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Production in advance versus production to order: Equilibrium and social surplus

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ABSTRACT

The mixed-strategy equilibrium of the production-in-advance type capacity-constrained Bertrand–Edgeworth duopoly game has not been derived analytically for the case of intermediate capacities in the literature. As in the case of the production-to-order version of the same game, the case of intermediate capacities turned out to be the most difficult one compared with the cases of small and large capacities. In this paper we derive analytically a symmetric mixed-strategy equilibrium of the production-in-advance version of this game for a large region of intermediate capacities. Nevertheless we show that in general the economic surplus within the production-to-order type environment is higher than in the respective production-in-advance type one, and, therefore, production-to-order should be preferred to production-in-advance if the mode of production can be influenced by the government.

1. Introduction

In one of the basic oligopoly games firms can set prices and quantities at the same time in a homogeneous good market. This framework was already introduced by Shubik (1955) and referred to by Maskin (1986) as the production-in-advance environment in which production takes place before sales are realized. Markets of perishable goods are usually mentioned as examples of advance production in a market. In contrast, in case of production-to-order, production takes place after prices are known.

Shubik (1955) indicated that the production-in-advance game might not have an equilibrium in pure strategies.[1] Maskin (1986) established the existence of a mixed-strategy equilibrium for the production-in-advance game under quite general conditions. Assuming unlimited capacities and linear demand, Levitan and Shubik (1978) computed the mixed-strategy equilibrium for the case of production in advance. In the same framework Gertner (1986) determined the mixed-strategy equilibrium under more general conditions. While comparing the production-in-advance equilibrium profits with that under production-to-order, Tasnádi (2004, Section 4) and Tasnádi (2019) determined the equilibrium profits of the production-in-advance game. Recently, Montez and Schutz (2018), as a part of a larger project on unsold inventories and exploring relations with other micro-theoretic models, determined the mixed-strategy equilibrium of the production-in-advance game for the case of unlimited capacities (or equivalently for the case of large capacities) and pointed out shortcomings of the previous solutions.[2]

In Tasnádi (2004) we showed that within the framework of a capacity-constrained Bertrand–Edgeworth duopoly the production-in-advance and the production-to-order environments result in the same profits. In obtaining this result we considered the small capacity, the intermediate capacity and the large capacity cases. Since the small capacity case has a simple solution in pure strategies (e.g. Tasnádi, 2004, Section 3) and the large capacity case has been solved completely by Montez and Schutz (2018), in this paper we focus on the most challenging case of intermediate capacities[3] for which we determine a symmetric mixed-strategy equilibrium on a large subregion. The latter case was only partially solved over the entire region of intermediate capacities in Tasnádi (2004, Section 5), which focused on the

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[1] Friedman (1988) established the non-existence of a pure-strategy equilibrium in case of differentiated goods, which does not include the homogeneous good case investigated in this paper.

[2] For recent theoretical results on the production-in-advance game we refer the reader to Bos and Vermeulen (2015) and van den Berg and Bos (2017). For related recent experimental results see Casaburi and Minerva (2011) and Davis (2013).

[3] The same three cases emerge in the production-to-order environment in which the cases of small and large capacities are simple, whereas the case of intermediate capacities is challenging (e.g. Osborne and Pitchik, 1986). In the production-in-advance environment the case of large capacities is far more complex than the case of small capacities; however, the case of intermediate capacities still appears to be the hardest.

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determination of the equilibrium profits. Calculating the bottom 'half' of the symmetric equilibrium price distribution for the case of intermediate capacities was already sufficient for the determination of the equilibrium profits. After one and a half decades the Montez and Schutz (2018) paper served as an inspiration to determine the top 'half' with respect to prices of the symmetric-mixed strategy equilibrium. The difficulty in calculating the mixed-strategy equilibrium was to realize that there is an area of best responses to the symmetric mixed-strategy equilibrium strategy of the other firm in the price-quantity space. Nevertheless, the support of the mixed-strategy equilibrium remains still one-dimensional, that is there is a two-dimensional set of indifferent pure strategies never played in a mixed-strategy equilibrium. In addition, we can show in general that the economic surplus is greater in the case where firms produce to order than when they produce in advance.

The remainder of the paper is organized as follows: Section 2 presents the framework, Section 3 determines a symmetric mixed-strategy equilibrium on a large subregion of intermediate capacities, Section 4 investigates economic surplus, and Section 5 concludes.

2. Preliminaries

In this section we introduce the necessary assumptions, notations and already available results.

Assumption 1. The demand curve $D : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly decreasing on $[0, b]$, identically zero on $[b, \infty)$, continuous at b and twice continuously differentiable on $(0, b)$. Furthermore, the revenue function $pD(p)$ is strictly concave on $[0, b]$.

We shall denote by a the horizontal intercept of D; i.e. $D(0) = a$. In addition, we shall denote by P the inverse demand function.

In our model two firms set their prices and quantities simultaneously.

Assumption 2. Firms 1 and 2 have identical positive unit costs $c \in (0, b)$ up to the same positive capacity constraint k. Each of them sets its price ($p_1, p_2 \in [0, b]$) and production quantity ($q_1, q_2 \in [0, k]$).

Throughout the paper i and j will be used to refer to the two firms; in particular, $i, j \in \{1, 2\}$ and $i \neq j$.

We employ the efficient rationing by the low-price firm, which occurs in a market if the consumers can costlessly resell the good to each other or if the consumers have heterogeneous unit demands and the consumers having higher reservation prices are served first (for more details we refer to Vives, 1999 and Wolfstetter, 1999), to determine the demand faced by the firms.

Assumption 3. The demand faced by firm i is given by

$$\Delta_i(p_1, q_1, p_2, q_2) = \begin{cases} D(p_i) & \text{if} \quad p_i < p_j \\ \frac{q_i}{q_i+q_j} D(p_i) & \text{if} \quad p_i = p_j \\ \left(D(p_i) - q_j\right)^+ & \text{if} \quad p_i > p_j. \end{cases}$$

Under Assumption 3 the low-price firm faces the entire demand, firms with identical prices split the demand in proportion to the firms' quantity decisions[4] and the high-price firm faces a so-called residual demand, which equals the demand minus the quantity produced by the low-price firm.

We define the firms' profit functions as follows:

$$\pi_i((p_1, q_1), (p_2, q_2)) = p_i \min\{\Delta_i(p_1, q_1, p_2, q_2), q_i\} - cq_i$$

for both $i \in \{1, 2\}$.

Maskin (1986, Theorem 1) demonstrated that the production-in-advance game possesses an equilibrium in mixed strategies. In the following, a mixed strategy is a probability measure defined on the σ-algebra of Borel measurable sets on $[0, b] \times [0, k]$, which can be restricted without loss of generality to $S = [c, b] \times [0, k]$. In equilibrium, each firm optimally chooses μ_i conditional on μ_j, $i \neq j$. Such an equilibrium is denoted by (μ_1^*, μ_2^*). A mixed-strategy equilibrium $\left(\mu_1^*, \mu_2^*\right)$ can be calculated by the following two conditions:

$$\pi_1\left((p_1, q_1), \mu_2^*\right) \leqslant \pi_1^*, \qquad \pi_2\left(\mu_1^*, (p_2, q_2)\right) \leqslant \pi_2^* \tag{1}$$

hold true for all $(p_1, q_1), (p_2, q_2) \in S$, and

$$\pi_1\left((p_1^*, q_1^*), \mu_2^*\right) = \pi_1^*, \qquad \pi_2\left(\mu_1^*, (p_2^*, q_2^*)\right) = \pi_2^* \tag{2}$$

holds true μ_1^*-almost everywhere and μ_2^*-almost everywhere, where π_1^*, π_2^* stand for the equilibrium profits corresponding to $\left(\mu_1^*, \mu_2^*\right)$.[5] In addition, it can be verified that a symmetric equilibrium in mixed strategies exists by applying Theorem 6* of Dasgupta and Maskin (1986).

Three special prices play an important role in the analysis. We define p^* to be the price that clears the firms' aggregate capacity from the market if such a price exists, and zero otherwise. That is,

$$p^* = \begin{cases} D^{-1}(2k) & \text{if} \quad D(0) > 2k \\ 0 & \text{if} \quad D(0) \leqslant 2k. \end{cases}$$

The function

$$\pi^r(p) = (p - c)(D(p) - k)$$

equals a firm's residual profit whenever its opponent sells k and $D(p) \geqslant k$. Let $\overline{p} = \arg\max_{p\in[c,b]} \pi^r(p)$ and $\overline{\pi} = \pi^r(\overline{p})$. Clearly, p^* and $\overline{p}$ are well defined whenever Assumptions 1 and 2 are satisfied. Finally, let $\underline{p} = c + \overline{\pi}/k$, that is $\underline{p}$ is the price at which a firm is indifferent between selling its entire capacity and maximizing profits on the residual demand curve.

For the case of small capacities, i.e. $p^* \geqslant \overline{p}$, the game has a unique equilibrium in pure strategies in which the firms produce at their capacity limits and set the market-clearing price (e.g. Tasnádi, 2004, Proposition 2). The mixed-strategy equilibrium for the case of large capacities, i.e. $D(c) \leqslant k$, has been determined recently by Montez and Schutz (2018) in which the firms charge prices above their common unit costs. Therefore, in this paper we focus on the open and most challenging case of intermediate capacities, i.e. $\overline{p} > \max\{p^*, c\}$, which was also the most difficult one in the production-to-order environment. We had established the following proposition earlier.

Proposition 1 (*Tasnádi, 2004, Proposition 4*)**.** *Let Assumptions 1–3 hold. If $\overline{p} > \max\{p^*, c\}$, then in a symmetric mixed-strategy equilibrium (μ^*, μ^*) of the production-in-advance game we have*

$$\mu^*\left(\left[\underline{p}, p\right] \times [0, k]\right) = \mu^*\left(\left[\underline{p}, p\right] \times \{k\}\right) = \frac{(p - c)k - \overline{\pi}}{p(2k - D(p))} \tag{3}$$

for any $p \in \left[\underline{p}, \overline{p}\right)$.

[4] The essential property of the tie-breaking rule employed in this paper is that firm i's demand is strictly increasing in firm i's own quantity (see also Maskin, 1986). In fact, any other tie-breaking rule satisfying the latter property does the job. Nevertheless, the tie-breaking rule specified in Assumption 3 reflects a larger visibility by consumers and a lower risk of being out-of-stock in case of a larger production.

[5] Conditions (1) and (2) are equivalent with the definition of an equilibrium in mixed strategies as stated, for instance, in Osborne and Rubinstein (1994, p. 34) as a generalization of their Lemma 33.2.

In the original statement of Proposition 4 in Tasnádi (2004) even the concavity of the demand curve was assumed. However, the concavity of the demand curve was imposed to exclude holes in the equilibrium distribution of prices and the proofs remain valid under weaker conditions. In particular, the last step at the end of the proof of Lemma 2 in Tasnádi (2004, 2019) holds also if a log-concave demand curve is assumed, while the last step at the end of the proof of Lemma 4 in Tasnádi (2004) works under the assumption of a concave revenue function $pD(p)$. These were the only two points where the concavity of the demand function was employed in the proofs leading to Proposition 4 in Tasnádi (2004).

Osborne and Pitchik (1986) showed already for the production-to-order environment that the assumption of a concave revenue function is indispensable to avoid holes in the support of the mixed-strategy equilibrium. Their solution is on $[\underline{p}, \overline{p})$ very similar to the equilibrium cumulative distribution of prices in Proposition 1.[6] Since finding the mixed-strategy equilibrium for the production-in-advance environment is much harder than for the respective production-to-order environment we avoid the complication related to handling possible price gaps in the equilibrium price distribution, and therefore we impose the assumption of a strictly concave revenue function $pD(p)$.

3. Mixed-strategy equilibrium

We build on Proposition 1 in that we try to extend the partially revealed symmetric mixed-strategy equilibrium. The main idea while finding the mixed-strategy equilibrium is to consider strategies which have in the best response correspondence an interval of quantities on an interval of prices starting at $\underline{p}$. However, in equilibrium they are just choosing one quantity from the interval of indifferent quantities resulting in the same equilibrium profit level.[7]

Before proceeding, we need to introduce several further notations. Let $F(p) = \mu^*\left(\left[\underline{p}, p\right] \times [0, k]\right)$ denote the cumulative distribution of equilibrium prices. From Proposition 1 we already know that $s(p) = k$ for all $p \in \left[\underline{p}, \overline{p}\right)$ and that F is atomless on $p \in \left[\underline{p}, \overline{p}\right)$. We shall denote by

$$\widehat{p} = \inf\{p \in [c, b] \mid \mu\left((p, b] \times [0, k]\right) = 0\}$$

the highest possible price set by a firm when playing an arbitrary strategy μ. We expect to find a symmetric mixed-strategy equilibrium in which at prices $p \in [c, \widehat{p}] \subset [c, b]$ at most one quantity $s(p) \in [0, k]$ will be produced in equilibrium. Therefore, a mixed-strategy equilibrium can be given by the triple $(\widehat{p}, s, F)$.

Proposition 2. *Let Assumptions 1–3 hold. If $\overline{p} > \max\{p^*, c\}$, then a symmetric mixed-strategy equilibrium (μ^*, μ^*) of the production-in-advance game is given by the following equilibrium price distribution*[8]

$$F(p) = \begin{cases} 0 & \text{if } 0 \leqslant p < \underline{p}, \\ \frac{(p-c)k-\overline{\pi}}{p(2k-D(p))} & \text{if } \underline{p} \leqslant p < \overline{p}, \\ 1 - \frac{c}{p} & \text{if } \overline{p} \leqslant p < \widehat{p}, \text{ and} \\ 1 & \text{if } \widehat{p} \leqslant p \leqslant b \end{cases} \tag{4}$$

and by the 'supply' function $s(p)$ given by $s(p) = k$ for all $p \in \left[\underline{p}, \overline{p}\right)$ and determined by

$$s(p) = D'(p)\left(\frac{p^2}{c} - p\right) + D(p) + \frac{\overline{\pi}}{c} \tag{5}$$

for all $p \in [\overline{p}, \widehat{p}]$ if

$$\widehat{p} \leqslant P(k), \tag{6}$$

where $\widehat{p}$ is the unique solution of $s(r) = D(r)/2$.

Proof. The proof also includes how the symmetric mixed-strategy equilibrium was derived and not just the mere verification of the statement.[9]

We search for a symmetric mixed-strategy equilibrium in a special form by assuming that at prices $p \in [c, \widehat{p}] \subset [c, b]$ at most one quantity $s(p) \in [0, k]$ will be produced in equilibrium. In addition, we assume that s is strictly decreasing and continuously differentiable on $[\overline{p}, \widehat{p})$. Furthermore, we assume that F is even atomless on $[\overline{p}, \widehat{p})$. From Proposition 1 we already know that $s(p) = k$ for all $p \in \left[\underline{p}, \overline{p}\right)$ and that F is atomless on $p \in \left[\underline{p}, \overline{p}\right)$. Some additional technical assumptions will be imposed during the process of determining the symmetric mixed-strategy equilibrium.

Assume that $(\widehat{p}, s, F)$ is associated with a symmetric mixed-strategy equilibrium (μ, μ). Since s and F are known for all $p \in \left[\underline{p}, \overline{p}\right)$ in what follows we consider without loss of generality only prices such that $p \geqslant \overline{p}$. Furthermore, let $f(p) = F'(p)$, where F is differentiable. We shall denote by $r^* \in [\overline{p}, \widehat{p}]$ the price at which $s(r^*) = D(r^*)/2$ and assume that such a price exists uniquely.[10] Furthermore, to arrive to the equilibrium given in the statement of Proposition 2 the inequality $r^* \leqslant P(k)$ (i.e. condition (6)) will be crucial.

Firm 1's profit equals

$$\begin{aligned} \pi_1\left((p, q), \mu\right) = {} & pq\left(1 - F(p)\right) \\ & + p\int_{\overline{p}}^{p} \min\left\{(D(p) - s(r))^+, q\right\} dF(r) + \\ & p\int_{\underline{p}}^{\overline{p}} \min\left\{(D(p) - k)^+, q\right\} dF(r) - cq \end{aligned} \tag{7}$$

for any $p \in (\overline{p}, \widehat{p})$ and any $q \in [0, \min\{k, D(p)\}]$, where we have already taken into account that $D(p) < s(p) = q$ does not make sense since then the firms produce a superfluous amount for sure. Note that we cannot have $q = s(p) < (D(p) - k)^+$ since this would result in even less profits than choosing pure-strategy $(\overline{p}, D(\overline{p}))$. Hence, in what follows we can assume that $q \geqslant (D(p) - k)^+$. Therefore, (7) simplifies to

$$\begin{aligned} \pi_1\left((p, q), \mu\right) = {} & pq\left(1 - F(p)\right) + p\int_{\overline{p}}^{p} \min\left\{D(p) - s(r), q\right\} dF(r) + \\ & p\int_{\underline{p}}^{\overline{p}} (D(p) - k) dF(r) - cq, \end{aligned} \tag{8}$$

where we can drop the positive part symbol in the first integral of (7) because we will speak only about a solution if finally (6) holds.

In determining $\frac{\partial \pi_1}{\partial q}\left((p, q), \mu\right)$ first let us consider the case in which $D(p) - s(p) < q$, and therefore $D(p) - s(r) < q$ for all

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6 In both the production-to-order and production-in-advance environments the firms produce at their capacity limits and there is just a slight difference in the shapes of the respective equilibrium price distributions.

7 On the interval on which Proposition 1 did not determine the equilibrium price distribution, the truncated Pareto distribution emerges as in other models with a lot of indifferent best responses (e.g. Ravid et al., 2019).

8 We conjecture that the derived mixed-strategy equilibrium is 'essentially' unique within the class of symmetric mixed-strategy equilibria. Clearly, it cannot be unique since the values of s can be altered on an F-null set. Showing that the symmetric mixed-strategy equilibrium is essentially unique, appears to be a very difficult task.

9 The verification of the equilibrium properties would not lead to a significantly shorter proof; however, we would loose the insightful steps to arrive to the equilibrium.

10 After deriving (19), we will verify in the proof that the s given by (5) is continuous and strictly decreasing on $p \in [\overline{p}, \widehat{p}]$ and that r^* is uniquely determined by the properties of D and this s.

$r \in [\overline{p}, p]$ since s is (assumed to be) strictly decreasing. Then it follows that

$$\frac{\partial \pi_1}{\partial q}((p,q),\mu) = p(1-F(p)) - c. \quad (9)$$

Second, we consider the case in which $D(p) - s(p) > q$. Since $D(p) - s(p) > q \geqslant D(p) - k = D(p) - s(\overline{p})$ and s is continuous and strictly decreasing on $[\underline{p}, \widehat{p}]$ there exists a unique $r \in [\underline{p}, p)$ such that $D(p) - s(r) = q$. Then $r = s^{-1}(D(p) - q)$. We denote the functional relationship between q and r by $r(q)$. Clearly, $r(q)$ is strictly increasing. Now (8) can be written as

$$\begin{aligned}\pi_1((p,q),\mu) &= pq(1-F(p)) + p\int_{r(q)}^{p} q dF(r) + \\ &\quad p\int_{\overline{p}}^{r(q)} D(p) - s(r) dF(r) + \\ &\quad p\int_{\underline{p}}^{\overline{p}} (D(p)-k) dF(r) - cq, \end{aligned} \quad (10)$$

from which we get

$$\begin{aligned}\frac{\partial \pi_1}{\partial q}((p,q),\mu) &= p(1-F(p)) + p\int_{r(q)}^{p} dF(r) - pqf(r(q))r'(q) + \\ &\quad p(D(p) - s(r(q)))f(r(q))r'(q) - c \\ &= p(1-F(p)) + p\int_{r(q)}^{p} dF(r) - pqf(r(q))r'(q) + \\ &\quad pqf(r(q))r'(q) - c \\ &= p(1-F(r(q))) - c. \end{aligned} \quad (11)$$

Summarizing (9) and (11), we get

$$\frac{\partial \pi_1}{\partial q}((p,q),\mu) = \begin{cases} p(1-F(p)) - c & \text{if} \quad D(p) - s(p) < q, \\ p(1-F(r(q))) - c & \text{if} \quad D(p) - s(p) > q \geqslant D(p) - k. \end{cases} \quad (12)$$

It can be verified that $\overline{p}(1 - F(\overline{p})) - c = 0$ and

$$p\left(1 - \frac{(p-c)k - \overline{\pi}}{p(2k - D(p))}\right) - c > 0 \quad (13)$$

for all $p \in [\underline{p}, b] \setminus \{\overline{p}\}$. Since F does not have an atom at price $\overline{p}$ we have

$$\pi_1((\overline{p}, q), \mu) = \overline{\pi}$$

for all $q \in [D(p) - k, k]$.

Assume that we have $p(1 - F(p)) - c = 0$ for all $p \in [\overline{p}, r^*)$ resulting for any $q \in [D(p) - s(p), k]$ in the same profits by (12). Then

$$F(p) = 1 - \frac{c}{p} \quad (14)$$

for all $p \in [\overline{p}, r^*)$, and therefore the firms never produce less than $D(p) - s(p)$ for any $p \in [\underline{p}, r^*)$ by $p(1 - F(r(q))) - c > 0$ and (12). Now from (8) and (14) we can derive s on the respective interval by solving

$$\begin{aligned}\overline{\pi} = \pi_1((p,q),\mu) &= pq\frac{c}{p} + p\int_{\overline{p}}^{p} (D(p) - s(r))\frac{c}{r^2} dr + \\ &\quad p\int_{\underline{p}}^{\overline{p}} (D(p)-k) f(r) dr - cq \\ &= pD(p)\left(1 - \frac{c}{p}\right) - pk\left(1 - \frac{c}{\overline{p}}\right) \\ &\quad - p\int_{\overline{p}}^{p} s(r)\frac{c}{r^2} dr, \end{aligned} \quad (15)$$

where we have taken into account (12) together with our observations from this paragraph. Let

$$S(p) = \int_{\overline{p}}^{p} s(r)\frac{c}{r^2} dr \quad (16)$$

for any $p \in [\overline{p}, r^*)$. Then we have

$$S(\overline{p}) = 0 \text{ and } S'(p) = s(p)\frac{c}{p^2} \quad (17)$$

for any $p \in [\overline{p}, r^*)$. From (15) we get

$$S(p) = \frac{pD(p)\left(1 - \frac{c}{p}\right) - pk\left(1 - \frac{c}{\overline{p}}\right) - \overline{\pi}}{p} \quad (18)$$

for any $p \in [\overline{p}, r^*)$ from which by differentiation we obtain S' and finally by simple rearrangements we get (5). By differentiation and rearrangements we get

$$s'(p) = D''(p)p\left(\frac{p}{c} - 1\right) + D'(p)\frac{2p}{c} \quad (19)$$

from which by Assumption 1 it follows that $s'(p)$ is negative, and thus s is indeed strictly decreasing. It can be verified that $s(p)$ is continuous at $\overline{p}$ by evaluating the expression in (5) at $\overline{p}$.

We verify that $s'(p) < D'(p)$ holds for prices higher than $\overline{p}$ by the following series of inequalities and a final rearrangement:

$$\begin{aligned} 0 &> \left[D''(p)p + 2D'(p)\right]\left(\frac{p}{c} - 1\right) = D''(p)p\left(\frac{p}{c} - 1\right) \\ &\quad + D'(p)\left(\frac{2p}{c} - 2\right) \\ &> D''(p)p\left(\frac{p}{c} - 1\right) + D'(p)\left(\frac{2p}{c} - 1\right) \Rightarrow \\ D'(p) &> D''(p)p\left(\frac{p}{c} - 1\right) + D'(p)\frac{2p}{c} = s'(p), \end{aligned}$$

where first, we employed Assumption 1, $p > c$, and finally (19). Taking $D(\overline{p})/2 < s(\overline{p})$ into account the desired uniqueness of r^* follows.

Clearly, both S and s can be extended through Eqs. (16) and (17) for prices higher than r^*, respectively, where for $p \geqslant r^*$ Eq. (15) takes the following form

$$\begin{aligned}\overline{\pi} = \pi_1((p,q),\mu) &= pq\frac{c}{p} + p\int_{r^*}^{p} s(r)\frac{c}{r^2} dr \\ &\quad + p\int_{\overline{p}}^{r^*} (D(p) - s(r))\frac{c}{r^2} dr + \\ &\quad p\int_{\underline{p}}^{\overline{p}} (D(p)-k) f(r) dr - cq \\ &= pD(p)\left(1 - \frac{c}{r^*}\right) - pk\left(1 - \frac{c}{\overline{p}}\right) - \\ &\quad p\int_{\overline{p}}^{r^*} s(r)\frac{c}{r^2} dr + p\int_{r^*}^{p} s(r)\frac{c}{r^2} dr. \end{aligned} \quad (20)$$

For any $p \geqslant r^*$ let

$$Q(p) = \int_{r^*}^{p} s(r)\frac{c}{r^2} dr. \quad (21)$$

Then we have

$$Q(r^*) = 0 \text{ and } Q'(p) = s(p)\frac{c}{p^2} \quad (22)$$

for any $p \in [r^*, r')$, where r' is uniquely defined by the implicit equation $s(r') = D(r') - k$. Clearly, setting prices above r' does no make sense, since playing these pure strategies against mixed-strategy $\mu_{s,F}$ will result in less profits than pure-strategy $(\overline{p}, D(\overline{p})-k)$. From (20) we get

$$Q(p)=\frac{pD(p)\left(1-\frac{c}{r^*}\right)-pk\left(1-\frac{c}{\overline{p}}\right)-pS(r^*)-\overline{\pi}}{p} \tag{23}$$

for any $p\in\left[r^*, r',\right)$ from which by differentiation we obtain Q' and finally by simple rearrangements $s(p)$. With a slight abuse of notation we will still denote the obtained function by $s(p)$ on $p\in\left(r^*, r'\right)$ though, as it will turn out, the firms will not produce at prices above r^*. These extensions will be helpful for us in the price interval $[r^*, r']$.

Now we will verify that having an atom at price r^* of mass $c/r^* = 1-F(r^*)$ completes a symmetric mixed-strategy equilibrium. We shall denote the price distribution that has just been completely specified by F. Assume that firm 2 plays the same mixed strategy. Then we already know that for any $p\in\left[\underline{p}, r^*\right)$ producing an amount of $q=s(p)$ results in $\overline{\pi}$ profit by Proposition 1 and the definition of s on $p\in[\overline{p}, r^*)$ by (17). Furthermore, for any $p\in\left[\underline{p}, \overline{p}\right)$ producing less than k results in less profits then $\overline{\pi}$, and for any $p\in[\overline{p}, r^*)$ and any quantity $[D(p)-s(p), k]$ profits equal $\overline{\pi}$, while they are strictly less for quantities less than $D(p)-s(p)$ by (12).

We claim that in the derived symmetric mixed-strategy equilibrium firms produce at price r^* an amount of $s(r^*)=D(r^*)/2$. Suppose that they would produce more than $D(r^*)/2$. Then there will be superfluous production at price r^*, and therefore by the continuity of profits for prices below r^* profits at price r^* would be less then at prices $r^*-\varepsilon$ if ε is sufficiently small. Suppose that they would produce an amount of q^* less than $D(r^*)/2$. Then $\pi_1\left((p,q),\mu_{s,F}\right)$ is continuous at (q^*, r^*), and therefore $\pi_1\left((r^*,q^*),\mu_{s,F}\right)<\overline{\pi}$; a contradiction. Thus, we must have indeed $s(r^*)=D(r^*)/2$. By the left continuity at price r^* it follows that $\pi_1\left((r^*, D(r^*)/2),\mu_{s,F}\right)=\overline{\pi}$.

To verify that the triple $(\widehat{p}, s, F)$ specified in the previous paragraphs specifies a strategy of a symmetric mixed-equilibrium it remains to be shown that prices above r^* combined with any quantity $q\in[0,k]$ result in less profits than $\overline{\pi}$.

The profit function of firm 1 in response to firm 2 playing the mixed strategy associated with $(\widehat{p}, s, F)$ for prices $p\geqslant r^*$ equals

$$\begin{aligned}\pi_1\left((p,q),\mu_{s,F}\right) &= p\min\left\{D(p)-\frac{D(r^*)}{2}, q\right\}\frac{c}{r^*}\\ &+p\int_{\overline{p}}^{r^*}\left(D(p)-s(r)\right)\frac{c}{r^2}dr+\\ &p\int_{\underline{p}}^{\overline{p}}\left(D(p)-k\right)f(r)dr-cq\end{aligned} \tag{24}$$

from which we get[11]

$$\frac{\partial\pi_1}{\partial q}((p,q),\mu)=\begin{cases}-c & \text{if}\quad D(p)-\frac{D(r^*)}{2}<q,\\ p\frac{c}{r^*}-c & \text{if}\quad D(p)-\frac{D(r^*)}{2}>q\geqslant D(p)-k\end{cases} \tag{25}$$

for any $p>\widehat{p}=r^*$. Since $pc/r^*-c>0$ we get that quantity $q=D(p)-\frac{D(r^*)}{2}$ results in the highest profit in (24) for any price $p>\widehat{p}=r^*$.

[11] Note that (12) is only valid for $(p,q)\in(\overline{p},\widehat{p})\times[0,k]$, while here we need the first order condition for $p>\widehat{p}$.

Hence, we define the profit function of firm 1 at the best quantities for prices $p\geqslant r^*$ by

$$\begin{aligned}\pi^*(p) &= p\left(D(p)-\frac{D(r^*)}{2}\right)\frac{c}{r^*}+p\int_{\overline{p}}^{r^*}\left(D(p)-s(r)\right)\frac{c}{r^2}dr+\\ &\quad p\int_{\underline{p}}^{\overline{p}}\left(D(p)-k\right)f(r)dr-c\left(D(p)-\frac{D(r^*)}{2}\right)\end{aligned} \tag{26}$$

It can be verified that $\pi^*(p)$ is strictly concave, and it would be straightforward to check that the derivative $\pi^*(p)$ is non-positive at r^*, which unfortunately does not result in a manageable inequality. Therefore, we consider the equality in (20) defining s and let us denote by

$$\begin{aligned}\pi^s(p) &= p\int_{r^*}^{p}s(r)\frac{c}{r^2}dr+p\int_{\overline{p}}^{r^*}\left(D(p)-s(r)\right)\frac{c}{r^2}dr+\\ &\quad p\int_{\underline{p}}^{\overline{p}}\left(D(p)-k\right)f(r)dr=\overline{\pi}\end{aligned} \tag{27}$$

for prices $p\in\left[r^*, r'\right]$. Clearly, $d\pi^s(p)/dp=0$ for any $p\in\left[r^*, r'\right]$ by the definition of s, which we will utilize by considering $\Delta(p)=$

$$\begin{aligned}\pi^*(p)-\pi^s(p) &= p\left(D(p)-\frac{D(r^*)}{2}\right)\frac{c}{r^*}-c\left(D(p)-\frac{D(r^*)}{2}\right)\\ &\quad -p\int_{r^*}^{p}s(r)\frac{c}{r^2}dr\\ &=\left(D(p)-\frac{D(r^*)}{2}\right)\left(p\frac{c}{r^*}-c\right)\\ &\quad -p\int_{r^*}^{p}s(r)\frac{c}{r^2}dr.\end{aligned} \tag{28}$$

Then

$$\begin{aligned}\Delta'(p) &= D'(p)\left(p\frac{c}{r^*}-c\right)+\left(D(p)-\frac{D(r^*)}{2}\right)\frac{c}{r^*}-\\ &\quad\int_{r^*}^{p}s(r)\frac{c}{r^2}dr-ps(p)\frac{c}{p^2}.\end{aligned} \tag{29}$$

By substituting r^* for p in (29) and taking $s(r^*)=D(r^*)/2$ into consideration we get $\Delta'(r^*)=0$, which implies $d\pi^*(p)/dp=0$, which completes the proof. □

The functional form of the equilibrium price distribution in Montez and Schutz (2019, Lemma IV) is identical with the equilibrium price distribution given in (4) on the interval $[\overline{p}, \widehat{p})$. However, besides the respective intervals the associated production levels are significantly different. In their equilibrium each firm produces the market demand $D(p)$ at price p, while in the equilibrium we have obtained each firm produces less than its capacity k (which is even less than $D(p)$).

The region of intermediate capacities not covered by Proposition 2 appears to be far more complex. We conjecture that the price distribution F specified in Proposition 2 remains still the equilibrium price distribution in the 'high range' of intermediate capacities. Furthermore, the expression on the right-hand side of (5) still specifies $s(p)$ on the interval $[\overline{p}, P(k)]$ since in this case in the proof of Proposition 2 in Eq. (7) $D(p)-s(r)$ is non-negative for any $p\in[\overline{p}, P(k)]$ and any $r\in[\overline{p}, p]$. We expect that s can be defined recursively and will be piecewise strictly decreasing and twice continuously differentiable on $\left[\underline{p}, \widehat{p}\right]$.

Providing more details about the idea, let s_1 be the expression on the right-hand side of (5). For notational convenience let $p_1=P(k)$ and $s_0(p)=k$ for any $p\in\left[\underline{p}, \widehat{p}\right]$. When extending function s to prices above $\overline{p}$ one needs to integrate in (7) $D(p)-s_1(r)$ only above prices r on which the integrand is non-negative. To determine the lowest price from which the integration of

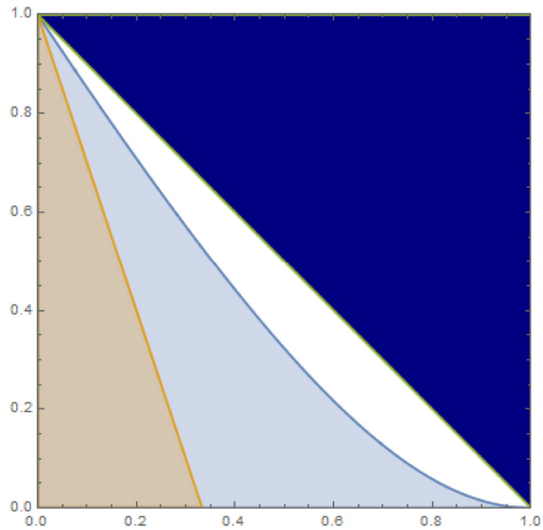

Fig. 1. Four different cases.

$D(p)-s_1(r)$ should start for a given p defining $t_1(p)=s_1^{-1}(D(p))=r$ would be useful. The strategy for constructing the mixed-strategy equilibrium would be to determine the next piece of s denoted by s_2. Then arriving either to r^* the top of the support of solution of F satisfying $s_2(r^*)=D(r^*)/2$ and $r^*\leqslant p_2=s_2(p_1)$ or repeating the same process to obtain the next piece of s denoted by s_3. The process should be repeated until an r^* obtains satisfying $s_n(r^*)=D(r^*)/2$ and $r^*\leqslant p_n=s_n(p_{n-1})$.

For the demand curve $D(p)=1-p$ Fig. 1 shows the four different cases we can have. On the horizontal axis we have k, while on the vertical axis we have c. The darkest shaded triangle depicts the case of large capacities ($c\geqslant 1-k$), the triangle on the bottom ($c\leqslant 1-3k$) depicts the case of small capacities, the shaded area in the middle ($s(1-k)\leqslant k/2$) depicts the case of intermediate capacities covered by Proposition 2, and the white area depicts the case for which we have not determined an equilibrium in mixed strategies.

To illustrate Proposition 2 we provide an example.

Example 1. Let $D(p)=1-p$, $k=0.4$ and $c=0.1$.

Then one can obtain that the price maximizing the residual profit function equals $\overline{p}=0.35$ and results in $\overline{\pi}=0.0625$ profit. The price at which a firm is indifferent between maximizing profits on the residual demand curve and selling its entire capacity equals $\underline{p}=0.25625$. By (4) a firm never sets prices below $\underline{p}$ and produces at its capacity limit $k=0.4$ when setting prices in $[\underline{p},\overline{p}]=[0.25625,0.35]$. Furthermore, by (4) in equilibrium at prices above $\overline{p}=0.35$ firms produce

$$s(p)=1.625-10p^2$$

for all $p\in[\overline{p},\widehat{p}]$, where $\widehat{p}$ is obtained as the solution of equation $s(p)=(1-p)/2$, and in particular, we get $\widehat{p}=0.36134060117684275$. The cumulative distribution function of prices set by a firm in equilibrium is given by

$$F(p)=\begin{cases}0 & \text{if } 0\leqslant p<\underline{p},\\ \frac{0.4p-0.12}{(p-0.2)p} & \text{if } \underline{p}\leqslant p<\overline{p},\\ 1-\frac{0.1}{p} & \text{if } \overline{p}\leqslant p<\widehat{p}, \text{ and}\\ 1 & \text{if } \widehat{p}\leqslant p\leqslant 1.\end{cases}$$

It is worthwhile mentioning that only the highest possible price $\widehat{p}$ is chosen with positive probability in equilibrium, in particular with probability $0.1/\widehat{p}\approx 0.27675$.

Firm 1's profits in response to firm 2 playing its equilibrium strategy given above can be seen in Fig. 2 in which prices range (x-axis) from $\underline{p}=0.25625$ to 0.45 (well beyond $\widehat{p}=0.36134060117684275$), moreover the full quantity range $[0,0.4]$ (y-axis) is admitted. The equilibrium profit $\overline{\pi}=0.0625$ is the highest value that can be seen on the vertical axis (z-axis). Basically, one can see four curves on the profit surface depicted in Fig. 2: the one starting at price level $\overline{p}=0.35$, the one starting at price level $\widehat{p}=0.36134060117684275$, the one corresponding to the residual demand $D(p)-s(p)$, and the one corresponding to the demand curve $1-p$. These four curves partition the profit surface into nine regions. Starting from the x-axis, one can see that profits are increasing in the three regions as we are getting closer to residual demand since the firm is only bounded by its production. Looking at the remaining two areas ending at the highest price, one can observe that setting higher prices than $\widehat{p}$ results in decreasing profits. Out of these two in the sharply declining area the firm produces more than total demand. The vertical plane at $q=0.4$ intersects the fourth curve (associated with $D(p)=1-p$) at price $\underline{p}$. Profits are the highest in the triangular area at the edge when the firm produces at its capacity limit, in the entire neighboring small rectangular area and in the rectangular area 'in the middle'.

We conclude this section with a remark on the relationship between the production in advance game and the respective classical Cournot and Bertrand games.

Remark 1. Let the assumptions of Proposition 2 hold. Then in the respective Bertrand game both firms set prices equal to their common unit cost, and therefore, firms set lower prices and achieve less profits in the Bertrand game than in the investigated production-in-advance game. Furthermore, since residual demand $D(p)-k$ is smaller than $D(p)-q^c$, where q^c stands for the equilibrium of the respective Cournot game, firms' profits in case of production-in-advance are smaller than Cournot profits.

4. Economic surplus

In this section we compare the production-in-advance game with the production-to-order game based on their economic (i.e. Marshallian) surpluses in equilibrium, which is given by

$$ES(p_1,q_1,p_2,q_2)=\begin{cases}\int_0^{\min\{D(p_j),q_1+q_2\}}P(q)dq-c(q_1+q_2) & \text{if } D(p_j)>q_i,\\ \int_0^{\min\{D(p_i),q_i\}}P(q)dq-c(q_1+q_2) & \text{if } D(p_j)\leqslant q_i,\end{cases}$$

where $0\leqslant p_i\leqslant p_j\leqslant b$.

Assuming linear demand $D(p)=1-p$ and $c=1/6$, we illustrate firms' profits and consumers' surplus in Fig. 3. The lightest gray triangle corresponds to the surplus realized by the consumers who purchase the product at the highest price, while the light-gray area depicts the surplus realized by the other consumers. On the producers' side, the low-price firm's surplus is given by the darkest-gray rectangle and the high-price firm's surplus by the dark-gray area. Note that economic surplus (i.e. the sum of the previously described four areas) is determined by the higher price, except when the residual demand equals zero at the higher price.

We would like to emphasize that if sales occur at the higher price, then the economic surplus is determined at the higher price.

It is well-known that for small capacities in the pure-strategy equilibrium of the production-to-order game the firms set the market-clearing price, and thus the production-to-order and the production-in-advance versions of the game have the same outcome. It also means that their economic surpluses are identical.

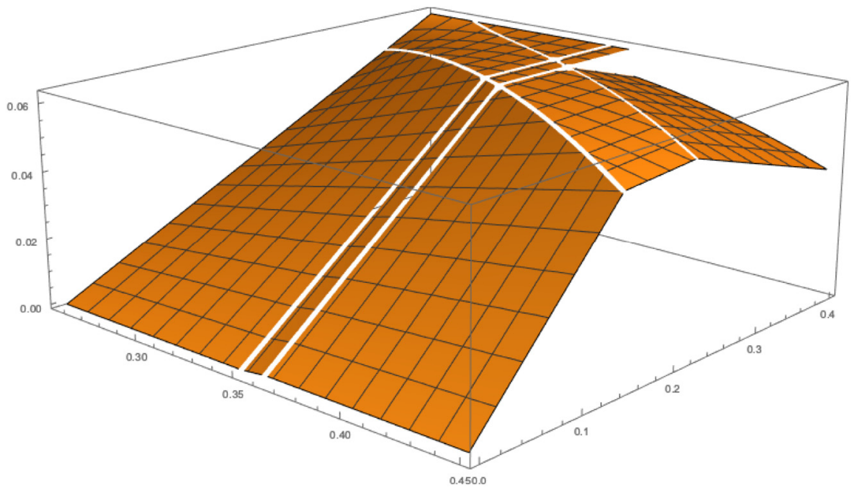

Fig. 2. Profit function $\pi_1\left((p, q), \mu_{s,F}\right)$.

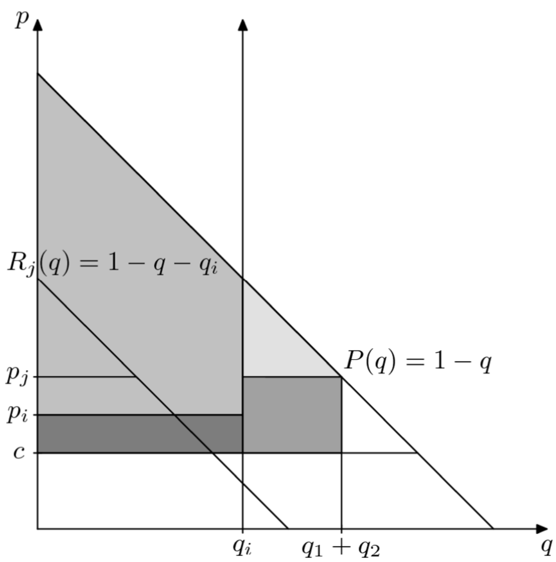

Fig. 3. Economic surplus.

For large capacities in the equilibrium of the production-to-order game firms set prices equal to unit costs, while in the equilibrium of the production-in advance game firms set prices above unit costs with positive probability (see for instance, Montez and Schutz, 2018). Therefore, the economic surplus is higher in the production-to-order game than in the production-in-advance game.

For the case of intermediate capacities (i.e. $\overline{p} > \max\{p^*, c\}$) it is well-known (see for instance Vives, 1986) that there is only an equilibrium in nondegenerated mixed strategies with cumulative distribution function

$$G(p) = \frac{(p-c)\,k - \overline{\pi}}{(p-c)\,(2k - D(p))} \tag{30}$$

for any $p \in \left[\underline{p}, \overline{p}\right]$. We will rely on (30) in the proof of our next proposition stating that in case of intermediate capacities economic surplus is higher in the production-to-order game than in the production-in-advance game.

Proposition 3. *Under Assumptions 1–3, $\overline{p} > \max\{p^*, c\}$ and that a symmetric equilibrium is played, the economic surplus is higher in the production-to-order game than in the production-in-advance game.*

Proof. First, note that the economic surplus related to the partially revealed symmetric mixed-strategy equilibrium by Proposition 1 is lower than in the case when both firms play mixed strategies F partially determined by (3) on the interval $\left[\underline{p}, \overline{p}\right)$ in the production-to-order game. By the latter modification the loss in economic surplus due to both underproduction and overproduction is eliminated.

Second, since F stochastically dominates G, the respective cumulative distribution function of the higher price F^2 also stochastically dominates G^2. Note that sales occur always at the higher price in case of production to order. Since the cumulative distribution functions F^2 and G^2 determine the economic surpluses if sales occur at the higher price and selling just at the lower price in case of production in advance results in less economic surplus than for the same price pair in case of production to order, the statement of the proposition holds true. □

Remark 2. Let the assumptions of Proposition 2 hold. Then economic surplus is the smallest in the production-in-advance game, second largest in the production-to-order game and the largest in the Bertrand game. The relationship between the economic surpluses of the Cournot game and the production-in-advance game is far from obvious and would need a thorough additional analysis.

5. Conclusions

In this paper we have derived analytically a symmetric mixed-strategy equilibrium of the production-in-advance game for a large region of intermediate capacities. This is the most difficult case to solve within the capacity-constrained framework compared with the case of small and large capacities.

In the case of small capacities the game has a simple pure-strategy equilibrium. The case of large capacities has also, as in the case of intermediate capacities, just a non-degenerated mixed-strategy equilibrium. In the mixed-strategy equilibrium of the large-capacity case firms produce at any price, with the entire demand emerging at that price (see Montez and Schutz, 2019, Lemma IV), while in the intermediate capacity-case there is a price region in which firms produce below their capacity constraints, i.e., they are not producing at the boundary of their maximum sales $D(p)$ or production possibilities k. Probably, this is the one of the main reasons why the intermediate-capacity case is more difficult to solve than the large-capacity case. The other reason might be that the lowest possible price in the support of the equilibrium price distribution equals unit costs in case of large capacities, while it lies above unit costs in the case of intermediate capacities.

Furthermore, we have demonstrated that economic surplus will be lower in case of production in advance than in case of production to order. This observation remains to be true also for the case of large capacities. However, for the case of small capacities the two games both have the same market-clearing outcome resulting in identical economic surpluses. To summarize, economic surplus is weekly lower in the production-in-advance environment than in the production-to-order one.

In a future research one could try to replace the efficient rationing of consumers imposed by Assumption 3 with another way (e.g. random rationing) of rationing consumers by the low-price firm. Comparing the available results for efficient rationing with random rationing in the production-to-order framework in case of intermediate capacities, one can expect far more complex calculations under random rationing than the ones carried out in the present paper. In particular, compare Levitan and Shubik (1972) with Beckmann (1965) on the mixed-strategy equilibrium in closed form or Vives (1986) with Allen and Hellwig (1986) on the limiting behavior of the production-to-order game when the number of firms tend to infinity.

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