y = x_i$) the projection set S_i does not only contain lines but also planes spanned by $y - x_i$ and by any $v \in S_i$. This result in particular restricts the possible support points of $\Phi(\mu)$, since support points can only exist if the set S_i (which, up to translation, is equal to the set $P_L^{-1}(0)$, and thus only depends on the metric and on the linear subspace L) contains these planes; in particular, if S_i does not contain any planes, then necessarily $y = x_i$.

In the third step, we show that any $y \in \text{supp}(\Phi(\mu))$ can only be associated with one support point of μ , that is, there exists for any y a unique $i \leq M$ such that $P_{L+v}(y) = P_{L+v}(x_i)$, which is independent of v .

To finish the proof we construct H as the span of all possible vectors $y - x_i$ such that for some measure $\mu \in \mathcal{G}$ and some isometry Φ , $x_i \in \text{supp}(\mu)$ and $y \in \text{supp}(\Phi(\mu))$, with $y \in S_i$. By the properties found in the first three steps, the linear subspace H will have the required properties of the lemma. In particular, if $P_L^{-1}(0)$ does not contain any planes, then for all $y \in \text{supp}(\Phi(\mu))$ there exists $x_i \in \text{supp}(\mu)$ such that $y = x_i$, and thus $H = \{0\}$ is trivial.

We now continue the proof with the following claim.

Claim 2. For $i \leq M$ fixed, take the affine subspace $L' = L + x_i$. Then for any $y \in \text{supp}(\Phi(\mu))$ such that $P_{L'}(y) = x_i$ and $v \in \mathbb{R}^n$ is small enough,

$$P_{L'}(y + v) = P_{L'}(y) \Leftrightarrow P_{L'}(x_i + v) = P_{L'}(x_i) = x_i. \quad (3.5)$$

Note that we trivially have $P_{L'}(x_i) = x_i$ and by the assumptions on $\mathcal{G}$ that $P_{L'}(x_j) \neq x_i$ for all $j \leq M$, $j \neq i$.

For $y = x_i$, this result is trivial. If $y \neq x_i$, this result follows by the continuity of the projection operator (Corollary 2.20 of [9]), Equation (3.4) and the previous lemmata.

Indeed, for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that, if $N(v) < \delta_0$, then

$$N(P_{L'}(y + v) - P_{L'}(y)) \leq \varepsilon/4.$$

By Lemma 3.3, we have for every v such that $N(v) < \delta := \min(\delta_0, \varepsilon/4)$ that

$$N(P_{L'-v}(y) - P_{L'}(y)) \leq \varepsilon/2.$$

Choosing ε such that $d_N(P_{L'}(x_i), P_{L'}(x_j)) > \varepsilon$ for any $j \neq i$, we can use Lemma 3.3 to show

$$\begin{aligned} N[P_{L'-v}(y) - P_{L'-v}(x_j)] &\geq N[P_{L'-v}(x_i) - P_{L'-v}(x_j)] - N[P_{L'-v}(y) - P_{L'-v}(x_i)] \\ &\geq N[P_{L'}(x_i) - P_{L'}(x_j)] - N[P_{L'}(x_i + v) - P_{L'}(x_i)] - N[P_{L'}(x_j + v) - P_{L'}(x_j)] \\ &\quad - N[P_{L'}(y + v) - P_{L'}(y)] - N[P_{L'}(x_i + v) - P_{L'}(x_i)] \\ &> \varepsilon - \varepsilon/4 - \varepsilon/4 - \varepsilon/4 - \varepsilon/4 = 0. \end{aligned}$$

Thus $P_{L'-v}(y) \neq P_{L'-v}(x_j)$ when $j \neq i$. But Equation (3.4) tells us that $P_{L'-v}(y) = P_{L'-v}(x_j)$ for some $j \leq M$. Thus we have that, for any v small enough, if $y \in \text{supp}(\Phi(\mu))$ and if $P_{L'}(y) = P_{L'}(x_i)$, then $P_{L'-v}(y) = P_{L'-v}(x_i)$. As a consequence of this, assume that v is small enough and that $P_{L'}(x_i + v) = P_{L'}(x_i)$. Then we have

$$P_{L'}(y + v) = P_{L'-v}(y) + v = P_{L'-v}(x_i) + v = P_{L'}(x_i) = P_{L'}(y).$$

The reverse is also true. Indeed, if $P_{L'}(y + v) = P_{L'}(y)$, then

$$P_{L'}(x_i + v) = P_{L'-v}(x_i) - v = P_{L'-v}(y) - v = P_{L'}(y) = P_{L'}(x_i),$$

showing Equation (3.5).

Claim 3. For $y \in \text{supp}(\Phi(\mu))$ such that $P_{L'}(y) = x_i$ and for any $v \in \mathbb{R}^n \setminus \{0\}$ such that $P_{L'}(x_i + v) = x_i$, we have that $P_{L'}(x_i + z) = x_i$ holds for any z in the set generated by $(y - x_i)$ and v .

Note that the set generated by $(y - x_i)$ and v is a plane, unless $y = x_i$ or v is colinear to $x_i - y$.

We first notice that, if $P_{L'}(x_i + v) = x_i$, then by Lemma 3.4 we get $P_{L'}(x_i + \lambda'v) = x_i$ for any $\lambda' \in \mathbb{R}$; thus we can assume w.l.o.g. that v is small enough. Then we have by Equation (3.5) that $P_{L'}(y + v) = x_i$. Again by Lemma 3.4, $P_{L'}(y + \lambda v) = x_i$ still holds for any $|\lambda| \leq 1$. A final application of Lemma 3.4 then gives that

$$P_{L'}(x_i + \alpha(y - x_i + \lambda v)) = x_i \tag{3.6}$$

for any $\alpha \in \mathbb{R}$.

Thus, if $z = \lambda_1(y - x_i) + \lambda_2 v \neq 0$ and $\delta_1 = \min\{\frac{\delta}{2N(z)}, \frac{1}{2|\lambda_1| + 2|\lambda_2|}\}$, set $z' = \delta_1 z$. Notice that by Lemma 3.4 and Equation (3.5),

$$P_{L'}(x_i + z) = x_i \Leftrightarrow P_{L'}(x_i + z') = x_i \Leftrightarrow P_{L'}(y + z') = x_i.$$

Since

$$y + z' = y + \delta_1(\lambda_1(y - x_i) + \lambda_2 v) = x_i + (1 + \delta_1 \lambda_1)(y - x_i) + \delta_1 \lambda_2 v,$$

we can write $y + z' = x_i + \alpha(y - x_i) + \lambda\alpha v$ (where the definition of δ_1 guarantees that $|\lambda| \leq 1$). Using Equation (3.6), this shows that $P_{L'}(y + z') = x_i$ and

$$P_{L'}(x_i + z) = x_i, \quad (3.7)$$

proving the claim.

From this, we will be able to easily deduce the following claim.

Claim 4. If for $y \in \text{supp}(\Phi(\mu))$ we have $P_{L+x_i}(y) = x_i$ and $P_{L+x_j}(y) = x_j$, then $i = j$. Thus we can write the support of $\Phi(\mu)$ as a disjoint union

$$\text{supp}(\Phi(\mu)) = \sqcup_{i=1}^M Y_{x_i},$$

with $y \in Y_{x_i}$ if $P_{L+x_i}(y) = x_i$ for some (and hence for any) $i \leq M$.

Indeed, if $P_{L+x_i}(y) = x_i$, since for $x_i^j := P_{L+x_j}(x_i)$, $P_{L+x_i}(x_i^j) = x_i$, using Claim 3,

$$P_{L+x_i}(x_i + y - x_i^j) = P_{L+x_i}(x_i + (y - x_i) - (x_i^j - x_i)) = x_i.$$

But then

$$P_{L+x_j}(y) = P_{L+x_j-(x_i^j-x_i)}(y - (x_i^j - x_i)) + (x_i^j - x_i) = P_{L+x_i}(x_i + y - x_i^j) + (x_i^j - x_i) = x_i^j,$$

where we used that $x_j - x_i^j \in L$. This shows that, for all $j \leq M$, if $P_{L+x_i}(y) = x_i$, then $P_{L+x_j}(y) = P_{L+x_j}(x_i) \neq x_j$ unless $i = j$. Thus, if we set $Y_{x_i} = P_{L+x_i}^{-1}(x_i) \cap \text{supp}(\Phi(\mu))$, then for any $i \neq j$, the intersection $Y_{x_i} \cap Y_{x_j}$ will be empty. From Equation (3.4) we also have that $\Phi(\mu)(Y_{x_i}) = \mu(x_i)$. Since

$$1 = \sum_{i=1}^M \mu(x_i) = \sum_{i=1}^M \Phi(\mu)(Y_{x_i}),$$

we have that $\sqcup_{i=1}^M Y_{x_i} = \text{supp}(\Phi(\mu))$. Thus, for all $y \in \text{supp}(\Phi(\mu))$, there exists $i \leq M$ with $P_{L+x_i}(y) = x_i$.

We are now ready to finish the argument. Since $P_{L+x_i}(z) = x_i \Leftrightarrow P_L(z - x_i) = 0$, for any isometry of the Wasserstein space Φ and any measure $\mu \in \mathcal{G}$, if $y \in \text{supp}(\Phi(\mu))$, then there exists $x_i \in \text{supp}(\mu)$ such that $y \in Y_{x_i}$ and

$$P_L(y - x_i) = 0. \quad (3.8)$$

Then, for any $v \in \mathbb{R}^n$ such that $P_L(v) = 0$, we have by Equation (3.7) that for any $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$P_L(\lambda_1(y - x_i) + \lambda_2 v) = 0. \quad (3.9)$$

We consider the set

$$V = \{v \in \mathbb{R}^n \mid \exists \Phi \in \text{Isom}(\mathcal{W}_p(\mathbb{R}^n, d_N)) \text{ which preserves measures supported on } L + v_0 \\ \forall v_0 \in \mathbb{R}^n, \exists \mu \in \mathcal{G}, \exists y \in \text{supp}(\Phi(\mu)), \exists x_i \in \text{supp}(\mu) \text{ such that } y \in Y_{x_i}, v = y - x_i\}.$$

In words, V is the set of vectors v such that there exists an isometry Φ of the Wasserstein space (which fixes measures supported on translated subspaces of L) and a measure $\mu \in \mathcal{G}$ such that, for some y in the support of $\Phi(\mu)$, if $P_{L+x_i}(y) = x_i$ for some $x_i \in \text{supp}(\mu)$, then $v = y - x_i$.

We now consider the set $H = \text{span}(V)$. It is clear that H is a linear subspace. To see that $P_L(H) = 0$, we first notice that if $v \in V$, there exists an isometry Φ , $\mu \in \mathcal{G}$, $x_i \in \text{supp}(\mu)$ and $y \in \text{supp}(\Phi(\mu))$ such that $y \in Y_{x_i}$, $v = y - x_i$ and (3.8) implies that $P_L(v) = 0$. Then, for $v_1, v_2 \in V$, setting $z = \lambda_1 v_1 + \lambda_2 v_2$, Equation (3.9) implies that $P_L(z) = 0$. Given a basis of H composed of $v_{y_1}, \dots, v_{y_m}$, an induction argument shows that for all $z \in H$ we have $P_L(z) = 0$.

Using the same method, we have that, for any $h \in H$,

$$P_L(h + x_i) = P_L(h + (x_i - P_L(x_i)) + P_L(x_i)) = P_L(h + (x_i - P_L(x_i))) + P_L(x_i) = P_L(x_i) \quad (3.10)$$

where we used Lemma 3.2 and that $P_L(x_i - P_L(x_i)) = P_L(x_i) - P_L(x_i) = 0$.

Thus we have a linear subspace H such that $P_L(H) = \{0\}$ and, for any isometry Φ and $\mu \in \mathcal{G}$, if $y \in \text{supp}(\Phi(\mu))$, then there exists $x_i \in \text{supp}(\mu)$ such that $y - x_i \in H$.

To finish the proof, we show that a density argument proves the lemma for general μ .

Claim 5. If $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$ is supported on $\text{supp } \mu$, then $\Phi(\mu)$ is supported on $\overline{H + \text{supp } \mu}$.

We have already proved that if $v \in \mathcal{G}$, then

$$\text{supp}(\Phi(v)) \subset H + \text{supp}(v). \quad (3.11)$$

The goal is to extend this conclusion by density and prove that

$$\text{supp}(\Phi(\mu)) \subset \overline{H + \text{supp}(\mu)} \quad (3.12)$$

for a general $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$, where the closure is taken in the topology induced by d_N .

Let $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$. As $\mathcal{G}$ is dense in $\mathcal{W}_p(\mathbb{R}^n, d_N)$, we can choose a sequence $(\mu_j)_{j \in \mathbb{N}} \subset \mathcal{G}$ such that

$$d_{W_p}(\mu_j, \mu) \longrightarrow 0.$$

Moreover, the proof of the density of $\mathcal{G}$ in $\mathcal{W}_p(\mathbb{R}^n, d_N)$ shows that we can choose the approximation so that

$$\text{supp } \mu_j \subset (\text{supp } \mu)_{\varepsilon_j} \quad (3.13)$$

for a sequence $\varepsilon_j \rightarrow 0$ of positive numbers, where

$$(\text{supp } \mu)_{\varepsilon_j} := \{x \in \mathbb{R}^n : d_N(x, \text{supp } \mu) < \varepsilon_j\}.$$

Since Φ is an isometry, we have

$$d_{W_p}(\Phi(\mu_j), \Phi(\mu)) = d_{W_p}(\mu_j, \mu) \longrightarrow 0.$$

In particular,

$$\Phi(\mu_j) \rightharpoonup \Phi(\mu)$$

weakly. By (3.11) we have

$$\text{supp } \Phi(\mu_j) \subset H + \text{supp } \mu_j, \quad (3.14)$$

and (3.13) and (3.14) imply that

$$\text{supp } \Phi(\mu_j) \subset H + (\text{supp } \mu)_{\varepsilon_j} \subset (H + \text{supp } \mu)_{\varepsilon_j}. \quad (3.15)$$

To show (3.12), let

$$y \notin \overline{H + \text{supp } \mu}.$$

Then there exists $r > 0$ such that

$$B(y, r) \cap (H + \text{supp } \mu) = \emptyset$$

where $B(y, r) = \{x \in \mathbb{R}^n : d_N(x, y) < r\}$. Consequently, for all j such that $\varepsilon_j < r/2$ we have

$$B(y, r/2) \cap (H + \text{supp } \mu)_{\varepsilon_j} = \emptyset.$$

Hence, for all sufficiently large j 's, by (3.15) one gets

$$\Phi(\mu_j)(B(y, r/2)) = 0.$$

Choose a continuous and bounded nonnegative function $f : \mathbb{R}^n \rightarrow [0, 1]$ such that

$$\text{supp}(f) \subset B(y, r/2) \quad \text{and} \quad f \equiv 1 \text{ on } B(y, r/4).$$

Then, for all sufficiently large j ,

$$\int_{\mathbb{R}^n} f \, d\Phi(\mu_j) = 0.$$

Passing to the weak limit gives

$$\int_{\mathbb{R}^n} f \, d\Phi(\mu) = 0.$$

Since $f \equiv 1$ on $B(y, r/4)$, and $f \geq 0$, it follows that

$$\Phi(\mu)(B(y, r/4)) = 0.$$

Thus $y \notin \text{supp } \Phi(\mu)$, as desired. □

To see an application of this lemma, we can look again at Example 3.7 and try to explicitly determine H . Consider an isometry Φ , a measure $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_4)$, $x_i \in \text{supp}(\mu)$ and a point $y \in \text{supp}(\Phi(\mu))$ such that $P_L(y - x_i) = 0$. Then, for any v such that $P_L(v) = 0$, if we consider the linear subspace $K = \text{span}((y - x_i), v)$, by Equation (3.9) we have $P_L(K) = 0$, and the preimage $P_L^{-1}(\{0\})$ must contain the linear subspace K . If $x_i \neq y$, we can choose v such that $x_i - y$ and v are not collinear, and K is a plane. Since in the case presented in Example 3.7 the preimage $P_L^{-1}(\{0\})$ does not contain any plane, this means that $y = x_i$; in other words, we have $V = \{0\} = H$, and Lemma 3.8 implies that if an isometry Φ of the Wasserstein space acts as the identity on measures supported on $L + v_0$, then for any measure μ , $\text{supp}(\Phi(\mu)) \subseteq \text{supp}(\mu)$ holds, that is, the support of $\Phi(\mu)$ is contained in the support of μ (and it is easy to see that actually μ and $\Phi(\mu)$ have the same support).

We are now ready to prove Proposition 3.1, in which we show that rigidity on certain subspaces of $\mathbb{R}^n$ is enough to prove rigidity on the whole space. This Proposition generalizes Proposition 2.1 from [3], which showed the result in two dimensions for the special case of the maximum norm. The main difference in our proof comes from the much more general nature of the preimage of a projection operator, which in 2 dimensions is simply a line. We recommend that readers who want to understand this proof in detail read first the Proposition 2.1 from [3], which presents the ideas of the proof in a simpler setting.

Proof of Proposition 3.1. We start with the case $H = \{0\}$. We build the subset $\mathcal{F}_0 \subset \mathcal{W}_p(\mathbb{R}^n, d_N)$ as follows:

$$\mathcal{F}_0 = \left\{ \sum_{k=1}^M a_k \delta_{x_k} : M \geq 1, \sum_{k=1}^M a_k = 1, x_k \in \mathbb{R}^n \text{ for all } 1 \leq k \leq M \right. \\ \left. \text{and if } k \neq k', \text{ then } a_k \neq a_{k'} \text{ and } P_L(x_k) \neq P_L(x_{k'}) \right\}.$$

It is well known (see, e.g., [17, Theorem 6.18]) that finite convex combinations of Dirac masses form a dense subset of $\mathcal{W}_p(\mathbb{R}^n, d_N)$.

Thus any measure of the Wasserstein space can be approximated by a finitely supported measure μ . Then, a simple perturbation can ensure all weights a_k are different. Indeed, for a very small vector v , the measure

$$\mu' = \sum_{k=1}^M (a_k - \alpha_k) \delta_{x_k} + \alpha \delta_{x_k+v}$$

can be chosen to be arbitrarily close to the original measure μ by choosing α_k small enough. Since M is finite, we can choose the variables α_k small so that all the weights of μ' are different. Finally, a second perturbation argument similar to the one of Claim 1 shows that μ' can be approximated by measures with all different weights and distinct projections of the supporting points; in particular the set $\mathcal{F}_0$ is dense.

Consider $\mu \in \mathcal{F}_0$, $\mu = \sum_{k=1}^M a_k \delta_{x_k}$. Since we consider the case $H = \{0\}$, Lemma 3.8 says that $\text{supp}(\mu) = \text{supp}(\Phi(\mu))$, and we can write $\Phi(\mu) = \sum_{k=1}^M b_k \delta_{x_k}$.

We now use Lemma 3.6 to show that

$$\sum_{k=1}^M b_k \delta_{P_L(x_k)} = P_{L\#}(\Phi(\mu)) = P_{L\#}(\mu) = \sum_{k=1}^M a_k \delta_{P_L(x_k)}.$$

Since, by the condition on $\mathcal{F}_0$, $\delta_{P_L(x_k)} \neq \delta_{P_L(x_{k'})}$ when $k \neq k'$, this implies that $b_k = a_k$ for any $1 \leq k \leq M$, and thus $\Phi(\mu) = \mu$. Since $\mathcal{F}_0$ is dense in $\mathcal{W}_p(\mathbb{R}^n, d_N)$, this shows the proposition when $H = \{0\}$.

When H is nontrivial, we want to use the same idea. However, we need to show that even in this case we have $\text{supp}(\mu) = \text{supp}(\Phi(\mu))$; this will be the main part of the rest of the proof.

We start by building the subset $\mathcal{F} \subset \mathcal{W}_p(\mathbb{R}^n, d_N)$ as follows:

$$\mathcal{F} = \left\{ \sum_{k=1}^M a_k \delta_{x_k} : M \geq 1, \sum_{k=1}^M a_k = 1, x_k \in \mathbb{R}^n \text{ for all } 1 \leq k \leq M \right. \\ \left. \text{and if } k \neq k', \text{ then } a_k \neq a_{k'} \text{ and } P_L(x_k) \neq P_L(x_{k'}), P_H(x_k) \neq P_H(x_{k'}) \right\}.$$

We will show that $\mathcal{F}$ has three properties: it is a dense set with respect to $\mathcal{W}_p(\mathbb{R}^n, d_N)$, the projection maps $P_{\#L}$ and $P_{\#H}$ restricted to $\mathcal{F}$ are injective, and $\mathcal{F}$ is globally invariant under the map Φ .

Claim 6. The set $\mathcal{F}$ is dense in $\mathcal{W}_p(\mathbb{R}^n, d_N)$.

The claim follows by a similar argument as for $\mathcal{F}_0$. Indeed, after a first perturbation to ensure that all weights are different and a second perturbation to ensure that all projections onto L are disjoint, we do a third perturbation similar to the one of Claim 1 by defining the third perturbation parameter m'_3 to be much smaller than the second perturbation parameter m'_2 ; then it is easy to see that after the third perturbation the measure is in $\mathcal{F}$. Since any perturbation can be made arbitrarily small, we thus have the claim.

Claim 7. The maps $P_{L\#}$ and $P_{H\#}$ from $\mathcal{F}$ to the set of measures supported on L and H , respectively, are injective.

We show the injectivity for the map $P_{L\#}$, the second case is similar. Assume that $\mu_1, \mu_2 \in \mathcal{F}$ with $P_{L\#}\mu_1 = P_{L\#}\mu_2$. We write

$$\mu_1 = \sum_{k=1}^{M_1} a_k^1 \delta_{x_k^1} \text{ and } \mu_2 = \sum_{k=1}^{M_2} a_k^2 \delta_{x_k^2}.$$

Using that $P_{L\#}\mu_1 = P_{L\#}\mu_2$, we get

$$\sum_{k=1}^{M_1} a_k^1 \delta_{P_L(x_k^1)} = \sum_{k=1}^{M_2} a_k^2 \delta_{P_L(x_k^2)}.$$

Due to the conditions of the set $\mathcal{F}$, we have that $M_1 = M_2$, $a_k^1 = a_k^2$ and $x_k^1 = x_k^2$ for $1 \leq k \leq M_1$. Therefore, $\mu_1 = \mu_2$, and $\mu \rightarrow P_{L\#}\mu$ (and by the same reasoning $\mu \rightarrow P_{H\#}\mu$) is injective on the set $\mathcal{F}$.

Claim 8. The set $\mathcal{F}$ is globally invariant under the isometry Φ , that is $\Phi(\mathcal{F}) \subset \mathcal{F}$.

This result will be the main part of the proof. To prove this, we start by showing the following claim.

Claim 9. Consider $\mu \in \mathcal{F}$. Then $\Phi(\mu)$ is finitely supported, and can be written as in Equation (3.18).

Take $\mu = \sum_{k=1}^M a_k \delta_{x_k} \in \mathcal{F}$. By Lemma 3.6,

$$P_{L\#}(\mu) = \Phi(P_{L\#}(\mu)) = P_{L\#}(\Phi(\mu)) \text{ and } P_{H\#}(\mu) = \Phi(P_{H\#}(\mu)) = P_{H\#}(\Phi(\mu)).$$

We can then write

$$P_{L\#}(\Phi(\mu)) = P_{L\#}(\mu) = \sum_{k=1}^M a_k \delta_{P_L(x_k)} \quad (3.16)$$

and similarly

$$P_{H\#}(\Phi(\mu)) = P_{H\#}(\mu) = \sum_{k=1}^M a_k \delta_{P_H(x_k)}. \quad (3.17)$$

By the Lemma 3.8 and Equation (3.17), we have that $\Phi(\mu)$ is supported on the set

$$S = \left(\bigcup_{i=1}^M (H + x_i) \right) \cap \left(\bigcup_{i=1}^M P_H^{-1}(P_H(x_i)) \right).$$

Since H is a linear subspace, given a pair (k, k') , the intersection $(H + x_k) \cap P_H^{-1}(P_H(x_{k'}))$ is a unique point, which we denote by $x_{k,k'}$. Indeed, if $z \in H + x_k$, then we can write $z = x_k + h$ with $h \in H$. On the other hand, if $z \in P_H^{-1}(P_H(x_{k'}))$, we have by Lemma 3.2 that

$$P_H(z) = P_H(x_k + h) = P_H(x_k) + h = P_H(x'_{k'}).$$

Thus $x_{k,k'}$ is uniquely given by $x_{k,k'} = x_k + P_H(x'_{k'}) - P_H(x_k)$.

By the definition of the set S and Equation (3.10) we have that, for any pair (k, k') , $P_L(x_{k,k'}) = P_L(x_k)$ and $P_H(x_{k,k'}) = P_H(x_{k'})$.

Thus $\Phi(\mu)$ is supported on the points of the form $x_{k,k'}$, and we can write

$$\Phi(\mu) = \sum_{k=1}^M \sum_{k'=1}^M b_{k,k'} \delta_{x_{k,k'}}. \quad (3.18)$$

Figure 2 gives an example of a measure μ and an (a priori) possible image measure.[†]

We now give a brief sketch of the rest of the proof. We will assume by contradiction that, as in Figure 2, there exist two distinct points z and z' in the support of $\Phi(\mu)$ that project onto $P_L(x_1)$; if

[†] We show in this proof that the right measure cannot actually be the image of the left measure under an isometry.

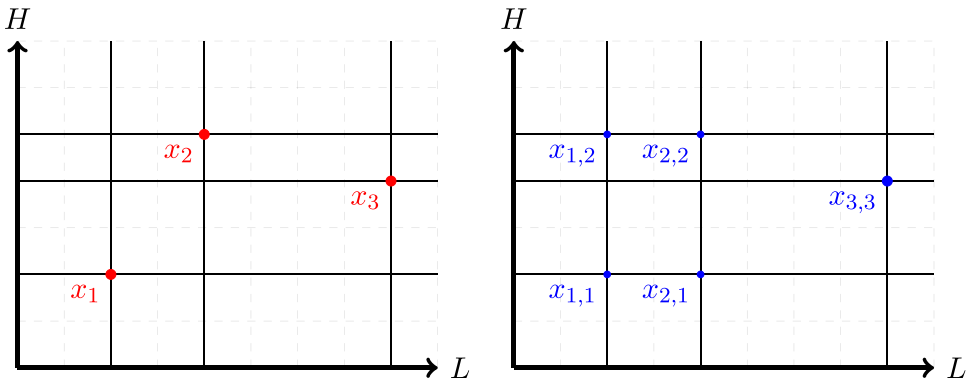


FIGURE 2 An example for a measure $\mu \in \mathcal{F}$ (left) and a possible image measure $\Phi(\mu)$ (right).

there are no such two points, a short argument then shows that $\Phi(\mu) = \mu \in \mathcal{F}$. We then slightly perturb the measure μ to create μ' , by moving a small weight from x_1 to some point x_0 close to x_1 . Similarly, we perturb $\nu = \Phi(\mu)$ twice, by shifting a small weight from either z or z' to a point close to either z or z' , creating two new and distinct measures ν'_1 and ν'_2 . These new measures are represented in Figure 3.

The constructed measure μ' has an important property. Indeed, if we call the *fingerprint* $\mathcal{R}$ of a measure ξ its projection onto L and H , $\mathcal{R}(\xi) = (P_{L\#}\xi, P_{H\#}\xi)$, then there exists no other measure ξ that a) has the same fingerprint as μ' and b) lies at the same distance from μ as μ' . In other words, if

$$\mathcal{R}(\xi) = \mathcal{R}(\mu') \text{ and } d_{\mathcal{W}_p}(\mu, \xi) = d_{\mathcal{W}_p}(\mu, \mu'),$$

then we have that $\xi = \mu'$. To get the desired contradiction, we will remark that both ν'_1 and ν'_2 have by their construction the same fingerprint as μ' , and are both at the same distance from ν as μ' is from μ ,

$$d_{\mathcal{W}_p}(\nu, \nu'_1) = d_{\mathcal{W}_p}(\nu, \nu'_2) = d_{\mathcal{W}_p}(\mu, \mu').$$

Also, by Lemma 3.6, the fingerprint of a measure is preserved under isometries, that is, $\mathcal{R}(\xi) = \mathcal{R}(\Phi(\xi))$. Thus, the images of ν'_1 and ν'_2 under the inverse isometry Φ^{-1} have the same fingerprint as μ' , and (since Φ^{-1} is also an isometry) lie at the same distance from $\Phi^{-1}(\nu) = \mu$ as μ' , or

$$\begin{aligned} \mathcal{R}(\Phi^{-1}(\nu'_1)) &= \mathcal{R}(\Phi^{-1}(\nu'_2)) = \mathcal{R}(\Phi^{-1}(\mu')), \\ d_{\mathcal{W}_p}(\mu, \Phi^{-1}(\nu'_1)) &= d_{\mathcal{W}_p}(\mu, \Phi^{-1}(\nu'_2)) = d_{\mathcal{W}_p}(\mu, \mu'). \end{aligned}$$

But then we necessarily have $\Phi^{-1}(\nu'_1) = \Phi^{-1}(\nu'_2) = \mu'$, and $\nu'_1 = \nu'_2$, giving the desired contradiction. Having given this brief sketch, we continue with the proof.

Assume for contradiction that $\Phi(\mu) \notin \mathcal{F}$. Then there exists $k_0 \leq M$ and $k_1 \neq k_2 \leq M$ such that both b_{k_0, k_1} and b_{k_0, k_2} are nonzero. Indeed, assume that for any $k \leq M$, there exists a unique $k' \leq M$

such that $b_{k,k'} \neq 0$. Since from Equation (3.16) we have that

$$\sum_{k'=1}^M b_{k,k'} = a_k,$$

this shows that $b_{k,k'} = a_k$, and $\Phi(\mu)$ is supported on M points. Again from (3.16) we can then deduce that actually $\Phi(\mu) = \mu \in \mathcal{F}$. Thus we have that there exists $k_0 \leq M$ such that, for some $k_1 \neq k_2 \leq M$, we have that b_{k_0,k_1} and b_{k_0,k_2} are nonzero. Without loss of generality, we can assume that $k_0 = 1$.

We set h to be the shortest distance between any two points $x_{k,k'}$, that is,

$$h = \min_{(k,k') \neq (\tilde{k},\tilde{k}')} d_N(x_{k,k'}, x_{\tilde{k},\tilde{k}'}).$$

Since the set $\{x_{k,k'} | k, k' \leq M\}$ is discrete and finite, we have that $h > 0$.

We build the point $x_0 = x_1 + \frac{h_0}{d_N(x_1, P_H(x_1))}(x_1 - P_H(x_1))$ where $h_0 \in \mathbb{R}$ is such that $0 < h_0 < \frac{h}{2}$. We then consider the points $x_{0,k}$ defined such that $P_H(x_{0,k}) = P_H(x_0)$ and $P_L(x_{0,k}) = P_L(x_k)$. Explicitly, these points are given by $x_{0,k'} = x_{k'} + P_H(x_0) - P_H(x_{k'})$. In particular, $x_{0,1} = x_0$. Also, we notice that $d_N(x_{0,k'}, x_{1,k'}) = h_0$; this follows easily from Lemma 3.2.

Consider the weight $a_0 = \frac{1}{2} \min(b_{1,k_1}, b_{1,k_2})$. By our assumption $b_{1,k_1}, b_{1,k_2} > 0$, and we thus have that $a_1 > a_0 > 0$. We build the following measures:

$$\begin{aligned} \mu' &= a_0 \delta_{x_0} + (a_1 - a_0) \delta_{x_1} + \sum_{k=2}^M a_k \delta_{x_k}, \\ \nu'_1 &= a_0 \delta_{x_{0,k_1}} + (b_{1,k_1} - a_0) \delta_{x_{1,k_1}} + \sum_{k=1, k \neq k_1} b_{1,k} \delta_{x_{1,k}} + \sum_{k=2}^M \sum_{k'=1}^M b_{k,k'} \delta_{x_{k,k'}}, \\ \nu'_2 &= a_0 \delta_{x_{0,k_2}} + (b_{1,k_2} - a_0) \delta_{x_{1,k_2}} + \sum_{k=1, k \neq k_2} b_{1,k} \delta_{x_{1,k}} + \sum_{k=2}^M \sum_{k'=1}^M b_{k,k'} \delta_{x_{k,k'}}. \end{aligned}$$

The measure μ' (resp. ν'_1 and ν'_2) is obtained by “shifting” a small portion of the weight of μ (resp. $\Phi(\mu)$) from x_1 to x_0 (resp. from $x_{1,k_1/2}$ to $x_{0,k_1/2}$). Thus the projections of μ' onto L and H are

$$P_{L\#}(\mu') = a_0 \delta_{P_L(x_0)} + (a_1 - a_0) \delta_{P_L(x_1)} + \sum_{k=2}^M a_k \delta_{P_L(x_k)}$$

and

$$P_{H\#}(\mu') = P_{H\#}(\mu) = \sum_{k=1}^M a_k \delta_{P_H(x_k)};$$

the measures ν'_1, ν'_2 have the same respective projections.

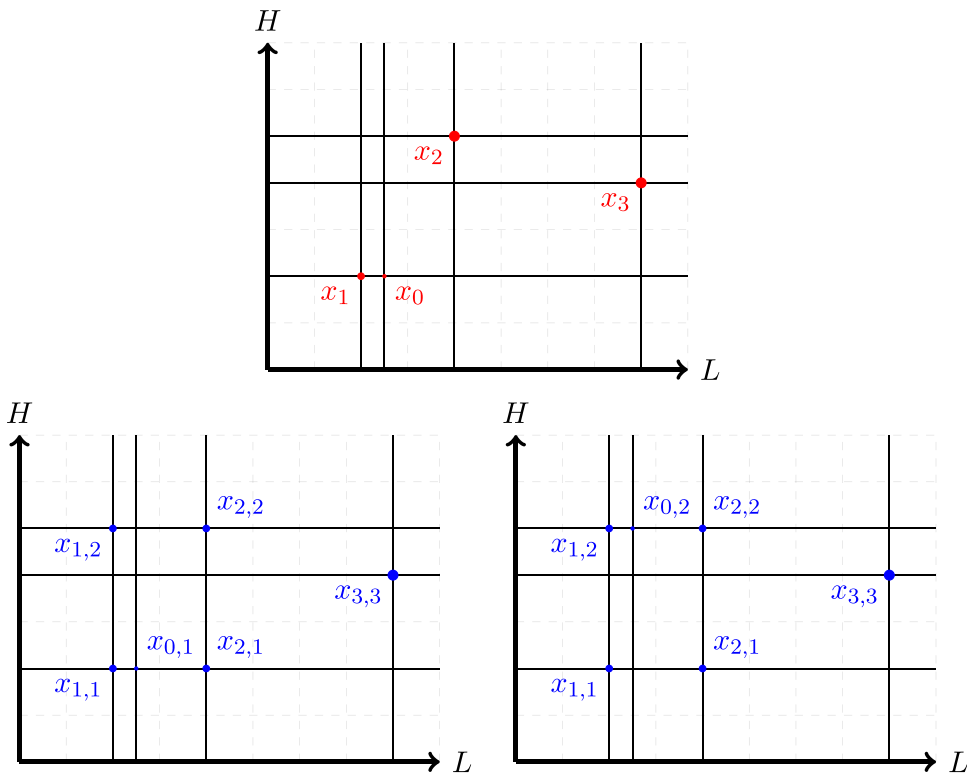


FIGURE 3 The constructions μ' (up), ν'_1 (down left), and ν'_2 (down right).

Claim 10. We have

$$d_{\mathcal{W}_p}^p(\mu, \mu') = d_{\mathcal{W}_p}(\Phi(\mu), \nu'_1) = d_{\mathcal{W}_p}(\Phi(\mu), \nu'_2) = a_0^{\frac{1}{p}} h_0.$$

For this, consider an optimal transport plan $\pi_0 \in \Pi(\mu, \mu')$. Then, as μ and μ' are a combination of Dirac masses, we have

$$d_{\mathcal{W}_p}^p(\mu, \mu') \geq \sum_{k=0}^M \pi_0(x_k, x_0) d_N^p(x_k, x_0) \geq \left(\sum_{k=1}^M \pi_0(x_k, x_0) \right) d_N^p(x_1, x_0) = a_0 h_0^p,$$

since for any $1 \leq k \leq M$, $d_N(x_k, x_0) \geq d_N(x_k, x_1) - d_N(x_1, x_0) \geq h - h_0 \geq d_N(x_1, x_0)$ (due to the definitions of h and h_0) and that $\pi_0(x_0, x_0) = 0$ (since μ is not supported on x_0). On the other hand, since μ' is obtained by shifting a weight a_0 from x_1 to x_0 , we have

$$d_{\mathcal{W}_p}^p(\mu, \mu') \leq a_0 d_N^p(x_0, x_1) = a_0 h_0^p.$$

Thus we have $d_{\mathcal{W}_p}(\mu, \mu') = a_0^{\frac{1}{p}} h_0$. The proof of the other cases is similar.

Claim 11. For $\xi \in \mathcal{W}_p(\mathbb{R}^n, d_N)$, if $d_{\mathcal{W}_p}(\mu, \xi) = a^{\frac{1}{p}} h_0$, $\mathcal{R}(\xi) = \mathcal{R}(\mu')$, and $\text{supp}(\xi) \subseteq \bigcup_{i=0}^M (H + x_i)$, then $\xi = \mu'$.

To see this, consider a measure $\xi \in \mathcal{W}_p(\mathbb{R}^n, d_N)$ such that we get $P_{H\#}(\xi) = P_{H\#}(\mu')$, $P_{L\#}(\xi) = P_{L\#}(\mu')$ and $\text{supp}(\xi) \in \bigcup_{i=0}^M (H + x_i)$. By the same argument as above we can write ξ as

$$\xi = \sum_{k=0}^M \sum_{k'=1}^M \tilde{b}_{ij} \delta_{x_{k,k'}}$$

with

$$\sum_{k'=1}^M \tilde{b}_{0k'} = a_0, \quad \sum_{k'=1}^M \tilde{b}_{1k'} = a_1 - a_0, \quad \sum_{k'=1}^M b_{kk'} = a_k \text{ for } 2 \leq k \leq M, \quad \sum_{k=0}^M \tilde{b}_{kk'} = a_{k'}.$$

Similar to above, we get that, for π_0 an optimal transport plan between μ and ξ , since $\pi_0(x_1, x_1) \leq \tilde{b}_{11} \leq a_1 - a_0$, we have that $\sum_{k=0}^M \sum_{k'=1}^M \pi_0(x_{k,k'}, x_1) - \pi_0(x_1, x_1) \geq a_0$, and thus

$$d_{\mathcal{W}_p}^p(\mu, \xi) \geq \sum_{k=0}^M \sum_{k'=1}^M \pi_0(x_1, x_{k,k'}) d_N^p(x_1, x_{k,k'}) \geq a_0 d_N^p(x_1, x_0) = a_0 h_0^p.$$

To have equality, we need that $\pi_0(x_{k,k'}, x_{k''}) = 0$ unless $k'' = 1$ or $k = k' = k''$. Since $d_N(x_0, x_1) < d_N(x_{k,k'}, x_1)$ unless $(k, k') = (0, 1)$, we also need that $\pi_0(x_{k,k'}, x_1) = 0$ for $(k, k') \neq (0, 1)$ or $k = k' = 1$. Thus the support of π_0 is exactly $\{(x_k, x_k)\} \cup \{(x_0, x_1)\}$. From the conditions on the weights we can thus conclude that $\xi = \mu$.

We are now ready to prove Claim 8 by contradiction. Indeed, we notice that, while $v'_1 \neq v'_2$,

$$d_{\mathcal{W}_p}(\Phi(\mu), v'_1) = d_{\mathcal{W}_p}(\Phi(\mu), v'_2) = a_0^{\frac{1}{p}} h_0.$$

Since Φ^{-1} is an isometry, we thus have that $d_{\mathcal{W}_p}(\mu, \Phi^{-1}(v'_1)) = d_{\mathcal{W}_p}(\mu, \Phi^{-1}(v'_2)) = a_0^{\frac{1}{p}} h_0$, with $\Phi^{-1}(v'_1) \neq \Phi^{-1}(v'_2)$. By Lemma 3.6, we have that $P_{H\#}(\Phi^{-1}(v'_1)) = P_{H\#}(v'_2) = P_{H\#}(\mu')$ and $P_{L\#}(\Phi^{-1}(v'_1)) = P_{L\#}(v'_2) = P_{L\#}(\mu')$. Lemma 3.8 shows that

$$\text{supp}(v'_1), \text{supp}(v'_2) \in \bigcup_{i=0}^M (H + x_i).$$

But then, by the previous remark, $\Phi^{-1}(v'_1) = \mu' = \Phi^{-1}(v'_2)$, giving the contradiction. Therefore we have that indeed $\Phi(\mu) \in \mathcal{F}$.

To finish the proof of the proposition, we combine the three properties of $\mathcal{F}$ to show that, if $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$, then $\Phi(\mu) = \mu$. By Lemma 3.6, we have that

$$P_{H\#}(\Phi(\mu)) = \Phi(P_{H\#}(\mu)) = P_{H\#}(\mu),$$

since Φ is assumed to be invariant on measures supported on H . By the injectivity of the projection map from the set $\mathcal{F}$, we have that $\Phi(\mu) = \mu$ whenever $\mu \in \mathcal{F}$. By the density of $\mathcal{F}$ we can extend the same result to any $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$. $\square$

4 | ISOMETRIC RIGIDITY USING C^2 -DIFFERENTIABILITY

In this section we prove Theorem 1.1, showing that the C^2 -smoothness and the strict convexity of a norm are sufficient conditions for the rigidity of the Wasserstein space $\mathcal{W}_p(\mathbb{R}^n, d_N)$ when $p \neq 2$. We consider $p \geq 1$, $p \neq 2$ fixed. We will work on the normed space $(\mathbb{R}^n, d_N)$, where the norm $N : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is C^2 -smooth on $\mathbb{R}^n \setminus \{0\}$, and N is strictly convex. By Corollary 2.4, for an isometry $\Phi : \mathcal{W}_p(\mathbb{R}^n, d_N) \rightarrow \mathcal{W}_p(\mathbb{R}^n, d_N)$, we can assume that $\Phi(\delta_x) = \delta_x$ for all $x \in \mathbb{R}^n$.

Our goal is to prove that $\Phi(\mu) = \mu$ for any measure $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$ using the so-called *potential functions* $\mathcal{T}_\mu^{(p)}$. For a measure μ , we define $\mathcal{T}_\mu^{(p)}$ as follows:

$$\mathcal{T}_\mu^{(p)} : \mathbb{R}^n \rightarrow \mathbb{R}_+, \quad x \mapsto d_{\mathcal{W}_p}^p(\mu, \delta_x) = \int_{\mathbb{R}^n} N^p(x - y) \, d\mu(y). \quad (4.1)$$

Observe that $\mathcal{T}_\mu^{(p)}(x) = \mathcal{T}_{\Phi(\mu)}^{(p)}(x)$ for all $x \in \mathbb{R}^n$. Indeed,

$$\mathcal{T}_\mu^{(p)}(x) = d_{\mathcal{W}_p}^p(\mu, \delta_x) = d_{\mathcal{W}_p}^p(\Phi(\mu), \Phi(\delta_x)) = d_{\mathcal{W}_p}^p(\Phi(\mu), \delta_x) = \mathcal{T}_{\Phi(\mu)}^{(p)}(x).$$

The question is whether $\mathcal{T}_\mu^{(p)} \equiv \mathcal{T}_\nu^{(p)}$ implies $\mu = \nu$? The answer in the general normed setting is no. To see an instructive example, consider the plane $\mathbb{R}^2$ equipped with the maximum norm $N_\infty(x_1, x_2) = \max\{|x_1|, |x_2|\}$. If we now take the measures

$$\mu = \frac{1}{2}(\delta_{(0,1)} + \delta_{(0,-1)}) \text{ and } \nu = \frac{1}{4}(\delta_{(0,1)} + \delta_{(0,-1)} + \delta_{(1,0)} + \delta_{(-1,0)}),$$

then the potential functions are given by

$$\mathcal{T}_\mu^{(1)}(x_1, x_2) = \frac{1}{2}(\max\{|x_1|, |x_2| + 1\} + \max\{|x_1| + 1, |x_2|\} - 1) = \mathcal{T}_\nu^{(1)}(x_1, x_2).$$

Thus, in a general normed setting, we cannot conclude from $\mathcal{T}_\mu^{(p)}(x) = \mathcal{T}_\nu^{(p)}(x)$ that $\mu = \nu$. However, we will now show that, if the norm N is at least C^2 -smooth, then $\mathcal{T}_\mu^{(p)}(x) = \mathcal{T}_\nu^{(p)}(x)$ implies $\mu = \nu$.

The proof of rigidity is split into two parts. We start by giving a direct proof in the case $p \in [1, 2)$. The second part will look at $p \in (2, \infty)$, where we show that measures supported on certain subspaces “remain” on these subspaces after an isometry, and we finish with an induction argument using the proposition of the previous section.

4.1 | First case: $p < 2$

We fix $p \in [1, 2)$, and we want to prove the following equality:

$$\lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{\mathcal{T}_\mu^{(p)}(x + h) - 2\mathcal{T}_\mu^{(p)}(x) + \mathcal{T}_\mu^{(p)}(x - h)}{2N^p(h)} = \mu(\{x\}).$$

As a first step, we show that

$$\lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{N^p(x+h) - 2N^p(x) + N^p(x-h)}{2N^p(h)} = \mathbb{1}_{\{0\}}(x). \quad (4.2)$$

If $x = 0$, the limit is trivially equal to 1. For $x \neq 0$, the function $y \rightarrow N^p(y)$ is twice differentiable at x , and Taylor's expansion gives $N^p(x \pm h) = N^p(x) + \langle \nabla N^p(x), \pm h \rangle + O(\|h\|^2)$ where $\|\cdot\|$ denotes the Euclidean norm. From here, we see that $N^p(x+h) - 2N^p(x) + N^p(x-h) = O(\|h\|^2)$, and thus

$$\lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{N^p(x+h) - 2N^p(x) + N^p(x-h)}{2N^p(h)} = \lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{O(\|h\|^2)}{2N^p(h)} = 0,$$

because N and $\|\cdot\|$ are equivalent norms (any two norms are bi-Lipschitz equivalent over a finite-dimensional vector space). Now integrating over $\mathbb{R}^n$ against μ we have

$$\mathcal{T}_\mu^{(p)}(x \pm h) = \int_{\mathbb{R}^n} N^p(x \pm h - y) d\mu(y),$$

and thus

$$\lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{\mathcal{T}_\mu^{(p)}(x+h) - 2\mathcal{T}_\mu^{(p)}(x) + \mathcal{T}_\mu^{(p)}(x-h)}{2N^p(h)}$$

can be written as

$$\lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \int_{\mathbb{R}^n} \frac{N^p(x+h-y) - 2N^p(x-y) + N^p(x-h-y)}{2N^p(h)} d\mu(y). \quad (4.3)$$

We want to show now that the function $G : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$G(x, h) = \begin{cases} \frac{N^p(x+h) - 2N^p(x) + N^p(x-h)}{2N^p(h)} & \text{if } h \neq 0 \\ \mathbb{1}_{\{0\}}(x) & \text{if } h = 0 \end{cases}$$

is bounded. To see this, we first notice that $G(\lambda x, \lambda h) = G(x, h)$ for any $\lambda \in \mathbb{R}$, thus we only need to prove that G is bounded on the set $E \times E$, with $E = \{y \in \mathbb{R}^n \text{ such that } N(y) \leq 1\}$. When $x = 0$, then $G(0, h) = 1$ is trivially bounded. When $x \neq 0$, we can use Taylor's expansion to write

$$G(x, h) = \frac{O(\|h\|^2)}{2N^p(h)}.$$

Since on $E \times E$ we have $N(h) \leq 1$, and since N and $\|\cdot\|$ are equivalent norms, G is bounded on $E \times E$, and thus is bounded on $\mathbb{R}^n \times \mathbb{R}^n$.

Since G is bounded, we can use the Lebesgue-dominated convergence theorem to interchange the limit and the integral of (4.3), and thus we get

$$\lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{\mathcal{T}_{\mu}^{(p)}(x+h) - 2\mathcal{T}_{\mu}^{(p)}(x) + \mathcal{T}_{\mu}^{(p)}(x-h)}{2N^p(h)} = \int_{\mathbb{R}^n} \mathbb{1}_{\{0\}}(x-y) d\mu(y) = \mu(\{x\}).$$

To prove rigidity, consider a finitely supported measure μ . Then, since $\mathcal{T}_{\mu}^{(p)} = \mathcal{T}_{\Phi(\mu)}^{(p)}$, we have that, for any $x_i \in \text{supp}(\mu)$,

$$\begin{aligned} \mu(\{x_i\}) &= \lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{\mathcal{T}_{\mu}^{(p)}(x+h) - 2\mathcal{T}_{\mu}^{(p)}(x) + \mathcal{T}_{\mu}^{(p)}(x-h)}{2N(h)} \\ &= \lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{\mathcal{T}_{\Phi(\mu)}^{(p)}(x+h) - 2\mathcal{T}_{\Phi(\mu)}^{(p)}(x) + \mathcal{T}_{\Phi(\mu)}^{(p)}(x-h)}{2N(h)} = \Phi(\mu)(\{x_i\}), \end{aligned}$$

and thus $\Phi(\mu) = \mu$. Since finitely supported measures are dense in $\mathcal{W}_p(\mathbb{R}^n, d_N)$, we have that $\Phi(\mu) = \mu$ for any $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$, proving Theorem 1.1 for the case $p \in [1, 2)$.

4.2 | Second case: $p > 2$

While we could, similarly to the method in [11], adapt the previous idea for higher p , this approach presents two problems. First, it would require a Taylor expansion of the order of $[p]$, while the norms we consider are only C^2 -smooth. Second, this method does not work when p is an even integer.

Instead, we show in the next proposition that, if we restrict the isometry to measures supported on a certain subspace, the image will be measures supported on the same subspace. The rigidity of the Wasserstein space then follows by an induction on the dimension n , using the result of the previous section.

Proposition 4.1. *If $p > 2$ and $N : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a C^2 -smooth norm, then there exists a proper linear subspace $L \subset \mathbb{R}^n$ such that for any $x_0 \in \mathbb{R}^n$, if μ is a measure supported on $x_0 + L$, then for any isometry Φ of the Wasserstein space $\mathcal{W}_p(\mathbb{R}^n, d_N)$ fixing Dirac masses, $\Phi(\mu)$ will also be supported on $x_0 + L$.*

To see this, we will start by proving in Proposition 4.2 the existence of an operator which, when applied to the norm N , gives a continuous nonconstant function over $\mathbb{R}^n \setminus \{0\}$ which attains its minimum at 0. In Lemma 4.4, we then use this operator to build the function $\mathbb{T}_{\mu}$ which is positive and vanishes at x only if μ is supported on the subspace $x + L$, where L is any subspace of points such that the operator of Proposition 4.2 vanishes. We finish by proving in Lemma 4.5 that, for a certain subspace L , the previous lemma implies that if Φ is an isometry preserving Dirac masses, then every measure μ supported on $x + L$ will be sent to another measure supported on $x + L$, which then implies Proposition 4.1.

Since N is a convex function, we have that the matrix $(\text{Hess}N^p)(x)$ is positive semi-definite for any $x \in \mathbb{R}^n \setminus \{0\}$. We start by showing the following proposition.

Proposition 4.2. *For any $N : \mathbb{R}^n \rightarrow \mathbb{R}_+$ that is a C^2 -smooth norm, there exists $v_1, v_2 \in S^{n-1}$ such that*

$$\min_{x \in \mathbb{R}^n \setminus \{0\}} v_2^\top (\text{Hess} N^P)(x) v_1 = 0$$

and the function $x \rightarrow v_2^\top (\text{Hess} N^P)(x) v_1$ is nonnegative and nonconstant on $\mathbb{R}^n \setminus \{0\}$.

Here S^{n-1} is the usual unit sphere with respect to the Euclidean metric, and we consider the unit sphere of the norm $S_N^{n-1} = \{x \in \mathbb{R}^n | N(x) = 1\}$.

Proof. If there exists $v_1 \in S^{n-1}$ and $x_0 \in S_N^{n-1}$ such that $v_1^\top (\text{Hess} N^P)(x) v_1 = 0$, this proves our claim. Indeed, using that $N^P(tv) = t^p N^P(v)$ when $t > 0$, we can easily calculate that $v_1^\top (\text{Hess} N^P)(v_1) v_1 = p(p-1)N^P(v_1) > 0$ and, due to positive semi-definiteness, that

$$v_1^\top (\text{Hess} N^P)(x) v_1 \geq 0$$

for all $x \in S_N^{n-1}$. Thus we assume that

$$v_1^\top (\text{Hess} N^P)(x) v_1 > 0 \quad (4.4)$$

for all $x \in S_N^{n-1}$ and all $v_1 \in S^{n-1}$.

Then we have in particular that $(-v_1)^\top (\text{Hess} N^P)(x) v_1 < 0$. If $v^1, \dots, v^n$ is a linearly independent basis of $\mathbb{R}^n$ with $v^1 = v_1$, consider for $2 \leq i \leq n$ the curves $\gamma_i : [0, 1] \rightarrow S^{n-1}$ given by $\gamma_i(t) = \tilde{n}((-1 + 2t)v^1 + (1 - |2t - 1|)v^i)$, where $\tilde{n}$ is the normalization operator $\tilde{n}(x) = \frac{x}{\|x\|_E}$, and $\|\cdot\|_E$ is the usual Euclidean norm.

We define the map $H_i : [0, 1] \times S_N^{n-1} \rightarrow \mathbb{R}$ (which depends implicitly on the choice of v_1) as

$$H_i(t, x) = \gamma_i(t)^\top (\text{Hess} N^P)(x) v_1,$$

define for any $x \in S_N^{n-1}$ the set $A_x^i = \{t \in [0, 1] | H_i(t, x) = 0\}$ and we take $t_x^i = \sup(A_x^i)$. Then $H_i(t, x) > 0$ for any $t \in [0, 1]$ with $t > t_x^i$. We also set $t_i = \sup\{t_x^i | x \in S_N^{n-1}\}$. Since H_i is continuous on a compact set, the preimage $H^{-1}(\{0\})$ is closed and compact, and thus $t_i \in (0, 1)$, and there exists $x_i \in S_N^{n-1}$ such that $t_i = t_{x_i}^i$. Then $H_i(t_i, x_i) = 0$, and for any $x \in S_N^{n-1}$, $H_i(t_i, x) \geq 0$.

If for some $v_1 \in S^{n-1}$ and for some $2 \leq i \leq n$ the function $H_i(t_i, \cdot)$ is not constant on S_N^{n-1} , choosing $v_2 = \gamma_i(t_i)$ proves the lemma. Thus we assume by contradiction that this never holds, that is, for any $v_1 \in S^{n-1}$ and any $2 \leq i \leq n$, $H_i(t_i, x) = 0$ for any $x \in S_N^{n-1}$.

Since by assumption (4.4) we have $\text{Hess} N^P(x) v_1 \neq 0$, the equality $H_i(t_i, x) = 0$ implies that, for any $x \in S_N^{n-1}$, $\text{Hess} N^P(x) v_1$ is a vector in the hyperplane orthogonal to $\gamma_i(t_i)$. As by construction the collection $\{\gamma_i(t_i)\}_{i=2}^n$ is a set of $n-1$ linearly independent vectors, if $\text{Hess} N^P(x) v_1$ is orthogonal to every vector in the collection, there exists $w_{v_1} \in S^{n-1}$ (which is orthogonal to all $\gamma_i(t_i)$) such that $\text{Hess} N^P(x) v_1 = \lambda_{v_1}(x) w_{v_1}$ for some function $\lambda_{v_1} : S_N^{n-1} \rightarrow \mathbb{R}$.

Choosing $v_1 = e_i$ a canonical vector, we thus have that $\partial_{ij} N^P(x) = \lambda_{e_i}(x) (w_{e_i})_j$; since assumption (4.4) guarantees that $\partial_i^2 N^P(x) > 0$ for all $x \in S_N^{n-1}$, $(w_{e_i})_i \neq 0$ and we get

$$\partial_{ij} N^P(x) = \frac{(w_{e_i})_j}{(w_{e_i})_i} \partial_i^2 N^P(x). \quad (4.5)$$

Since N^p is p -homogeneous, this still holds true for any $x \in \mathbb{R}^n \setminus \{0\}$. If $\partial_{ij} N^p(x) = 0$ for all $i, j \leq n$, $i \neq j$, then using partial integration we show that we can write $N^p(x) = \sum_{i=1}^n a_i(x_i)$ for some functions $a_i : \mathbb{R} \rightarrow \mathbb{R}$. As $\partial_{ij} N^p$ is $(p-2)$ -homogeneous, we see that, for example, $(\partial_1^2 a_1)(0) = 0$ and $\partial_1^2 N^p(e_2) = 0$ contradicting assumption (4.4).

Thus, we take $i \neq j$ such that $\partial_{ij} N^p(x) \neq 0$. Then

$$\frac{(w_{e_i})_j}{(w_{e_i})_i} \partial_i^2 N^p(x) = \partial_{ij} N^p(x) = \frac{(w_{e_j})_i}{(w_{e_j})_j} \partial_j^2 N^p(x). \quad (4.6)$$

We set $c = \frac{(w_{e_i})_j}{(w_{e_i})_i}$ and $d = \frac{(w_{e_j})_i}{(w_{e_j})_j}$. Using $v_1 = \tilde{n}(\frac{1}{\sqrt{c}}e_i - \frac{1}{\sqrt{d}}e_j)$ (where $\tilde{n}$ is the normalization operator), assumption (4.4) gives the condition that $cd > 1$.

We first assume that $c, d > 0$ (and thus $\partial_{ij} N^p(x) > 0$) and consider the vector $v_1(\vartheta) = \cos(\vartheta)e_i - \sin(\vartheta)e_j$ for $\vartheta \in [0, 2\pi)$. Then (4.4) implies that

$$\cos^2(\vartheta) \partial_i^2 N^p(x) - 2 \sin(\vartheta) \cos(\vartheta) \partial_{ij} N^p(x) + \sin^2(\vartheta) \partial_j^2 N^p(x) > 0.$$

Using (4.6), we rewrite this as

$$\left(\frac{1}{c} \cos^2(\vartheta) - 2 \sin(\vartheta) \cos(\vartheta) + \frac{1}{d} \sin^2(\vartheta) \right) \partial_{ij} N^p(x) > 0$$

which implies

$$d \cos^2(\vartheta) - 2cd \sin(\vartheta) \cos(\vartheta) + c \sin^2(\vartheta) > 0.$$

This is in turn equivalent to

$$\left(\sqrt{d} \cos(\vartheta) - \sqrt{c} \sin(\vartheta) \right)^2 - 2(cd - \sqrt{cd}) \sin(\vartheta) \cos(\vartheta) > 0.$$

By choosing $\vartheta \in (0, \frac{\pi}{2})$ such that $\tan(\vartheta) = \sqrt{\frac{d}{c}}$, the square of the previous equation vanishes, and since $\sin(\vartheta), \cos(\vartheta) > 0$, we get the condition

$$cd - \sqrt{cd} < 0,$$

contradicting our assumption that $cd > 1$.

If $c, d < 0$, we instead define $v_1(\vartheta) = \cos(\vartheta)e_i + \sin(\vartheta)e_j$. By the same argument we then get

$$d \cos^2(\vartheta) + 2cd \sin(\vartheta) \cos(\vartheta) + c \sin^2(\vartheta) < 0$$

or

$$|d| \cos^2(\vartheta) - 2cd \sin(\vartheta) \cos(\vartheta) + |c| \sin^2(\vartheta) > 0,$$

which gives the same contradiction as in the previous case. $\square$

For v_1, v_2 given by Proposition 4.2, we consider the set

$$A = \{x \in \mathbb{R}^n \setminus \{0\} \mid v_2^\top (\text{Hess} N^p)(x) v_1 = 0\},$$

and $A_0 = A \cup \{0\}$. We also define the function $T : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ as

$$T(x) = v_2^\top (\text{Hess} N^p)(x) v_1.$$

Then, for $x \neq 0$, we have $T(x) = 0$ if $x \in A$, and $T(x) > 0$ if $x \notin A$. Since $\text{Hess} N^p$ is $(p-2)$ -homogeneous, with $p > 2$, we have that $\lim_{x \rightarrow 0} T(x) = 0$, and we can continuously extend T to the whole $\mathbb{R}^n$. For a general measure $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$, we define $\mathbb{T}_\mu : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$\mathbb{T}_\mu(x) = \int_{\mathbb{R}^n} T(x-y) d\mu(y).$$

We now show that the function $y \rightarrow T(x-y)$ is integrable with respect to elements of $\mathcal{W}_p$ which implies that $\mathbb{T}_\mu$ is well defined and we can exchange the integral and the Hessian of the function T .

Lemma 4.3. *Under the assumptions of Proposition 4.2, we have that*

$$\mathbb{T}_\mu(x) = v_2^\top \text{Hess} \left(\int_{\mathbb{R}^n} N^p(x-y) d\mu(y) \right) v_1 = v_2^\top \text{Hess} \left(\mathcal{T}_\mu^{(p)}(x) \right) v_1. \quad (4.7)$$

In particular, $\mathbb{T}_\mu(x) = \mathbb{T}_{\Phi(\mu)}(x)$ for any isometry Φ which fixes Dirac masses.

Proof. We first prove that

$$\int_{\mathbb{R}^n} \nabla N^p(x-y) \cdot v_1 d\mu(y) = \nabla \left(\int_{\mathbb{R}^n} N^p(x-y) d\mu(y) \right) \cdot v_1 \quad (4.8)$$

holds for all $x \in \mathbb{R}^n$ and $v_1 \in \mathbb{R}^n$.

Let us consider the right-hand side of (4.8). Let $x \in \mathbb{R}^n$ and $v_1 \in \mathbb{R}^n$ be fixed, and consider the parametric family of functions

$$F_h : \mathbb{R}^n \rightarrow \mathbb{R}; y \mapsto F_h(y) := \frac{1}{h} (N^p(x + h v_1 - y) - N^p(x - y)) \quad (4.9)$$

where the parameter h runs over $(0,1]$. By Lagrange's mean value theorem, for all $h \in (0,1]$ and for all $y \in \mathbb{R}^n$ there is an $h_0 = h_0(h, y) \in [0, h]$ such that

$$F_h(y) = \frac{1}{h} (N^p(x + h v_1 - y) - N^p(x - y)) = \nabla N^p(x + h_0 v_1 - y) \cdot v_1. \quad (4.10)$$

Note that $h_0(h, y) \in [0,1]$ for all $h \in (0,1]$ and $y \in \mathbb{R}^n$, and hence

$$|x + h_0 v_1 - y| \leq |x| + |v_1| + |y|. \quad (4.11)$$

Note also that ∇N^p is homogeneous of order $p - 1$ on $\mathbb{R}^n$, it is continuous on the compact set S_N^{n-1} as N^p is C^2 on $\mathbb{R}^n \setminus \{0\}$, and hence $+\infty > C_1 := \max_{\theta \in S_N^{n-1}} |\nabla N^p(\theta)|$. Therefore, the parametric family of functions $F_h(\cdot)$ admits the integrable majorant

$$\begin{aligned} |F_h(y)| &= |\nabla N^p(x + h_0 v_1 - y) \cdot v_1| \leq |\nabla N^p(x + h_0 v_1 - y)| |v_1| \\ &\leq C_1 |x + h_0 v_1 - y|^{p-1} |v_1| \leq C_1 (|x| + |v_1| + |y|)^{p-1} |v_1|. \end{aligned} \quad (4.12)$$

The integrability of $y \mapsto (|x| + |v_1| + |y|)^{p-1}$ with respect to μ follows from the direct computation

$$(|x| + |v_1| + |y|)^{p-1} \leq (2(|x| + |v_1|))^{p-1} + (2|y|)^{p-1} \leq (2(|x| + |v_1|))^{p-1} + 2^{p-1} (1 + |y|^p)$$

and from the assumption $\mu \in W_p(R^n, d_N)$ which implies that $\{y \mapsto |y|^p\} \in L^1(\mathbb{R}^n, \mu)$. Thus we can apply Lebesgue's dominated convergence theorem to conclude that

$$\lim_{h \rightarrow 0} F_h(y) = \nabla N^p(x - y) \cdot v_1 \quad (4.13)$$

is integrable with respect to μ , the limit $\lim_{h \rightarrow 0} \int_{\mathbb{R}^n} F_h(y) d\mu(y)$ exists, and

$$\begin{aligned} \int_{\mathbb{R}^n} \nabla N^p(x - y) \cdot v_1 d\mu(y) &= \int_{\mathbb{R}^n} \lim_{h \rightarrow 0} F_h(y) d\mu(y) = \lim_{h \rightarrow 0} \int_{\mathbb{R}^n} F_h(y) d\mu(y) \\ &= \lim_{h \rightarrow 0} \left(\frac{1}{h} \left(\int_{\mathbb{R}^n} N^p(x + h v_1 - y) d\mu(y) - \int_{\mathbb{R}^n} N^p(x - y) d\mu(y) \right) \right) \\ &= \nabla \left(\int_{\mathbb{R}^n} N^p(x - y) d\mu(y) \right) \cdot v_1. \end{aligned} \quad (4.14)$$

Let us turn back to (4.7) and note that

$$v_2^T \text{Hess} N^p(x - y) v_1 = \lim_{h \rightarrow 0} \frac{1}{h} (\nabla N^p(x + h v_2 - y) \cdot v_1 - \nabla N^p(x - y) \cdot v_1). \quad (4.15)$$

From this point on, we can use an almost verbatim copy of the previous argument to justify (4.7). Indeed, let us consider the parametric family of functions

$$G_h : \mathbb{R}^n \rightarrow \mathbb{R}; y \mapsto G_h(y) := \frac{1}{h} (\nabla N^p(x + h v_2 - y) \cdot v_1 - \nabla N^p(x - y) \cdot v_1) \quad (4.16)$$

where the parameter h runs over $(0, 1]$. By Lagrange's mean value theorem, for all $h \in (0, 1]$ and for all $y \in \mathbb{R}^n$ there is an $h_0 = h_0(h, y) \in [0, h]$ such that

$$G_h(y) = \frac{1}{h} (\nabla N^p(x + h v_2 - y) \cdot v_1 - \nabla N^p(x - y) \cdot v_1) = v_2^T \text{Hess} N^p(x + h_0 v_2 - y) v_1. \quad (4.17)$$

Note that $h_0(h, y) \in [0, 1]$ for all $h \in (0, 1]$ and $y \in \mathbb{R}^n$, and hence

$$|x + h_0 v_2 - y| \leq |x| + |v_2| + |y|. \quad (4.18)$$

Note also that $\text{Hess}N^p$ is homogeneous of order $p - 2$ on $\mathbb{R}^n$, it is continuous on the compact set S_N^{n-1} as N^p is C^2 on $\mathbb{R}^n \setminus \{0\}$, and hence $+\infty > C_2 := \max_{\theta \in S_N^{n-1}} \|\text{Hess}N^p(\theta)\|$. Therefore, the parametric family of functions $G_h(\cdot)$ admits the integrable majorant

$$\begin{aligned} |G_h(y)| &= \left| v_2^T \text{Hess}N^p(x + h_0 v_2 - y) v_1 \right| \leq |v_2| \|\text{Hess}N^p(x + h_0 v_2 - y)\| |v_1| \\ &\leq C_2 |x + h_0 v_2 - y|^{p-2} |v_1| |v_2| \leq C_2 (|x| + |v_2| + |y|)^{p-2} |v_1| |v_2|. \end{aligned} \quad (4.19)$$

The integrability of $y \mapsto (|x| + |v_2| + |y|)^{p-2}$ with respect to μ follows from the direct computation

$$(|x| + |v_2| + |y|)^{p-2} \leq (2(|x| + |v_2|))^{p-2} + (2|y|)^{p-2} \leq (2(|x| + |v_2|))^{p-2} + 2^{p-2} (1 + |y|^p)$$

and from the assumption $\mu \in W_p(R^n, d_N)$ which implies that $\{y \mapsto |y|^p\} \in L^1(\mathbb{R}^n, \mu)$. Thus we can apply Lebesgue's dominated convergence theorem to conclude that

$$\lim_{h \rightarrow 0} G_h(y) = v_2^T \text{Hess}N^p(x - y) v_1 \quad (4.20)$$

is integrable with respect to μ , the limit $\lim_{h \rightarrow 0} \int_{\mathbb{R}^n} G_h(y) d\mu(y)$ exists, and

$$\begin{aligned} \int_{\mathbb{R}^n} v_2^T \text{Hess}N^p(x - y) v_1 d\mu(y) &= \int_{\mathbb{R}^n} \lim_{h \rightarrow 0} G_h(y) d\mu(y) = \lim_{h \rightarrow 0} \int_{\mathbb{R}^n} G_h(y) d\mu(y) \\ &= \lim_{h \rightarrow 0} \left(\frac{1}{h} \left(\int_{\mathbb{R}^n} \nabla N^p(x + h v_2 - y) \cdot v_1 d\mu(y) - \int_{\mathbb{R}^n} \nabla N^p(x - y) \cdot v_1 d\mu(y) \right) \right). \end{aligned} \quad (4.21)$$

We know by (4.8) that the right-hand side of (4.21) is further equal to

$$\lim_{h \rightarrow 0} \left(\frac{1}{h} \left(\nabla \left(\int_{\mathbb{R}^n} N^p(x + h v_2 - y) d\mu(y) \right) \cdot v_1 - \nabla \left(\int_{\mathbb{R}^n} N^p(x - y) d\mu(y) \right) \cdot v_1 \right) \right) \quad (4.22)$$

which by definition is equal to

$$v_2^T \text{Hess} \left(\int_{\mathbb{R}^n} N^p(x - y) d\mu(y) \right) v_1, \quad (4.23)$$

as desired. $\square$

We now have the following lemma.

Lemma 4.4. *Let $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$ be a measure. If there exists a point $x_0 \in \mathbb{R}^n$ such that $\mathbb{T}_\mu(x_0) = 0$, then μ is supported on the set of points $x_0 + A_0$. Furthermore, if μ is a measure supported on an affine subset $x_0 + L \subset x_0 + A_0$ for a proper linear subset $L \subset \mathbb{R}^n$, then $\mathbb{T}_\mu(x) = 0$ for all $x \in x_0 + L$.*

Proof. We start by showing the first part of the statement. Assume that $\mathbb{T}_\mu(x_0) = 0$, but μ is not supported on $x_0 + A_0$, that is, there exists $x'_0 \in \text{supp}(\mu)$ such that $x'_0 \notin x_0 + A_0$. By the

definition of the support of a measure, for any ball $B(\varepsilon, x'_0)$ around x'_0 with radius $\varepsilon > 0$, we have that $\mu(B(\varepsilon, x'_0)) > 0$.

We choose $\varepsilon > 0$ such that $x_0 + A_0 \cap B(\varepsilon, x'_0) = \emptyset$ (since N^p is C^2 -continuous, the set A_0 is closed, and such an ε exists). Then, for any $y \in \overline{B(\varepsilon/2, x'_0)} =: B_1$, we have $y \notin x_0 + A_0$ and

$$T(x_0 - y) > 0. \quad (4.24)$$

Since B_1 is compact, (4.24) attains its minimum at a point $y_0 \in B_1$, at which we still have $T(x_0 - y_0) > 0$. Thus, since $T(x)$ is a nonnegative function, we have that

$$\mathbb{T}_\mu(x_0) \geq \int_{B_1} T(x_0 - y) d\mu(y) \geq T(x_0 - y_0) \mu(B_1) > 0.$$

Therefore, if there exists $x_0 \in \mathbb{R}^n$ with $\mathbb{T}_\mu(x_0) = 0$, then μ is supported on the set $x_0 + A_0$.

For the second part of the lemma, consider a measure μ supported on $x_0 + L$. Then, if $x, y \in x_0 + L$, since by linearity $x - y \in A_0$, we have that $T(x - y) = 0$ and

$$\mathbb{T}_\mu(x) = \int_{x_0+L} T(x - y) d\mu(y) = 0$$

for any $x \in x_0 + L$. □

Lemma 4.5. *Consider for any $x_0 \in \mathbb{R}^n$ a proper maximal affine subset $x_0 + L \subset x_0 + A_0$, in the sense that, for every $y \notin L$, we have that $\text{span}(\{L, y\}) \not\subset A_0$. If $\mu \in \mathcal{W}_p(\mathbb{R}^n, d_N)$ is a measure supported on $x_0 + L$ and $\Phi : \mathcal{W}_p(\mathbb{R}^n, d_N) \rightarrow \mathcal{W}_p(\mathbb{R}^n, d_N)$ is an isometry fixing Dirac masses, then $\Phi(\mu)$ is also supported on $x_0 + L$.*

Proof. Consider μ as a measure supported on $x_0 + L$. Then, by the previous lemma, $\mathbb{T}_\mu(x) = 0$ for all $x \in x_0 + L$. By the same lemma, since we have $\mathbb{T}_\mu(x) = \mathbb{T}_{\Phi(\mu)}(x)$, we get that $\Phi(\mu)$ is supported on the set $x + A_0$ for any $x \in x_0 + L$.

Assume for contradiction that there exists $x_1 \in \text{supp}(\Phi(\mu))$ such that $x_1 \notin x_0 + L$. Then, for any $x \in x_0 + L$, we still have that $x_1 - x \in A_0$. Consider $y \in \text{span}(L, x_1 - x_0)$, $y \notin L$ and write $y = \lambda_1(x_1 - x_0) - \lambda_2 x_2$ for $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 \neq 0$ and $x_2 \in L$. We can rewrite this as $y = \lambda_0((x_1 - x_0) - (\frac{\lambda_1}{\lambda_0} x_1))$, and thus $y \in A_0$. But then $x_0 + \text{span}(\{L, x_1 - x_0\}) \subset x_0 + A_0$, contradicting the maximality of the subset L . □

Notice that we can always find a proper maximal subset. Indeed, by Proposition 4.2, A_0 is a proper subset of $\mathbb{R}^n$, and thus a maximal set is also proper. Furthermore, since A is not empty, we have $L_1 = x_0 + tx_1 \subset x_0 + A_0$ for any $x_1 \in A$. Then, if L_1 is not maximal, there exists $y \notin L_1$ such that $\text{span}(\{L_1, y\}) \subset x_0 + A_0$, and we consider the set $L = \text{span}(\{L_1, y\})$. If L is not maximal, we proceed in the same way, until we find a set L' that is maximal.

Therefore this lemma also proves Proposition 4.1.

Proof of Theorem 1.1 if $p > 2$. Using Proposition 4.1, we now have that, for any strictly convex, C^2 -smooth norm N , if $p > 2$, there exists a proper linear subspace L such that for every $x_0 \in \mathbb{R}^n$, if a measure ν is supported on $L_{x_0} := x_0 + L$, then $\Phi(\nu)$ is also supported on L_{x_0} . Note that for $p = 2$, this is in general not true, as the example in Figure 4 and Remark 4.6 will show.

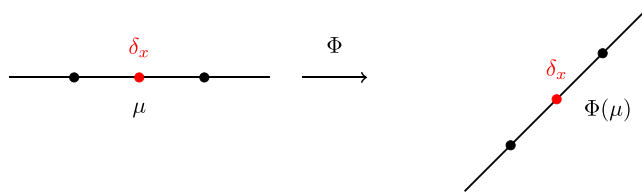


FIGURE 4 In the Euclidean plane for $p = 2$, there exist shape-preserving isometries of $\mathcal{W}_2(\mathbb{R}^2, d_E)$ that send a measure supported on a line to a measure supported on a different line.

We first look at the case $n = 2$. Since L is proper, L is a line and $(L_{x_0}, d_{N, L_{x_0}})$ is isometric to $(\mathbb{R}, |\cdot|)$, thus we can use [10] to assume that for any measure ν supported on L_{x_0} , $\Phi(\nu) = \nu$. By Lemma 3.2, the set $P_L^{-1}(0)$ is again a line, and $P_L^{-1}(0) = H$, the linear subspace given by Lemma 3.8. By Lemma 3.6, we know that $P_{L\#}(\mu) = P_{L\#}(\Phi(\mu))$, which implies in particular that for every measure ν supported on H , we have that $P_{L\#}\nu = \delta_0$, which implies that $\Phi(\nu)$ is also supported on H , and we can again assume that $\Phi(\nu) = \nu$ when ν is supported on H . Since the norm N is strictly convex, it projects uniquely onto L and H , and we can apply Proposition 3.1. Then we have rigidity of the Wasserstein space $\mathcal{W}_p(\mathbb{R}^2, d_N)$ for $p > 2$.

We now make the induction assumption on n , that for any $n \leq n_0$, the space $\mathcal{W}_p(\mathbb{R}^n, d_N)$ is rigid for $p \neq 2$ when N is a C^2 strictly convex norm, and we consider the Wasserstein space $\mathcal{W}_p(\mathbb{R}^{n_0+1}, d_N)$.

For L given by Proposition 4.1, since L is a proper linear subspace of $\mathbb{R}^{n_0+1}$, $(L_{x_0}, d_{N, L_{x_0}})$ is isometric to $(\mathbb{R}^n, d_{N, L_{x_0}})$ for some $n < n_0 + 1$. Since for any measure ν supported on L_{x_0} we have that $\Phi(\nu)$ is also supported on L_{x_0} , we can use the induction assumption to assume that $\Phi(\nu) = \nu$ for any measure ν supported on L_{x_0} .

We take the linear subspace $H \subset P_L^{-1}(0)$ given by Lemma 3.8. Then, if ν is a measure supported on H , again by Lemma 3.8 we have that $\text{supp}(\Phi(\nu)) \subset H + \text{supp}(\nu) = H$, and as above we can use the induction argument to assume that $\Phi(\nu) = \nu$ whenever ν is supported on H . Then, by Proposition 3.1 we have rigidity of the space $\mathcal{W}_p(\mathbb{R}^{n_0+1}, d_N)$. $\square$

4.3 | Special case: $p = 2, N = l_q, q > 2$

Note that in the proof of Theorem 1.1, we only use in a few places the fact that $p \neq 2$. In particular, we used in Proposition 4.2 that $\partial_{ij} N^p$ is $(p - 2)$ -homogeneous to show that $\partial_i^2 N^p(e_2) = 0$ if the mixed derivatives all vanish. In the definition of T , the $(p - 2)$ -homogeneity again allows us to define $T(0)$. Finally, in the proof of Theorem 1.1, we use that in [10], Gehér et al. showed rigidity of the Wasserstein space over the real line when $p \neq 2$.

Remark 4.6. In general, we know that these limitations are strict, since we have for $p = 2$ examples of nonrigid Wasserstein spaces $\mathcal{W}_2(\mathbb{R}^n, d_N)$, for instance if the norm N is the standard Euclidean norm.

For example, in the case of $\mathbb{R}^2$, we can take the shape-preserving isometry which rotates measures around their center of mass with an angle of $\frac{\pi}{4}$, according to the construction given in Proposition 6.1 of Kloeckner [12]. Then, even if this isometry preserves Dirac masses, any

measure supported on a line will be sent to a measure supported on a different line, namely on the line rotated by $\pi/4$, as seen in Figure 4. Thus we cannot even hope to find a replacement of Proposition 4.1 for general C^2 -norms.

Nevertheless, in the case of the l_q norms with $q > 2$, we can adapt the argument of the proof of Theorem 1.1 to show the rigidity of the Wasserstein space $\mathcal{W}_2(\mathbb{R}^n, d_q)$. Indeed, for the l_q norm given by

$$N_q(x) = \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}},$$

since $q > 2$, we have that N_q is C^2 -smooth and the second derivatives are given by

$$\partial_i^2 N_q^2(x) = 2(2-q)(N(x))^{2-2q} |x_i|^{2q-2} + 2(q-1)(N(x))^{2-q} |x_i|^{q-2}.$$

In particular, we have $\partial_i^2 N_q^2(x) \geq 0$ for $x \in \mathbb{R}^n \setminus \{0\}$, with equality if and only if $x_i = 0$. Thus, instead of using Proposition 4.2 to define T , we can simply choose $i \leq n$, and set $T(x) = \partial_i^2 N^2(x)$, which has the property that $T(x) = 0$ if $x_i = 0$, $x \neq 0$ and $T(x) > 0$ for $x_i \neq 0$. While T is not necessarily defined at $x = 0$, the proofs of Lemmata 4.4 and 4.5 still hold for measures that are absolutely continuous with respect to the Lebesgue measure of a subspace L of dimension $k \geq 1$. Indeed, we then have that $\int_{\mathbb{R}^n} T(x-y) d\mu(y) = \int_{\mathbb{R}^n \setminus \{x\}} T(x-y) d\mu(y)$, which is well defined. Thus Proposition 4.1 holds for measures that are absolutely continuous on subspaces; since a hypersurface of $\mathbb{R}^n$ is closed and since absolutely continuous measures are dense in $\mathcal{W}_2(\mathbb{R}^n, d_q)$, we can analogously to Proposition 4.1 state the following proposition:

Proposition 4.7. *Consider $q > 2$, and d_q the metric induced by the l_q norm. For any $i \leq n$, consider the hypersurface L_i given by $L_i = \{x \in \mathbb{R}^n | x_i = 0\}$. If μ is a measure supported on $x_0 + L_i$, then for any isometry Φ of the Wasserstein space $\mathcal{W}_2(\mathbb{R}^n, d_q)$ fixing Dirac masses, $\Phi(\mu)$ will also be supported on $x_0 + L_i$.*

To finish the proof of Theorem 1.2, we want to again use an induction argument and Proposition 3.1; however, we first need to show that, if an isometry Φ of the Wasserstein space fixes Dirac masses and globally fixes measures supported on a line, then it acts as the identity on measures supported on the line.

To see this, consider in $\mathbb{R}^n$ the line $L_1 = \{te_1 | t \in \mathbb{R}\}$ (the other cases can be handled similarly). If an isometry $\Phi : \mathcal{W}_2(\mathbb{R}^n, d_q) \rightarrow \mathcal{W}_2(\mathbb{R}^n, d_q)$ globally preserves measures supported on L_1 , since (L_1, d_{q,L_1}) is isometric to $(\mathbb{R}, |\cdot|)$, we can use the result from Kloeckner (Lemma 5.2 [12]) to characterize Φ . Indeed, we recall Kloeckner's notation

$$\mu = \mu(x, \sigma, p) = \frac{e^{-p}}{e^{-p} + e^p} \delta_{(x-\sigma e^p)e_1} \frac{e^p}{e^{-p} + e^p} \delta_{(x+\sigma e^{-p})e_1}$$

to represent measures supported on two points on L_1 , then Lemma 5.2 of [12] gives us that an isometry which fixes Dirac masses and globally fixes L_1 acts on measures supported on two points of L_1 in the following way:

$$\Phi(\mu(x, \sigma, p)) = \mu(x, \sigma, \varphi(p)),$$

where φ is an isometry of $(\mathbb{R}, |\cdot|)$. It is well known that the isometries of $(\mathbb{R}, |\cdot|)$ are given by $\varphi(x) = sx + t$ for some $t \in \mathbb{R}$ and $s \in \{-1, 1\}$. We will show that if a map Φ does not leave measures supported on 2 points of L_1 invariant (i.e., if $s = -1$ or $t \neq 0$), then Φ cannot be an isometry of the Wasserstein space $\mathcal{W}_2(\mathbb{R}^n, d_q)$.

To start, assume by contradiction that the isometry Φ^t is given by

$$\Phi^t(\mu(x, \sigma, p)) = \mu(x, \sigma, p + t)$$

for some $t \neq 0$.

If we consider $\mu_0 = \mu(0, 1, 0) = \frac{1}{2}\delta_{-e_1} + \frac{1}{2}\delta_{e_1}$, then

$$\Phi^t(\mu_0) = \frac{e^{-t}}{e^t + e^{-t}}\delta_{-e^te_1} + \frac{e^t}{e^t + e^{-t}}\delta_{e^{-t}e_1}.$$

We consider the Dirac mass $\nu = \delta_{e_2}$. Since Φ^t fixes Dirac masses, $\Phi^t(\nu) = \nu$, and

$$d_{\mathcal{W}_2}^2(\mu_0, \nu) = d_{\mathcal{W}_2}^2(\Phi^t(\mu_0), \Phi^t(\nu)) = d_{\mathcal{W}_2}^2(\Phi^t(\mu_0), \nu).$$

We can explicitly calculate both distances to get the following equality

$$2^{\frac{2}{q}} = d_{\mathcal{W}_2}^2(\mu_0, \nu) = d_{\mathcal{W}_2}^2(\Phi^t(\mu_0), \nu) = \frac{e^{-t}}{e^t + e^{-t}}(e^{tq} + 1)^{\frac{2}{q}} + \frac{e^t}{e^t + e^{-t}}(e^{-tq} + 1)^{\frac{2}{q}}.$$

We can rewrite this equality as

$$2^{\frac{2}{q}}(e^t + e^{-t}) = (e^{tq/2} + e^{-tq/2})^{\frac{2}{q}} + (e^{-tq/2} + e^{tq/2})^{\frac{2}{q}} = 2(e^{-tq/2} + e^{tq/2})^{\frac{2}{q}}$$

which is equivalent to

$$\frac{1}{2}(e^{qt/2})^{\frac{2}{q}} + \frac{1}{2}(e^{-qt/2})^{\frac{2}{q}} = \left(\frac{1}{2}e^{-tq/2} + \frac{1}{2}e^{tq/2}\right)^{\frac{2}{q}}.$$

Setting $A := e^{\frac{tq}{2}}$, the equation is written

$$\frac{1}{2}A^{\frac{2}{q}} + \left(\frac{1}{A}\right)^{\frac{2}{q}} = \left(\frac{1}{2}A + \frac{1}{2}\frac{1}{A}\right)^{\frac{2}{q}}.$$

But, since the function $s \rightarrow s^{\frac{2}{q}}$ is strictly concave when $q > 2$, the equality can only hold if $A = \frac{1}{A}$, that is, $e^{\frac{qt}{2}} = 1$, which only holds for $t = 0$, giving the contradiction.

Now assume by contradiction that the isometry Φ^* is given by

$$\Phi^* : \mu(x, \sigma, p) \rightarrow \mu(x, \sigma, -p),$$

and consider

$$\mu_1 = \mu(0, \sqrt{2}, -\ln(2)/2) = \frac{2}{3}\delta_{-e_1} + \frac{1}{3}\delta_{2e_1}.$$

Then the image of μ_1 is given by

$$\Phi^*(\mu_1) = \Phi^*\left(\mu(0, \sqrt{2}, -\ln(2)/2)\right) = \mu(0, \sqrt{2}, \ln(2)/2) = \frac{1}{3}\delta_{-2e_1} + \frac{2}{3}\delta_{e_1}.$$

We consider the Dirac mass $\nu = \delta_{e_1+e_2}$ and, since we still have $\Phi^*(\nu) = \nu$, we have

$$3d_{\mathcal{W}_2}^2(\mu_1, \nu) = 2(2^q + 1)^{2/q} + 2^{2/q} = 2 + (3^q + 1)^{2/q} = 3d_{\mathcal{W}_2}^2(\Phi^*(\mu_1), \nu)$$

We can rewrite the left side as

$$\frac{3}{4}d_{\mathcal{W}_2}^2(\mu_1, \nu) = \left(2^{-q/2}\right)^{2/q} (2^q + 2^0)^{2/q} + 2^{2/q-2} = \left(2^{q/2} + 2^{-q/2}\right)^{2/q} + 2^{2/q-2} \quad (4.25)$$

and the right side as

$$\begin{aligned} \frac{3}{4}d_{\mathcal{W}_2}^2(\Phi^*(\mu_1), \nu) &= 2^{-1} + \frac{1}{4}(3^q + 3^0)^{2/q} = 2^{-1} + (2^{-q})^{2/q}(3^q + 3^0)^{2/q} \\ &= 2^{-1} + \left(\left(\frac{3}{2}\right)^q + \left(\frac{1}{2}\right)^q\right)^{2/q} = 2^{-1} + \left(\left(\frac{9}{4}\right)^{q/2} + \left(\frac{1}{4}\right)^{q/2}\right)^{2/q}. \end{aligned} \quad (4.26)$$

Let us compare the right-hand sides of (4.25) and (4.26). Clearly, $2^{2/q-2} < 2^{-1}$ if $q > 2$, and the key observation is that there exists a $\lambda \in (0, 1)$ such that

$$2^{-1} = \lambda \cdot \frac{1}{4} + (1 - \lambda) \cdot \frac{9}{4} \text{ and } 2 = (1 - \lambda) \cdot \frac{1}{4} + \lambda \cdot \frac{9}{4}. \quad (4.27)$$

(The exact value of λ is $\lambda = 7/8$, but this is not important.) Therefore,

$$\begin{aligned} 2^{q/2} + 2^{-q/2} &= \left((1 - \lambda) \cdot \frac{1}{4} + \lambda \cdot \frac{9}{4}\right)^{q/2} + \left(\lambda \cdot \frac{1}{4} + (1 - \lambda) \cdot \frac{9}{4}\right)^{q/2} \\ &\leq (1 - \lambda) \cdot \left(\frac{1}{4}\right)^{q/2} + \lambda \cdot \left(\frac{9}{4}\right)^{q/2} + \lambda \cdot \left(\frac{1}{4}\right)^{q/2} + (1 - \lambda) \cdot \left(\frac{9}{4}\right)^{q/2} = \left(\frac{1}{4}\right)^{q/2} + \left(\frac{9}{4}\right)^{q/2} \end{aligned} \quad (4.28)$$

where we used the convexity of the function $s \mapsto s^{q/2}$ on $[0, \infty)$. The map $t \mapsto t^{2/q}$ is monotone increasing for any $q > 0$ on $[0, \infty)$, and hence (4.28) implies that the right-hand side of (4.25) is smaller than that of (4.26). That is, $d_{\mathcal{W}_2}(\mu_1, \nu) < d_{\mathcal{W}_2}(\Phi^*(\mu_1), \nu) = d_{\mathcal{W}_2}(\Phi^*(\mu_1), \Phi^*(\nu))$, which shows that Φ^* is not an isometry.

Note that if $q < 2$, then the inequality (4.28) is reversed by the concavity of $s \mapsto s^{q/2}$, and $2^{2/q-2} > 2^{-1}$, so in this case one gets $d_{\mathcal{W}_2}(\mu_1, \nu) > d_{\mathcal{W}_2}(\Phi^*(\mu_1), \nu) = d_{\mathcal{W}_2}(\Phi^*(\mu_1), \Phi^*(\nu))$ which rules out Φ^* in the case $q < 2$ as well.

These proofs show that, if Φ is an isometry that fixes Dirac masses and sends measures supported on L_1 to measures supported on L_1 , then Φ acts as the identity on the space $\mathcal{W}_2(L_1, |\cdot|)$, that is, $\Phi(\mu) = \mu$ for any measure μ supported on L_1 . Using this result, we now prove Theorem 1.2:

Proof of Theorem 1.2. We again start with the case $n = 2$. By Proposition 4.7, we know that if a measure is supported on a translated canonical axis and Φ is an isometry of $\mathcal{W}_2(\mathbb{R}^2, d_q)$, then $\Phi(\nu)$ is supported on the same translated canonical axis. By the argument presented above, we can also

assume that $\Phi(\nu) = \nu$ for any measure supported on $x_0 + L_i$, with $x_0 \in \mathbb{R}^2$, $i \in \{1, 2\}$. Since the norm l_q is strictly convex, as $q > 2$, it projects uniquely onto L_1 and L_2 ; furthermore, it is easy to verify that the projection of L_2 onto L_1 is simply $P_{L_1}(L_2) = \{0\}$. Thus we can apply Proposition 3.1 to prove the rigidity of the Wasserstein space $\mathcal{W}_2(\mathbb{R}^2, d_q)$.

We now make the induction assumption on n , that for any $n \leq n_0$, the space $\mathcal{W}_2(\mathbb{R}^n, d_q)$ is rigid, and we consider the Wasserstein space $\mathcal{W}_p(\mathbb{R}^{n_0+1}, d_q)$.

By Proposition 4.7, if a measure ν is supported on $x_0 + L_i$ for $x_0 \in \mathbb{R}^{n_0+1}$, $i \in \{1, \dots, n_0 + 1\}$ (where L_i is the hyperplane given by $x_i = 0$), then $\Phi(\nu)$ will also be supported on $x_0 + L_i$. Since the space $(x_0 + L_i, d_{q, x_0 + L_i})$ is isometric to $(\mathbb{R}^{n_0}, d_q)$, we can use the induction assumption to show that $\Phi(\nu) = \nu$ for any measure ν supported on $x_0 + L_i$.

Since the line $H_i = \{te_i\}$ is the intersection of all hyperplanes L_j with $j \neq i$, if a measure μ is supported on H_i , we can use Proposition 4.7 to show that its image $\Phi(\mu)$ is also supported on every hyperplane L_j with $j \neq i$. Thus its image is again supported on H_i , and by the argument presented above, we can assume that $\Phi(\mu) = \mu$ for any measure μ supported on H_i . Since we still have that the l_q norm projects uniquely onto L_i and H_i , and since it is easy to see that $P_{L_i}^{-1}(0) = H_i$, we can apply Proposition 3.1 to prove the rigidity of the Wasserstein space $\mathcal{W}_2(\mathbb{R}^{n_0+1}, d_q)$. $\square$

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