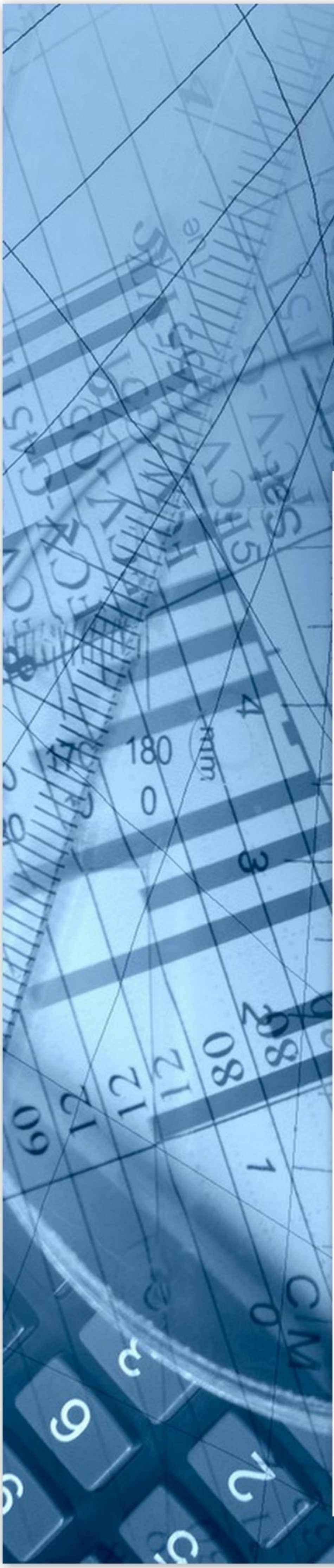

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Weighted nucleoli and dually essential coalitions

by
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Weighted nucleoli and dually essential coalitions *

Tamás Solymosi †

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Abstract We consider linearly weighted versions of the least core and the (pre)nucleolus and investigate the reduction possibilities in their computation. We slightly extend some well-known related results and establish their counterparts by using the dual game. Our main results imply, for example, that if the core of the game is not empty, all dually inessential coalitions (which can be weakly minorized by a partition in the dual game) can be ignored when we compute the per-capita least core and the per-capita (pre)nucleolus from the dual game. This could lead to the design of polynomial time algorithms for the per-capita (and other monotone nondecreasingly weighted versions of the) least core and the (pre)nucleolus in specific classes of balanced games with polynomial many dually esential coalitions.

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1 Introduction

The nucleolus (Schmeidler, 1969) is one of the major single-valued solution concepts for transferable utility cooperative games. It seemingly depends on all coalitional values, but a closer look reveals the inherent high redundancy in its definition. Indeed, as Brune (1983), and later Reijnierse and Potters (1998) have proved: in any n-player game there are at most $2n-2$ coalitions which actually determine the nucleolus. Unfortunately, the identification of these nucleolus-defining coalitions is no less laborious as computing the nucleolus itself. On the other hand, if special properties of the game enable us to specify a priori a polynomial size characterization family of coalitions for the nucleolus, then we can compute it in polynomial time (in the number of players).

There are several classes of balanced games for which polynomial time nucleolus algorithms are available in the literaure. The key to the efficiency of some of these algorithms (e.g. in case of assignment games) is Huberman's (1980) theorem stating that in a balanced game all nucleolus-defining coalitions are essential (cannot be weakly majorized by a partition) in the game, hence the inessential ones can be ignored. Although typically not explicitly mentioned, but several other known polytime algorithms (e.g. in case of fixed-tree games) rely on the dual counterpart of Huberman's result: in computing the nucleolus of a balanced game, all dually inessential coalitions (which can be weakly minorized by a partition in the dual game) can be ignored.

Our aim is to investigate what kinds of weighted versions of the nucleolus can also be computed by taking into account only coalitions in these families. Since the above mentioned reducibility results require nonemptiness of the core, our domain will also be the class of balanced games. We are mostly concerned about the per-capita nucleolus (Grotte, 1970, 1972), so we restrict our study to linear weight systems. On the other hand, we allow weights that depend not only on the size, but also on the value of the coalitions. In particular, we allow the weight of a coalition to be equal to its value (provided it is positive), thus some of our results also apply to the proportional nucleolus of a balanced game (with only positive coalitional values).

The nucleolus is based on the coalitional surplusses (the difference between the payoff to and the value of the coalition). This measure, however, does not take into account neither the size, nor the value (or any other characteristic that maybe important for an application) of the coalitions. Various weighted nucleoli (based on weighted surplus measures) were considered by several authors, but mostly from an axiomatization point of view, see e.g. (Derks and Peters, 1998), (Derks and Haller, 1999), (Kleppe, 2010), (Kleppe et al., 2016). We address issues in connection with their computation.

In general, a linearly weighted nucleolus can be determined by the very same methods as the (standard) nucleolus, only straightforward adjustments are needed that only negligibly effect the performance. This is particularly true for the most frequently applied sequential linear programming approach pioneered by Kopelowitz (1967) (for a recent implementation finely tuned even for large games, see (Nguyen and Thomas, 2016)). On the other hand, and in contrast to the rich literature on the computation of the (standard) nucleolus in specific classes of games, we can only mention the algorithm by Huijink et al. (2015) that computes the per-capita nucleolus in bankruptcy games. One of our results might shed light on a possible reason for this phenomenon. We demonstrate (in Example 3) that the family of essential coalitions is not sufficient to determine the per-capita nucleolus, not even in a balanced game, so Huberman's (1980) reducibility result cannot be used in the computation of the per-capita nucleolus.

We find, however, that if we compute the nucleolus of a balanced game from the dual coalitional values, Huberman's idea works, not just for the (standard) nucleolus (that is implicitly the basis for many known efficint algorithms), but also for the per-capita and other monotone nondecreasingly weighted nucleoli. We prove (in Theorem 4) that if the core of the game is not empty, all dually inessential coalitions (those which can be weakly minorized by a partition in the dual game) can be ignored when we compute the per-capita (or other monotone nondecreasingly weighted versions of the) nucleolus from the dual game. We believe that this observation could become the theoretical basis for various polynomial time algorithms (yet to be) designed for the per-capita nucleolus in specific classes of balanced games known to have polynomial many dually essential coalitions (e.g. assignment games, fixed-tree games). Other candidates for this endeavour might be the well-known classes of games whose duality relations are discussed by Oishi and Nakayama (2009). The usefulness of looking at the dual games also in the axiomatizations of solutions is underlined by Oishi et al. (2016).

The organization of the paper is as follows. We collect the necessary general preliminaries and introduce the linear weight systems in the next section. In section 3, we discuss weighted least cores, since computing them is the first step in finding the weighted (pre)nucleoli. We present properties of the weight function under which the family of essential coalitions is sufficient to determine the weighted least core, and also when the family of dually essential coalitions is sufficient to determine the weighted least core in the dual game of a balanced game. In section 4, we present the weighted primal and dual versions of the lexicographic center procedure (Maschler, Peleg, Shapley, 1979) that sequentially reduces the set of allowable payoffs until it shrinks to the (pre)nucleolus allocation, and discuss which properties of the weight system make the inessential coalitions, or in the dual version the dually inessential coalitions redundant in these sequential optimization processes when applied to balanced games.

2 Preliminaries

A transferable utility cooperative game on the non-empty finite *set N of players* is defined by a *coalitional function* $v : 2^N \longrightarrow \mathbb{R}$ that satisfies $v(\emptyset) = 0$. The function v specifies the worth of every *coalition* $S \subseteq N$. We shall denote by

$$\mathcal{N} = \{S \subseteq N : S \neq \emptyset, N\}$$

the collection of *non-trivial* coalitions. The player set N will be fixed throughout the paper, so we drop it from the notation and refer to v as the game. The game v is called *superadditive*, if $S \cap T = \emptyset$ implies $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$; and *subadditive*, if its negative $-v$ is superadditive.

Given a game v, a *payoff vector* $x \in \mathbb{R}^N$ is called *efficient*, if $x(N) = v(N)$; *coalitionally rational*, if $x(S) \geq v(S)$ for all $S \subseteq N$; where, by the standard notation, $x(S) = \sum_{i \in S} x_i$ if $S \neq \emptyset$, and $x(\emptyset) = 0$. We denote by $\mathbf{Ef}(v)$ the set of efficient payoff vectors called the *preimputation set*, and by $\mathbf{Co}(v)$ the set of efficient and coalitionally rational payoff vectors called the *core* of the game v. Games with a non-empty core are called *balanced*.

The *excess* $e(S, x, v) = v(S) - x(S)$ is the usual measure of gain (or loss if negative) to coalition $S \subseteq N$ in game v if its members depart from allocation $x \in \mathbb{R}^N$ in order to form their own coalition. Note that in any game v, $e(\emptyset, x, v) = 0$ for all $x \in \mathbb{R}^N$, and the core is the set of efficient allocations which yield non-positive excess for all non-trivial coalitions. It will be more convenient to use the negative excess $f(S, x, v) := -e(S, x, v)$, we call it the *surplus* of coalition S at allocation x in game v.

The *dual game* (N, v^*) of game (N, v) is defined by $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$. Notice that $v^*(\emptyset) = 0$, so v^* is indeed a game, and $v^*(N) = v(N)$, so $\mathbf{Ef}(v^*) = \mathbf{Ef}(v)$ for any game v. The name dual is explained by the relation $v^{**}(S) = v(S)$ for all $S \subseteq N$.

Since $N \setminus S \in \mathcal{N}$ for each $S \in \mathcal{N}$, and

$$f(S, x, v) = -f(N \setminus S, x, v^*) \quad \text{for all } x \in \mathbf{Ef}(v) = \mathbf{Ef}(v^*), \tag{1}$$

the core of a game coincides with the *anticore* (where the inequalities are reversed) of its dual game, that is,

$$\mathbf{Co}(v) = \mathbf{Co}^*(v^*) := \{x \in \mathbf{Ef}(v^*) : f(T, x, v^*) \leq 0 \ \ \forall T \in \mathcal{N}\}. \tag{2}$$

We call (2) the dual description of the core.

We will investigate which families of non-trivial coalitions are sufficient to determine a solution in a game and which coalitions are redundant. Two types of coalitions will be considered.

Coalition $S \subseteq N$ is called *inessential* in game (N, v), if its value can be weakly majorized by a proper partition, i.e. if $v(S) \leq v(S_1) + \ldots + v(S_k)$ for some partition

$S = S_1 \cup \ldots \cup S_k$ with $k \geq 2$. A coalition is *essential* in a game if it is not inessential. Observe that an inessential coalition has a weakly majorizing partition consisting only of essential coalitions. Notice that all 1-player coalitions are essential in any game. We denote by $\mathcal{E}(v) \subseteq \mathcal{N}$ the family of essential coalitions in game v. It is straightforward that all inessential coalitions are redundant for the core, i.e.

$$\mathbf{Co}(v) = \mathbf{Co}(\mathcal{E}(v), v) := \{x \in \mathbf{Ef}(v) : f(T, x, v) \geq 0 \ \ \forall T \in \mathcal{E}(v)\} . \tag{3}$$

Observe that the core $\mathbf{Co}(v) = \mathbf{Co}(\mathcal{N}, v)$ is described by $1 + |\mathcal{N}| = 2^{|N|} - 1$ linear constraints but in the restricted description $\mathbf{Co}(\mathcal{E}(v), v)$ the number of linear constraints is $1 + |\mathcal{E}(v)|$ that could be significantly smaller than $2^{|N|} - 1$.

The dual description (2) of the core also has a reduced form. Coalition $S \subseteq N$ is called *dually inessential* in game (N, v), if it is anti-inessential in the dual game, i.e. it has a proper partition $S = S_1 \cup \ldots \cup S_k$ with $k \geq 2$ such that $v^*(S_1) + \ldots + v^*(S_k) \leq v^*(S)$. A coalition is *dually essential* in a game if it is not dually inessential. Observe that a dually inessential coalition has a minorizing partition in the dual game that consists only of dually essential coalitions. Notice that all 1-player coalitions are always dually essential. We denote by $\mathcal{E}^*(v^*) \subseteq \mathcal{N}$ the family of dually essential coalitions. It is straightforward that all dually inessential coalitions are redundant for the core, i.e.

$$\mathbf{Co}(v) = \mathbf{Co}^*(\mathcal{E}^*(v^*), v^*) := \{x \in \mathbf{Ef}(v^*) : f(T, x, v^*) \leq 0 \ \ \forall T \in \mathcal{E}^*(v^*)\} . \tag{4}$$

The above remark on the significant reduction possibility in the size of the (dual) core description applies here too.

The standard surplus (excess) does not take into account neither the size, nor the value (or any other characteristic that maybe important for an application) of the coalitions. More general excess functions were considered by several authors, but we restrict ourselves to the weighted versions that preserve the linearity of the measure with respect to the payoff variables.

In the sequel we assign a (maybe coalition specific) positive *weight* $q(S) > 0$ to each non-trivial coalition $S \in \mathcal{N}$, and define the *q-weighted surplus* (q-surplus for short) of non-trivial coalition $S \in \mathcal{N}$ at allocation $x \in \mathbb{R}^N$ in game v to be

$$f_q(S, x, v) = \frac{x(S) - v(S)}{q(S)} \ \ \forall S \in \mathcal{N}. \tag{5}$$

Note that no matter which system $\{q(S) > 0 : S \in \mathcal{N}\}$ of weights is used,

$$\mathbf{Co}(v) = \{x \in \mathbf{Ef}(v) \ : \ f_q(S, x, v) \geq 0 \ \ \forall S \in \mathcal{N}\},$$

i.e., the core is the set of efficient allocations which yield non-negative q-surplus for all non-trivial coalitions.

We say that a weight function is *subadditive*, if $S \cap T = \emptyset$ implies $q(S) + q(T) \geq q(S \cup T)$ for all $S, T \in \mathcal{N}$; *superadditive*, if the inequality is reversed; *additive*, if both subadditive and superadditive; *monotone nondecreasing*, if $S \subset T$ implies $q(S) \leq q(T)$ for all $S, T \in \mathcal{N}$; and *monotone nonincreasing*, if the inequality is reversed.

We consider two surplus-based solutions: the least core and the (pre)nucleolus. The weighted versions of both solutions (formally defined later) are obtained if we replace the standard surplus measure f with the weighted surplus f_q in their respective definitions. As special cases we get

- the (standard) least core and nucleolus, if we take the monotone and subadditive (but not superadditive) weight function $q(S) = 1$ for all $S \in \mathcal{N}$;
- the per-capita least core and nucleolus, if we take the monotone and additive weight function $q(S) = |S|$ for all $S \in \mathcal{N}$;
- for positive-valued game v (i.e. $v(S) > 0$ for all $S \neq \emptyset$), the proportional least core and nucleolus, if we take the weight function $q(S) = v(S)$ for all $S \in \mathcal{N}$.

3 Weighted least cores

The *least core* $\mathbf{LC}(v)$ of a game v was first formally treated by Maschler, Peleg and Shapley (1979) as the set of all efficient allocations that maximize the minimum surplus of non-trivial coalitions, i.e.,

$$\mathbf{LC}(v) := \arg \max_{x \in \mathbf{Ef}(v)} \min_{S \in \mathcal{N}} f(S, x, v).$$

Recall that in any game the least core is a non-empty polytope.

Given a positive weight function q, the q-weighted least core $\mathbf{LC}_q(v)$ (q-least core for short) is defined analogously as the set of all efficient allocations that maximize the minimum q-surplus of non-trivial coalitions, i.e.,

$$\begin{array}{rcl} \alpha_q^1(v) & := & \max_{x \in \mathbf{Ef}(v)} \min_{S \in \mathcal{N}} f_q(S, x, v) \\ \mathbf{LC}_q(v) & := & \{x \in \mathbf{Ef}(v) : f_q(S, x, v) \geq \alpha_q^1(v) \ \ \forall\, S \in \mathcal{N}\}. \end{array} \tag{6}$$

Observe that for any game v and weight function q, the uniformly guaranteed q-surplus level $\alpha_q^1(v)$ is well defined, the q-least core is a non-empty polytope, and $\mathbf{Co}(v) \neq \emptyset$ if and only if $\alpha_q^1(v) \geq 0$. In a balanced game v, $\mathbf{LC}_q(v) \subseteq \mathbf{Co}(v)$, and $\mathbf{LC}_q(v) = \mathbf{Co}(v)$ if and only if $\alpha_q^1(v) = 0$.

The linearity of the q-surplus in the payoff variables allows us to compute $\alpha_q^1(v)$ with the following LP with all variables $x \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$ unrestricted in sign:

$$\begin{array}{lll} & \alpha \to \max & \\ \hline x(N) & = v(N) & \\ x(S) - q(S)\alpha & \geq v(S) & \forall\, S \in \mathcal{N} \\ \hline \end{array} \tag{7}$$

Clearly, this LP has optimal solution(s), its optimum value equals to $\alpha_q^1(v)$, and its optimal solutions are of the form $(x, \alpha_q^1(v))$ with some q-weighted least core payoff vector $x \in \mathbf{LC}_q(v)$.

Since $v(N) = v^*(N)$ and $\mathcal{N}$ also contains $N \setminus S$ for each $S \in \mathcal{N}$, if we subtract from the efficiency equation the inequalities related to the subcoalitions and reverse the direction of optimization by substituting $\alpha = -\beta$, we get an equivalent LP in terms of the dual game:

$$\begin{array}{lll} & \beta \to \min & \\ \hline x(N) & = v^*(N) & \\ x(T) - q(N \setminus T)\beta & \leq v^*(T) & \forall T \in \mathcal{N} \\ \hline \end{array} \tag{8}$$

Notice that unless $q(T) = q(N \setminus T)$ for all $T \in \mathcal{N}$, the inequalities in (8) can not be expressed in terms of the q-surplus in the dual game, since in the inequality related to T variable β is multiplied by the weight $q(N \setminus T)$ of the complement coalition. Thus, unlike for the core, the q-least core of a game typically can not be obtained by simply reversing the inequalities in the definition of the q-weighted least core of the dual game. Since the general weighted version of relation (1) is

$$q(S)f_q(S, x, v) = -q(N \setminus S)f_q(N \setminus S, x, v^*) \quad \text{for all } x \in \mathbf{Ef}(v) = \mathbf{Ef}(v^*), \tag{9}$$

and $f(S, x, v) = q(S)f_q(S, x, v)$, we introduce a transformed version of the weighted surplus in the dual game:

$$g_q(T, x, v^*) := \frac{f(T, x, v^*)}{q(N \setminus T)} = \frac{x(T)}{q(N \setminus T)} - \frac{v^*(T)}{q(N \setminus T)}.$$

Then the dual description of the q-weighted least core is

$$\begin{array}{rcl} \beta_q^1(v^*) & := & \min_{x \in \mathbf{Ef}(v^*)} \max_{T \in \mathcal{N}} g_q(T, x, v^*) \\ \mathbf{LC}_q^*(v^*) & := & \{x \in \mathbf{Ef}(v^*) : g_q(T, x, v^*) \leq \beta_q^1(v^*) \;\; \forall T \in \mathcal{N}\}. \end{array} \tag{10}$$

Clearly, $\beta_q^1(v^*) = -\alpha_q^1(v)$. Observe that for the standard least core $\mathbf{LC}(v)$ (when $q(S) = 1$ for all $S \in \mathcal{N}$) the dual description simplifies to

$$\mathbf{LC}^*(v^*) = \arg \min_{x \in \mathbf{Ef}(v^*)} \max_{S \in \mathcal{N}} f(S, x, v^*),$$

that is a straightforward counterpart of its definition.

The following characterizations of weighted least-core allocations in terms of balanced collections can be easily obtained by standard LP duality arguments applied to the LP descriptions (7) or (8).

Proposition 1. *An efficient payoff allocation x belongs to the q-weighted least core of game v if and only if the family of non-trivial coalitions that satisfy either type of the following two properties contains a (minimal) balanced collection*

1. *at x, the coalition minimizes $f_q(S, x, v)$ over all coalitions $S \in \mathcal{N}$ in game v;*

2. *at x, the coalition maximizes $g_q(T, x, v^*)$ over all coalitions $T \in \mathcal{N}$ in the dual game v^*.*

For the standard least core, the first type of characterization (in terms of v) is well-known (cf. e.g. Peleg and Sudhölter, 2003, p.183).

There is a close relationship between the two types of (minimal) balanced collection(s) mentioned in Proposition 1. If, at a q-weighted least core allocation x, we replace all coalitions of a (minimal) balanced collection contained in the family $\mathcal{S}^1(x)$ of coalitions with minimum q-surplus $f_q(S, x, v)$ with their complements, we get a (minimal) balanced collection contained in the family $\mathcal{T}^1(x)$ of coalitions with maximum transformed dual q-surplus $g_q(N \setminus S, x, v^*)$, and vice versa.

We now identify families of redundant coalitions for weighted least cores.

Theorem 1. 1. *In a balanced game v, all inessential coalitions are redundant for $\mathbf{LC}_q(v)$ with a subadditive weight function q. In particular, for the standard least core $\mathbf{LC}(v) = \mathbf{LC}(\mathcal{E}(v), v)$, and for the per-capita least core $\mathbf{LC}_{pc}(v) = \mathbf{LC}_{pc}(\mathcal{E}(v), v)$.*

2. *In a non-balanced game v, all inessential coalitions are redundant for $\mathbf{LC}_q(v)$ with a superadditive weight function q. In particular, for the per-capita least core $\mathbf{LC}_{pc}(v) = \mathbf{LC}_{pc}(\mathcal{E}(v), v)$.*

3. *In any game v, all inessential coalitions are redundant for $\mathbf{LC}_q(v)$ with an additive weight function q. In particular, for the per-capita least core $\mathbf{LC}_{pc}(v) = \mathbf{LC}_{pc}(\mathcal{E}(v), v)$.*

Proof. For all three claims, let $S \in \mathcal{N} \setminus \mathcal{E}(v)$ be inessential in game v, because of the partition $S = S_1 \cup S_2$ with $S_1, S_2 \in \mathcal{E}(v)$ and $v(S) \leq v(S_1) + v(S_2)$. For simplicity of notation, we assume (without loss of generality) that the weakly majorizing partition consists only of $k = 2$ subcoalitions. Then at any $x \in \mathbb{R}^N$, we have the inequalities

$$\begin{array}{cccc} v(S_1) & + & q(S_1)\alpha & \leq x(S_1) \\ v(S_2) & + & q(S_2)\alpha & \leq x(S_2) \\ \hline v(S_1) + v(S_2) & + & [q(S_1) + q(S_2)]\alpha & \leq x(S) \\ \hline v(S) & + & q(S)\alpha & \leq x(S) \end{array} \tag{11}$$

where the third one is the sum of the first two. By the above assumption, $v(S) \leq v(S_1) + v(S_2)$, so the third inequality implies the last one, hence that is redundant for the system in (7), if $[q(S_1) + q(S_2)]\alpha \geq q(S)\alpha$. This condition clearly holds, if

1. $\alpha \geq 0$ (i.e. v is balanced) and q is a subadditive weight function.

2. $\alpha < 0$ (i.e. v is not balanced) and q is a superadditive weight function.

3. q is an additive weight function.

Since $S_1, S_2 \in \mathcal{E}(v)$, the above argument can be independently done for any inessential $S \in \mathcal{N} \setminus \mathcal{E}(v)$. The claims for the special least cores follow from the properties of their respective weight functions. □

As the following example demonstrates, the second and third statements in Theorem 1 are not true for the standard least core **LC**.

Example 1. Consider the following game on player set $N = \{1, 2, 3, 4\}$ given by $v(N) = v(14) = v(24) = v(124) = v(134) = v(234) = 18$, $v(34) = 12$, $v(12) = v(123) = 6$, and $v(R) = 0$ for all other coalitions $R \in \mathcal{N}$.

It is easily checked that v is superadditive, but not balanced, e.g. $\frac{1}{2}v(12) + \frac{1}{2}v(134) + \frac{1}{2}v(234) = 21 > 18 = v(N)$. The maximum uniformly guaranteed surplus is $\alpha^1(v) = -2$, the standard least core is a singleton $\mathbf{LC}(v) = \{x = (2, 2, 0, 14)\}$. Indeed, at allocation x, the family of coalitions with smallest surplus $(= -2)$ is $\mathcal{S}^1(x) = \{12, 14, 24, \ 123, 134, 234\}$ that is the union of the (minimal) balanced collections $\{12, 134, 234\}$, $\{14, 123, 234\}$, and $\{24, 123, 134\}$, so by Proposition 1, $x \in \mathbf{LC}(v)$. The uniqueness of this least core allocation comes from the "full rank nature" of $\mathcal{S}^1(x)$.

On the other hand, if we take into account only the essential coalitions $\mathcal{E}(v) = \{1, 2, 3, 4, 12, 14, 24, 34\}$ in (6), we get another uniform surplus level $\alpha^1(\mathcal{E}(v), v) = -\frac{6}{5}$, and another (singleton) least core $\mathbf{LC}(\mathcal{E}(v), v) = \{y = (\frac{12}{5}, \frac{12}{5}, -\frac{6}{5}, \frac{72}{5})\}$. Indeed, at allocation y, the family of essential coalitions with smallest surplus $(= -\frac{6}{5})$ is $\mathcal{S}^1(\mathcal{E}(v), y) = \{3, 12, 14, 24\}$ that is itself a (minimal) balanced collection of "full rank", so by Proposition 1, $y \in \mathbf{LC}(\mathcal{E}(v), v)$, and y is the unique $\mathcal{E}(v)$-restricted least core allocation.

In contrast, and as an illustration of the second statement in Theorem 1, the uniformly guaranteed per-capita surplus is $\alpha^1_{pc}(v) = -\frac{3}{4}$, the (singleton) per-capita least core is $\mathbf{LC}_{pc}(v) = \{z = (\frac{9}{4}, \frac{9}{4}, -\frac{3}{4}, \frac{57}{4})\}$. Indeed, at allocation z, the family of coalitions with smallest per-capita surplus $(= -\frac{3}{4})$ is $\mathcal{S}^1_{pc}(z) = \{3, 12, 14, 24, 123, 134, 234\}$ that is the union of the balanced collections $\{3, 12, 14, 24\}$ and the above $\mathcal{S}^1(x)$. so by Proposition 1, $z \in \mathbf{LC}_{pc}(v)$. Since $\{3, 12, 14, 24\} \subset \mathcal{E}(v)$ and it is itself a "full rank" (minimal) balanced collection, the restriction to the family of essential coalitions gives the same $\alpha^1_{pc}(\mathcal{E}(v), v) = -\frac{3}{4}$ and (singleton) least core $\mathbf{LC}_{pc}(\mathcal{E}(v), v) = \{(\frac{9}{4}, \frac{9}{4}, -\frac{3}{4}, \frac{57}{4})\}$ as in the unrestricted case. ——— ◇

Let us see redundant coalitions in the dual descriptions of weighted least cores.

Theorem 2. *In a balanced game v, all dually inessential coalitions are redundant for $\mathbf{LC}_q(v) = \mathbf{LC}^*_q(v^*)$ with a monotone nondecreasing weight function q. In particular, for the standard least core $\mathbf{LC}(v) = \mathbf{LC}^*(\mathcal{E}^*(v^*), v^*)$, and for the per-capita least core $\mathbf{LC}_{pc}(v) = \mathbf{LC}^*_{pc}(\mathcal{E}^*(v^*), v^*)$.*

Proof. Let $T \in \mathcal{N} \setminus \mathcal{E}^*(v^*)$ be dually inessential, because of the partition $T = T_1 \cup T_2$ with $T_1, T_2 \in \mathcal{E}^*(v^*)$ and $v^*(T_1) + v^*(T_2) \leq v^*(T)$. For simplicity of notation, we assume (without loss of generality) that the weakly minorizing partition consists only of $k = 2$ subcoalitions. Then at any $x \in \mathbb{R}^N$, we have the inequalities

$$\begin{array}{rcccc} x(T_1) \leq & v^*(T_1) & + & q(N \setminus T_1)\beta \\ x(T_2) \leq & v^*(T_2) & + & q(N \setminus T_2)\beta \\ \hline x(T) \leq & v^*(T_1) + v^*(T_2) & + & [q(N \setminus T_1) + q(N \setminus T_2)]\beta \\ \hline x(T) \leq & v^*(T) & + & q(N \setminus T)\beta \end{array} \qquad (12)$$

where the third one is the sum of the first two. By the above assumption, $v^*(T_1) + v^*(T_2) \leq v^*(T)$, so the third inequality implies the last one, hence that is redundant for the system in (8), if $[q(N \setminus T_1) + q(N \setminus T_2)]\beta \leq q(N \setminus T)\beta$. This condition clearly holds if $\beta = -\alpha \leq 0$ (i.e. v is balanced) and the weight function q is monotone nondecreasing, because then $(N \setminus T_1) \cap (N \setminus T_2) = N \setminus T \neq \emptyset$ implies $q(N \setminus T) \leq \min\{q(N \setminus T_1), q(N \setminus T_2)\} \leq q(N \setminus T_1) + q(N \setminus T_2)$.

Since $T_1, T_2 \in \mathcal{E}^*(v^*)$, the above argument can be independently done for any dually inessential $T \in \mathcal{N} \setminus \mathcal{E}^*(v^*)$. The claims for the special least cores follow from the fact that their respective weight functions are monotone nondecreasing. □

As the following example demonstrates, balancedness of the game in Theorem 2 is needed for both the standard least core $\mathbf{LC}$ and the per-capita least core $\mathbf{LC}_{pc}$.

Example 2. Consider the dual game v^* of the 4-player non-balanced game v in Example 1: $v^*(N) = v^*(14) = v^*(24) = v^*(123) = v^*(124) = v^*(134) = v^*(234) = 18$, $v^*(4) = v^*(34) = 12$, $v^*(12) = 6$, and $v^*(R) = 0$ for all other coalitions $R \in \mathcal{N}$.

In the dual description (10) for the standard least core, the minimum uniformly guaranteed transformed dual surplus is $\beta^1(v^*) = 2$, and the set of optimal solutions is the singleton $\mathbf{LC}^*(v^*) = \{x = (2, 2, 0, 14)\}$, that is, of course, the same as $\mathbf{LC}(v)$ in Example 1. We can also check it directly by the second characterization in Proposition 1. At allocation x, the family of coalitions with largest transformed dual surplus ($= 2$) is $\mathcal{T}^1(x) = \{1, 2, 4, 13, 23, 34\}$ that is the union of the partitions $\{1, 2, 34\}$, $\{1, 4, 23\}$, and $\{2, 4, 13\}$, so $x \in \mathbf{LC}^*(v^*)$ indeed. The uniqueness comes from the "full rank nature" of $\mathcal{T}^1(x)$. Notice that $\mathcal{T}^1(x)$ consists of the complements of the coalitions in $\mathcal{S}^1(x)$ in Example 1 and $\beta^1(v^*) = -\alpha^1(v)$.

On the other hand, if we take into account only the dually essential coalitions $\mathcal{E}^*(v^*) = \{1,2,3,4\}$ in (10), we get another uniform transformed dual surplus level $\beta^1(\mathcal{E}^*(v^*), v^*) = \frac{3}{2}$, and another (singleton) optimal solution set $\mathbf{LC}^*(\mathcal{E}^*(v^*), v^*) = \{s = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{27}{2})\}$. At this allocation, $\mathcal{T}^1(\mathcal{E}^*(v^*), s) = \{1,2,3,4\}$ that is itself a partition of "full rank", so by Proposition 1, $s \in \mathbf{LC}^*(\mathcal{E}^*(v^*), v^*)$, and s is the unique such allocation.

In the dual description (10) for the per-capita least core, the minimum uniformly guaranteed transformed dual surplus is $\beta^1_{pc}(v^*) = \frac{3}{4}$, and the set of optimal solutions is the singleton $\mathbf{LC}^*_{pc}(v^*) = \{z = (\frac{9}{4}, \frac{9}{4}, -\frac{3}{4}, \frac{57}{4})\}$, that is, of course, the same as $\mathbf{LC}_{pc}(v)$ in Example 1. We can also confirm this by the second characterization in Proposition 1. Indeed, at allocation z, the family of coalitions with largest transformed dual per-capita surplus ($= \frac{3}{4}$) is $\mathcal{T}^1_{pc}(z) = \{1,2,4,13,23,34,124\}$ that is the union of the partitions $\{1,2,34\}$, $\{1,4,23\}$, $\{2,4,13\}$, and the (minimal) balanced collection $\{13,23,34,124\}$, so $t \in \mathbf{LC}^*_{pc}(v^*)$ indeed. The uniqueness comes again from the "full rank nature" of $\mathcal{T}^1_{pc}(z)$. Notice also here that $\mathcal{T}^1_{pc}(z)$ consists of the complements of the coalitions in $\mathcal{S}^1_{pc}(z)$ in Example 1 and $\beta^1_{pc}(v^*) = -\alpha^1_{pc}(v)$.

On the other hand, since only the single-player coalitions are dually essential and $g_{pc}(k,.,v^*) = \frac{f(k,.,v^*)}{3} = \frac{1}{3}g(k,.,v^*)$ for each $k \in N$, the $\mathcal{E}^*(v^*)$-restricted optimization in the per-capita case gives the same set of optimal solutions as in the standard case. Thus, $\mathbf{LC}^*_{pc}(\mathcal{E}^*(v^*), v^*) = \{s = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{27}{2})\}$. Only the optimum value is scaled $\beta^1_{pc}(\mathcal{E}^*(v^*), v^*) = \frac{1}{2} = \frac{1}{3}\beta^1(\mathcal{E}^*(v^*), v^*)$. At this allocation, $\mathcal{T}^1_{pc}(\mathcal{E}^*(v^*), s) = \{1,2,3,4\}$ that is itself a partition of "full rank", so by Proposition 1, $s \in \mathbf{LC}^*_{pc}(\mathcal{E}^*(v^*), v^*)$, and s is the unique such allocation. ——— $\diamond$

4 Weighted nucleoli

The (pre)nucleolus (Schmeidler, 1969) is a non-empty set of (pre)imputations that consists of a single element, called the (pre)nucleolus allocation. The following alternative definition (Maschler, Peleg, Shapley, 1979) will serve us better here.

For game (N, v) and weight function q, the q-weighted prenucleolus $\mathbf{Nu}_q(v)$

(q-prenucleolus for short) is defined as the outcome of the following procedure:

Let $X^0 := \mathbf{Ef}(v)$ and $\Sigma^0 := \mathcal{N}, \Delta^0 := \{N\}$.
For $r = 1, \ldots, \varrho$ define recursively

$$\left| \begin{array}{ll} \alpha_q^r := & \max_{x \in X^{r-1}} \min_{S \in \Sigma^{r-1}} f_q(S, x, v), \\ X^r := & \{x \in X^{r-1} : \min_{S \in \Sigma^{r-1}} f_q(S, x, v) = \alpha_q^r\}, \\ \Delta_r := & \{S \in \Sigma^{r-1} : \max_{x \in X^r} f_q(S, x, v) = \alpha_q^r\}, \\ \Sigma^r := & \Sigma^{r-1} \setminus \Delta_r, \quad \Delta^r := \Delta^{r-1} \cup \Delta_r \end{array} \right. \tag{13}$$

where ϱ is the first value of r for which $\Sigma^r = \emptyset$.

The final set X^ϱ is the *q-prenucleolus* $\mathbf{Nu}_q(v)$ of game v. We refer to the unique vector η_q in X^ϱ as the *q-prenucleolus-allocation.*

By straightforward adjustments of the arguments given by Maschler, Peleg, and Shapley (1979) one can easily see that

- ϱ is well defined and finite;
- $\alpha_q^1(v) = \alpha_q^1 < \alpha_q^2 < \ldots < \alpha_q^\varrho$ are well defined;
- $\mathbf{LC}_q(v) = X^1 \supseteq X^2 \supseteq \ldots \supseteq X^\varrho$ are non-empty polytopes;
- $\Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_\varrho$ forms a partition of $\mathcal{N}$,
- if $S \in \Delta_k$ then $v(S) + q(S)\alpha_q^k = \eta_q(S)$,

where $\alpha_q^1(v)$ and $\mathbf{LC}_q(v)$ are defined in (6). Notice the difference between $\alpha_q^1(v)$, a characteristic of the game, and α_q^1, a number determined by the algorithm (13).

We now identify families of redundant coalitions for weighted prenucleoli. The following theorem is a slight generalization of Huberman's (1980) theorem on the standard (pre)nucleolus that is fundamental for the efficient computability of the (pre)nucleolus in various types of balanced games with polynomially many essential coalitions, as it is the case e.g. in assignment games (Solymosi, Raghavan, 1994).

Huberman (1980) proves that

- *in a balanced game, all inessential coalitions are redundant for the nucleolus.*

Recall that for balanced games the prenucleolus is the same as the nucleolus.

Theorem 3. *In a balanced game v, all inessential coalitions are redundant for $\mathbf{Nu}_q(v)$ with a monotone nonincreasing weight function q. In particular, for the standard prenucleolus $\mathbf{Nu}(v) = \mathbf{Nu}(\mathcal{E}(v), v)$.*

Proof. Let $S \in \mathcal{N} \setminus \mathcal{E}(v)$ be inessential in game v, because of the partition $S = S_1 \cup S_2$ with $S_1, S_2 \in \mathcal{E}(v)$ and $v(S) \leq v(S_1) + v(S_2)$. For simplicity of notation, we assume (without loss of generality) that the weakly majorizing partition consists only of $k = 2$ subcoalitions.

We prove that in all iterations $r = 1, \ldots, \varrho$ of algorithm (13) the (inequality or equality) constraint related to S is redundant, because it is implied by the constraints related to S_1 and S_2.

Since iteration $r = 1$ determines $\alpha_q^1(v)$ and the q-weighted least core, the redundancy of inequalities related to inessential coalitions in a balanced game was shown in Theorem 1.1 using (11) even under the weaker assumption of subadditivity on the weight function. Observe that the same argument proves our claim in any other iteration $r > 1$ in which all subcoalitions in the weakly majorizing essential partition are still unsettled (i.e. $S_i \in \Sigma^{r-1}$ for all $i = 1, \ldots, k$), hence all related constraints are inequalities like in (11).

Suppose now that at the beginning of iteration $r > 1$ coalition S is still not settled (i.e. $S \in \Sigma^{r-1}$), but there are both settled and unsettled subcoalitions in the weakly majorizing essential partition. For simplicity, let $S_1 \in \Delta^{r-1}$ be settled, and $S_2 \in \Sigma^{r-1}$ be still unsettled. If S_1 got settled at the end of iteration $j \leq r - 1$, i.e. $S_1 \in \Delta_j$, then the related constraints in the optimization problem of iteration r are

$$\begin{array}{l} v(S_1) + q(S_1)\alpha_q^j \qquad\qquad\qquad\quad = x(S_1) \\ v(S_2) + q(S_2)\alpha_q^j + q(S_2)(\alpha - \alpha_q^j) \leq x(S_2) \\ \hline v(S) \ + \ q(S)\alpha_q^j \ + \ q(S)(\alpha - \alpha_q^j) \ \leq \ x(S) \end{array} \tag{14}$$

By the above assumption, $v(S) \leq v(S_1) + v(S_2)$, so the sum of the first two constraints implies the third one, because (i) in iteration r variable α to be maximized satisfies $\alpha \geq \alpha_q^{r-1} \geq \alpha_q^j$; (ii) for a balanced game we have $\alpha_q^j \geq \alpha_q^1 \geq 0$; and (iii) in case of a monotone nonincreasing weight function, $q(S) \leq \min\{q(S_1), q(S_2)\} \leq q(S_1) + q(S_2)$.

Finally, suppose that at the end of some iteration $r \geq 1$ coalition S becomes settled (i.e. $S \in \Delta_r$). This is equivalent to saying that all subcoalitions in the weakly majorizing essential partition have become settled by the end of that iteration. For simplicity, let $S_2 \in \Delta_r$ be the last one to become settled. Then all related constraints become equalities in (14), the redundancy of the last constraint, however, follows in the same way, for all subsequent iterations.

In all three cases the constraints related to S_1 and S_2 imply the constraint related to S, hence algorithm (13) yields the same outcomes even if we discard S from all considerations. Since $S_1, S_2 \in \mathcal{E}(v)$, the above arguments can be independently repeated for any inessential $S \in \mathcal{N} \setminus \mathcal{E}(v)$, and the theorem follows.

The constant $q(S) = 1$ for all $S \in \mathcal{N}$ weight function is monotone nonincreasing, so we get Huberman's (1980) theorem on the redundancy of inessential coalitions for the standard (pre)nucleolus in balanced games as a corollary. $\square$

For completeness, we present (without proof) a characterization of weighted prenucleoli in terms of balanced collections. It is a slight generalization of the characterization given by Wallmeier (1984) for q-prenucleoli with monotone nondecreasing and symmetric (i.e. $q(S) = q(|S|)$ for all $S \in \mathcal{N}$) weight function, that, in turn is a straightforward generalization of Kohhlberg's (1971) criterion for the standard prenucleolus. Streamlined versions of Kohlberg's (1971) characterization are given in (Groote Schaarsberg et al., 2012) and (Nguyen, 2016) for the standard (pre)nucleolus, and in (Huijink et al., 2015) for the per-capita (pre)nucleolus. Similar characterizations in more general and abstract settings that accomodate the weighted versions discussed here can be found in (Maschler et al., 1992) and (Potters and Tijs, 1992).

Proposition 2. *An efficient payoff allocation x belongs to the q-weighted prenucleolus of game v if and only if the family of non-trivial coalitions whose q-surplus at x is at least t is a balanced (or an empty) collection for any $t \in \mathbb{R}$.*

The following example demonstrates that Huberman's (1980) redundancy result cannot be applied for the per-capita (pre)nucleolus, we can not only use essential coalitions, not even in a balanced game (in which case the per-capita prenucleolus coincides with the per-capita nucleolus). This could partly explain why there are much fewer special-purpose algorithms proposed in the literature for the per-capita (pre)nucleolus than for the standard (pre)nucleolus. A recent exception is the algorithm by Huijink et al. (2015) for the per-capita nucleolus of bankruptcy games.

Example 3. Consider the 4-player balanced superadditive game: $v(N) = 12$, $v(12) = v(34) = v(123) = v(124) = v(134) = v(234) = 6$, $v(14) = 4$, and $v(R) = 0$ for all other coalitions $R \in \mathcal{N}$. Let the weight function be $q(S) = |S|$ for all $S \in \mathcal{N}$.

The first iteration of algorithm (13) gives $\alpha_q^1 = 0$, so $X^1 = \mathbf{LC}_q(v) = \mathbf{Co}(v)$, and $\Delta_1 = \{12, 34\}$. The second iteration gives $\alpha_q^2 = 1$ and $\Delta_2 = \{14, 123, 124, 134, 234\}$. The third iteration gives $\alpha_q^3 = 3$ and $\Delta_3 = \{1, 2, 3, 4, 13, 23, 24\}$, and the algorithm stops. Thus, $\varrho = 3$. The only allocation in X^3 (in fact, already in X^2) is $(3, 3, 3, 3)$, it is the per-capita prenucleolus. It is easily checked also by Proposition 2. Indeed, $\mathcal{S}^1 = \Delta_1$, $\mathcal{S}^2 = \Delta_1 \cup \Delta_2$, $\mathcal{S}^3 = \Delta_1 \cup \Delta_2 \cup \Delta_3$ are all balanced families.

Let us now consider only the essential coalitions $\mathcal{E}(v) = \{1, 2, 3, 4, 12, 14, 34\}$ and initiate algorithm (13) with $\Sigma^0 := \mathcal{E}(v)$ instead of $\mathcal{N}$. Then the first iteration gives again $\alpha_q^1 = 0$, so $X^1 = \mathbf{LC}_q(\mathcal{E}(v), v) = \mathbf{Co}(\mathcal{E}(v), v) = \mathbf{Co}(v)$, and $\Delta_1 = \{12, 34\}$. On the other hand, the second iteration gives $\alpha_q^2 = 2$ and $\Delta_2 = \{2, 3, 14\}$. The third iteration gives $\alpha_q^3 = 4$ and $\Delta_3 = \{1, 4\}$, and the algorithm stops. Thus, $\varrho = 3$. The only allocation in X^3 (in fact, already in X^2) is $(4, 2, 2, 4)$, it is the per-capita prenucleolus of the $\mathcal{E}(v)$-restricted game. It is easily checked also by the restricted version of Proposition 2. Indeed, $\mathcal{S}^1 = \Delta_1$, $\mathcal{S}^2 = \Delta_1 \cup \Delta_2$, $\mathcal{S}^3 = \Delta_1 \cup \Delta_2 \cup \Delta_3$ are all balanced families, consisting only of essential coalitions. ——— $\diamond$

Analogously to how we obtained the dual description (10) of the q-weighted least core from its definition (6), given a game (N, v) and weight function q, we can alternatively get the q-weighted prenucleolus $\mathbf{Nu}_q(v)$ from the dual game (N, v^*) as the outcome of the following procedure, that we call the dual description of the q-prenucleolus:

Let $X^0 := \mathbf{Ef}(v^*)$ and $\widehat{\Sigma}^0 := \mathcal{N}, \widehat{\Delta}^0 := \{N\}$.
For $r = 1, \dots, \varrho$ define recursively

$$\left| \begin{array}{ll} \beta_q^r := & \min_{x \in X^{r-1}} \max_{T \in \widehat{\Sigma}^{r-1}} g_q(T, x, v^*), \\ X^r := & \{x \in X^{r-1} : \max_{T \in \widehat{\Sigma}^{r-1}} g_q(T, x, v^*) = \beta_q^r\}, \\ \widehat{\Delta}_r := & \{T \in \widehat{\Sigma}^{r-1} : \min_{x \in X^r} g_q(T, x, v^*) = \beta_q^r\}, \\ \widehat{\Sigma}^r := & \widehat{\Sigma}^{r-1} \setminus \widehat{\Delta}_r, \quad \widehat{\Delta}^r := \widehat{\Delta}^{r-1} \cup \widehat{\Delta}_r \end{array} \right. \tag{15}$$

where ϱ is the first value of r for which $\widehat{\Sigma}^r = \emptyset$.

It is easily seen that

- ϱ is the same well-defined finite number as in procedure (13);
- $\beta_q^1(v^*) = \beta_q^1 > \beta_q^2 > \dots > \beta_q^\varrho$ are well defined, and for all $r = 1, \dots, \varrho$ we have $\beta_q^r = -\alpha_q^r$, the optimum values obtained in procedure (13);
- $\mathbf{LC}_q^*(v^*) = X^1 \supseteq X^2 \supseteq \dots \supseteq X^\varrho$ is the same sequence of non-empty polytopes as in procedure (13); in particular, X^ϱ consists of the unique q-prenucleolus-allocation η_q defined in (13).
- $\widehat{\Delta}_1 \cup \widehat{\Delta}_2 \cup \dots \cup \widehat{\Delta}_\varrho$ forms a partition of $\mathcal{N}$; moreover, for each $r = 1, \dots, \varrho$, the family $\widehat{\Delta}_r$ consists of the complements of the coalitions in Δ_r generated by procedure (13);
- if $T \in \widehat{\Delta}_k$ then $\eta_q(T) - q(N \setminus T)\beta_q^k = v^*(T)$,

where $\beta_q^1(v^*)$ and $\mathbf{LC}_q^*(v^*)$ are defined in (10).

We now identify a family of coalitions which are redundant in the dual description of weighted prenucleoli of a balanced game.

Theorem 4. *In a balanced game v, all dually inessential coalitions are redundant for $\mathbf{Nu}_q(v)$ with a monotone nondecreasing weight function q. In particular, for the standard prenucleolus $\mathbf{Nu}(v) = \mathbf{Nu}(\mathcal{E}^*(v^*), v^*)$ and for the per-capita prenucleolus $\mathbf{Nu}_{pc}(v) = \mathbf{Nu}_{pc}(\mathcal{E}^*(v^*), v^*)$.*

Proof. Let $T \in \mathcal{N} \setminus \mathcal{E}^*(v^*)$ be dually inessential in game v, because of the partition $T = T_1 \cup T_2$ with $T_1, T_2 \in \mathcal{E}^*(v^*)$ and $v^*(T_1) + v^*(T_2) \leq v^*(T)$. For simplicity of notation, we assume (without loss of generality) that the weakly minorizing partition consists only of $k = 2$ subcoalitions.

We prove that in all iterations $r = 1, \ldots, \varrho$ of algorithm (15) the (inequality or equality) constraint related to T is redundant, because it is implied by the constraints related to T_1 and T_2.

Since iteration $r = 1$ determines $\beta_q^1(v^*)$ and the q-weighted least core, the redundancy of inequalities related to dually inessential coalitions in a balanced game was shown in Theorem 2. Observe that the same argument proves our claim in any other iteration $r > 1$ in which all subcoalitions in the weakly minorizing dually essential partition are still unsettled (i.e. $T_i \in \widehat{\Sigma}^{r-1}$ for all $i = 1, \ldots, k$), hence all related constraints are inequalities like in (12).

Suppose now that at the beginning of iteration $r > 1$ coalition T is still not settled (i.e. $T \in \widehat{\Sigma}^{r-1}$), but there are both settled and unsettled subcoalitions in the weakly minorizing dually essential partition. For simplicity, let $T_1 \in \widehat{\Delta}^{r-1}$ be settled, and $T_2 \in \widehat{\Sigma}^{r-1}$ be still unsettled. If T_1 became settled at the end of iteration $j \leq r - 1$, i.e. $T_1 \in \widehat{\Delta}_j$, then the related constraints in the optimization problem of iteration r are the following:

$$\begin{array}{l} x(T_1) = v^*(T_1) + q(N \setminus T_1)\beta_q^j \\ x(T_2) \leq v^*(T_2) + q(N \setminus T_2)\beta_q^j + q(N \setminus T_2)(\beta - \beta_q^j) \\ \hline x(T) \ \leq v^*(T) \ + \ q(N \setminus T)\beta_q^j \ + \ q(N \setminus T)(\beta - \beta_q^j) \end{array} \tag{16}$$

By the above assumption, $v^*(T_1) + v^*(T_2) \leq v^*(T)$, so the sum of the first two constraints implies the third one, because (i) in iteration r variable β to be minimized satisfies $\beta \leq \beta_q^{r-1} \leq \beta_q^j$; (ii) for a balanced game we have $\beta_q^j \leq \beta_q^1 \leq 0$; and (iii) in case of a monotone nondecreasing weight function, $(N \setminus T_1) \cap (N \setminus T_2) = N \setminus T \neq \emptyset$ implies $q(N \setminus T) \leq \min\{q(N \setminus T_1), q(N \setminus T_2)\} \leq q(N \setminus T_1) + q(N \setminus T_2)$.

Finally, suppose that at the end of some iteration $r \geq 1$ coalition T becomes settled (i.e. $T \in \widehat{\Delta}_r$). This is equivalent to saying that all subcoalitions in the weakly minorizing dually essential partition have become settled by the end of that iteration. For simplicity, let $T_2 \in \widehat{\Delta}_r$ be the last one to become settled. Then all related constraints become equalities in (16), the redundancy of the last constraint, however, follows in the same way, for all subsequent iterations.

In all three cases the constraints related to T_1 and T_2 imply the constraint related to T, hence algorithm (15) yields the same outcomes even if we discard T from all considerations. Since $T_1, T_2 \in \mathcal{E}^*(v^*)$, the above arguments can be independently repeated for any dually inessential $T \in \mathcal{N} \setminus \mathcal{E}^*(v^*)$, and the theorem follows.

The claims for the particular (pre)nucleoli follow immediately from the monotone nondecreasing nature of the respective weight functions. □

Note that since the constant $q(S) = 1$ for all $S \in \mathcal{N}$ weight function is monotone nondecreasing, we get the dual counterpart of Huberman's (1980) theorem that states the redundancy of dually inessential coalitions for the standard (pre)nucleolus in balanced games. This is the implicit basis of various known efficient nucleolus algorithms, e.g. (Megiddo, 1978), (Granot et al., 1996), (Brânzei et al., 2005), (van den Brink et al., 2011).

We emphasize that in Theorem 4, balancedness of the game is a necessary condition. To make the point, let us consider the non-balanced dual game in Example 2: in that game the (standard / per-capita) least core consists of a unique allocation that is precisely the (standard / per-capita) prenucleolus.

For completeness, we present a Kohlberg-type characterization of weighted prenucleoli in terms of the dual game. It is the dual counterpart of the characterization in Proposition 2, and the analogue of the second characterization of weighted least core allocations in Proposition 1.

Proposition 3. *An efficient payoff allocation x belongs to the q-weighted prenucleolus of game v if and only if the family of non-trivial coalitions whose transformed dual q-surplus $g_q(., x, v^*)$ at x is at most t is a balanced (or an empty) collection for any $t \in \mathbb{R}$.*

We omit the proof, since the standard LP duality arguments that prove Proposition 2 can be straightforwardly adjusted to the dual description (15).

We use the balanced game in Example 3 to illustrate how Theorem 4 can help in calculating, for example, the per-capita (pre)nucleolus. Recall that in that game we could not use only the essential coalitions, discarding all inessential coalitions lead to a different allocation. Now we demonstrate that we, however, can omit all coalitions that are inessential in the dual game.

Example 4. Consider the dual game of the 4-player balanced superadditive game in Example 3: $v^*(N) = v^*(13) = v^*(14) = v^*(24) = v^*(123) = v^*(124) = v^*(134) = v^*(234) = 12$, $v^*(23) = 8$, and $v^*(R) = 6$ for all other coalitions $R \in \mathcal{N}$. Let the weight function be $q(S) = |S|$ for all $S \in \mathcal{N}$.

The first iteration of algorithm (15) gives $\beta_q^1 = 0$, so $X^1 = \mathbf{LC}_q^*(v^*) = \mathbf{Co}(v)$, and $\widehat{\Delta}_1 = \{12, 34\}$. The second iteration gives $\beta_q^2 = -1$ and $\widehat{\Delta}_2 = \{23, 1, 2, 3, 4\}$. The third iteration gives $\beta_q^3 = -3$ and $\widehat{\Delta}_3 = \{13, 14, 24, 123, 124, 134, 234\}$, and the algorithm stops. Thus, $\varrho = 3$. The only allocation in X^3 (in fact, already in X^2) is $(3, 3, 3, 3)$, it is the per-capita prenucleolus (cf. Example 3). It is easily checked also by Proposition 3. Indeed, $\mathcal{T}^1 = \widehat{\Delta}_1$, $\mathcal{T}^2 = \widehat{\Delta}_1 \cup \widehat{\Delta}_2$, $\mathcal{T}^3 = \widehat{\Delta}_1 \cup \widehat{\Delta}_2 \cup \widehat{\Delta}_3$ are all balanced families. Notice that $\widehat{\Delta}_1$, $\widehat{\Delta}_2$, and $\widehat{\Delta}_3$ consists of, respectively, the complements of Δ_1, Δ_2, and Δ_3, computed in Example 3. Furthermore, $\beta_q^r = -\alpha_q^r$ for $r = 1, 2, 3$.

Let us now take only the dually essential coalitions and initiate algorithm (15) with $\widehat{\Sigma}^0 := \mathcal{E}^*(v^*) = \{1, 2, 3, 4, 12, 23, 34\}$ instead of $\mathcal{N}$. Then the first iteration gives again $\beta_q^1 = 0$, so $X^1 = \mathbf{LC}_q^*(\mathcal{E}^*(v^*), v^*) = \mathbf{Co}(\mathcal{E}^*(v^*), v^*) = \mathbf{Co}(v)$, and $\widehat{\Delta}_1 = \{12, 34\}$. The second iteration gives $\beta_q^2 = -1$ and $\widehat{\Delta}_2 = \{1, 2, 3, 4, 23\}$, and the algorithm stops. Thus, now $\varrho = 2$. The only allocation in X^2 is $(3, 3, 3, 3)$, it is the per-capita prenucleolus of the $\mathcal{E}^*(v^*)$-restricted game, that is the same as the above output of algorithm (15) run with the unrestricted dual input, that, in turn, coincides with the per-capita prenucleolus of the game v (cf. Example 3). It is easily checked also by the restricted version of Proposition 3. Indeed, $\mathcal{T}^1 = \widehat{\Delta}_1$, $\mathcal{T}^2 = \widehat{\Delta}_1 \cup \widehat{\Delta}_2$ are both balanced families, consisting only of dually essential coalitions. ——— $\diamond$

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