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Predictor-corrector interior-point algorithm for $P_*(\kappa)$-linear complementarity problems based on a new type of algebraic equivalent transformation technique

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ABSTRACT

We propose a new predictor-corrector (PC) interior-point algorithm (IPA) for solving *linear complementarity problem* (LCP) with $P_*(\kappa)$-matrices. The introduced IPA uses a new type of *algebraic equivalent transformation* (AET) on the centering equations of the system defining the central path. The new technique was introduced by Darvay and Takács (2018) for linear optimization. The search direction discussed in this paper can be derived from positive-asymptotic kernel function using the function $\varphi(t) = t^2$ in the new type of AET. We prove that the IPA has $O\left((1+4\kappa)\sqrt{n}\log\frac{3n\mu^0}{4\epsilon}\right)$ iteration complexity, where κ is an upper bound of the handicap of the input matrix. To the best of our knowledge, this is the first PC IPA for $P_*(\kappa)$-LCPs which is based on this search direction.

1. Introduction

The linear complementarity problem (LCP) is a well-known problem which includes linear programming (LP) and linearly constrained (convex) quadratic programming problem (QP), as special cases. The most important basic results related to LCPs are summarized in the books of Cottle, Pang, & Stone (1992) and Kojima, Megiddo, Noma, & Yoshise (1991). Many classical applications of LCPs can be found in different fields, such as optimization theory, game theory, economics, engineering, etc. Cottle et al. (1992); Ferris & Pang (1997). For example, bimatrix games can be transformed into LCPs under specific assumptions Lemke & Howson (1964). Kojima & Saigal (1979) used the degree theory in order to study LCPs. Furthermore, the Arrow–Debreu competitive market equilibrium problem with linear and Leontief utility functions can be also given as LCP (Ye, 2008). More recent work of Brás, Eichfelder, & Júdice (2016) connected the copositivity testing of matrices and solvability of special LCPs. Darvay, Illés, Povh, & Rigó (2020b) published a PC IPA for sufficient LCPs using the function $\bar{\varphi}(t) = t - \sqrt{t}$ for AET, but tested numerically their algorithm beyond the class of sufficient matrices, too. Numerical results produced by the developed PC IPA for testing copositivity of matrices using LCPs were very promising. Sloan & Sloan (2020) showed that solvability of LCPs related to quitting games ensures the existence of different ε-equilibrium solutions. There is no reported computational study on this type of application of LCPs, yet.

In the LCP we want to find vectors $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$, that satisfy the constraints

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \;\; \mathbf{xs} = \mathbf{0}, \;\; \mathbf{x}, \mathbf{s} \geq \mathbf{0}, \tag{LCP}$$

where $M \in \mathbb{R}^{n\times n}$, $\mathbf{q} \in \mathbb{R}^n$ and $\mathbf{xs}$ denotes the Hadamard product of vectors $\mathbf{x}$ and $\mathbf{s}$. The following notations are used to denote the feasible region, the interior and the solutions set of LCP:

$$\begin{aligned} \mathcal{F} &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^n_{\oplus} \times \mathbb{R}^n_{\oplus} : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \\ \mathcal{F}^+ &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^n_{+} \times \mathbb{R}^n_{+} : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \quad \text{and} \\ \mathcal{F}^* &:= \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F} : \mathbf{xs} = \mathbf{0}\}. \end{aligned}$$

We denoted by $\mathbb{R}^n_{\oplus}$ the n-dimentional nonnegative orthant and by $\mathbb{R}^n_{+}$ the positive orthant, respectively. We call a problem $P_*(\kappa)$-LCP if the problem's matrix of (LCP) is $P_*(\kappa)$-matrix, i.e.

$$(1+4\kappa)\sum_{i\in I_+(\mathbf{x})} x_i(M x)_i + \sum_{i\in I_-(\mathbf{x})} x_i(M x)_i \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

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where

$I_+(\mathbf{x}) = \{1 \le i \le n : x_i(Mx)_i > 0\}$ and
$I_-(\mathbf{x}) = \{1 \le i \le n : x_i(Mx)_i < 0\}$

and $\kappa \ge 0$ is a nonnegative real number. We will assume throughout the paper that $\mathcal{F}^+ \ne \emptyset$, there is an initial point $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^+$ and M is a $P_*(\kappa)$-matrix. The class of P_* matrices is the set of all $P_*(\kappa)$-matrices, where $\kappa \ge 0$. Väliaho (1996) showed that the class of P_*-matrices is equivalent to the class of sufficient matrices given by Cottle, Pang, & Venkateswaran (1989). The handicap of M (Väliaho, 1996) is the smallest value of $\hat{\kappa}(M) \ge 0$ such that M is $P_*(\hat{\kappa}(M))$-matrix. Väliaho (1996) also proved that a matrix M is P_* if and only if the handicap $\hat{\kappa}(M)$ of M is finite.

There are several methods for solving LCPs with different matrices, such as simplex (Csizmadia, Csizmadia, & Illés, 2018; van de Panne & Whinston, 1964; 1969; Wolfe, 1959), criss-cross (Csizmadia & Illés, 2006; Csizmadia, Illés, & Nagy, 2013; den Hertog, Roos, & Terlaky, 1993; Fukuda, Namiki, & Tamura, 1998; Fukuda & Terlaky, 1997) or other pivot (Lemke, 1968; van de Panne, 1974) algorithms. However, the IPAs for solving LCPs received more attention in last decades (Kojima et al., 1991). It should be mentioned that LCPs belong to the class of NP-complete problems (Chung, 1989). In spite of this fact, due to the results of Kojima et al. (1991), if we suppose that the problem's matrix has $P_*(\kappa)$-property, the IPAs solving these kind of LCPs usually have polynomial complexity in the handicap of the problem's matrix, the size of the problem and the bitsize of the data. However, note that the worst-case iteration complexity of the IPAs for LCP depends on the upper bound of the handicap of the matrix M. de Klerk & Nagy (2011) showed that the handicap of a $P_*(\kappa)$-matrix may be exponential in its bit size. This means that if the handicap of the matrix is exponentially large in the size and bit size of the problem, then the known complexity bounds of IPAs may not be polynomial in the input size of the LCP.

Potra & Liu (2005) proposed an IPA for sufficient LCPs which uses a wide neighbourhood of the central path and the algorithm does not depend on the handicap of the problem. There are several known IPAs not depending on the handicap of the sufficient matrix, such as the IPAs given by Potra & Sheng (1997), Potra & Liu (2005), Illés & Nagy (2007), Liu & Potra (2006) and Lešaja & Potra (2019). The IPAs for solving sufficient LCPs have been also extended to general LCPs (Illés, Nagy, & Terlaky, 2010a; 2010b). Illés, Nagy, & Terlaky (2009, 2010a) generalized large-update, affine scaling and PC IPAs for solving LCPs with general matrices.

The PC IPAs perform a predictor and one or more corrector steps in a main iteration. The aim of the predictor step is to reach optimality, hence after an affine-scaling step a certain amount of deviation from the central path is allowed. The goal of the corrector step is to return in the neighbourhood of the central path. The PC IPAs turned out to be efficient in practice. The first PC IPA for LO was given by Mehrotra (1992) and Sonnevend, Stoer, & Zhao (1991). Potra & Sheng (1996, 1997) defined PC IPAs for sufficient LCPs. Mizuno, Todd, & Ye (1993) gave the first PC IPA for LO which uses only one corrector step in a main iteration and these IPAs were named Mizuno–Todd–Ye (MTY) type PC IPAs. Miao (1995) extended the MTY IPA given in Mizuno et al. (1993) to $P_*(\kappa)$-LCPs. Following this result, several MTY type PC IPAs have been proposed among others by Illés & Nagy (2007), Kheirfam (2014) and Darvay et al. (2020b). In Darvay et al. (2020b) the authors gave a unified framework to determine the Newton systems and scaled systems in case of PC IPAs using the AET technique.

Barrier functions are often used for the determination of search directions in case of IPAs. By considering self-regular kernel functions, Peng, Roos, & Terlaky (2002) reduced the theoretical complexity of large-update IPAs. Later on, Lešaja & Roos (2010) provided a unified analysis of IPAs for $P_*(\kappa)$-LCPs that are based on eligible kernel functions. Tunçel & Todd (1997) considered for the first time a reparametrization of the central path system. Karimi, Luo, & Tunçel (2017) used entropy-based search directions for LP working in a wide neighbourhood of the central path. Darvay (2003) proposed the AET technique for defining search directions in case of IPAs for LO. He divided both sides of the nonlinear equation of the central path system by the barrier parameter μ. After that he applied a continuously differentiable, invertible, monotone increasing function $\bar{\varphi} : (\xi^2, \infty) \to \mathbb{R}$, where $0 \le \xi < 1$, on the modified nonlinear equation of the central path problem. The majority of the published IPAs for sufficient LCPs does not use any transformation of the central path equations, which means that these IPAs use the identity map in the AET technique in order to define the search directions. Darvay (2003, 2005) used the square root function in the AET technique for LO. Later on, Darvay, Papp, & Takács (2016) introduced an IPA for LO based on the direction using the function $\bar{\varphi}(t) = t - \sqrt{t}$. In her Ph.D. thesis, Rigó (2020) presented several IPAs that use the function $\bar{\varphi}(t) = t - \sqrt{t}$ in the AET technique. Recently, Kheirfam & Haghighi (2016) have proposed an IPA for $P_*(\kappa)$-LCPs which uses the function $\bar{\varphi}(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ in the AET technique. Haddou, Migot, & Omer (2019) have recently introduced a family of smooth concave functions which leads to IPAs with the best known iteration bound. The AET technique has been also extended to LCPs (Achache, 2010; Asadi & Mansouri, 2012; 2013; Asadi, Mansouri, & Darvay, 2017; Asadi, Zangiabadi, & Mansouri, 2016; Kheirfam, 2014; Mansouri & Pirhaji, 2013).

Zhang & Xu (2011) used the equivalent form $\mathbf{v}^2 = \mathbf{v}$ of the centering equation, where $\mathbf{v} = \sqrt{\frac{\mathbf{xs}}{\mu}}$, $\mu > 0$. They considered the $\mathbf{xs} = \mu\mathbf{v}$ transformation. Darvay & Takács (2018) introduced a new method for determining class of search directions using a new type of AET of the centering equations. They modified the nonlinear equation $\mathbf{v}^2 = \mathbf{v}$ by applying componentwisely a continuously differentiable function $\varphi : (\xi^2, \infty) \to \mathbb{R}$, $0 \le \xi < 1$ to the both sides of this equation. The properties of this function φ will be presented in Section 2.2. The relationship between the functions φ and $\bar{\varphi}$ will be discussed later as a novelty of this paper. In Darvay & Takács (2018) the authors considered the function $\varphi(t) = t^2$ in order to determine the new search directions. Zhang, Huang, Li, & Lv (2020) extended the feasible IPA given in Darvay & Takács (2018) to $P_*(\kappa)$-LCPs. Furthermore, Takács & Darvay (2018) generalized the approach for determining search directions proposed in Darvay & Takács (2018) to SO and they showed that the corresponding kernel function is a positive-asymptotic kernel function. The positive-asymptotic kernel function was introduced by Darvay & Takács (2018) and differs from the class of kernel functions introduced by Bai, El Ghami, & Roos (2004).

In this paper we introduce a new PC IPA for solving $P_*(\kappa)$-LCPs which uses the new type of AET given in Darvay & Takács (2018) for LO. The proposed IPA applies the function $\varphi(t) = t^2$ on the modified nonlinear equation $\mathbf{v}^2 = \mathbf{v}$ in order to obtain the search directions. In this sense, the corresponding kernel function is a positive-asymptotic kernel function. Similar to Darvay et al. (2020b) we present the method for determining the Newton systems and scaled systems in case of PC IPAs using this new type of AET. We also present the complexity analysis of the proposed PC IPA. Due to the used search direction we have to ensure during the whole process of the IPA that the components of the vector $\mathbf{v}$ are greater than $\frac{\sqrt{2}}{2}$, which makes the analysis more difficult. In spite of this fact, we show that the introduced IPA has $O((1+4\kappa)\sqrt{n}\log\frac{3n\mu^0}{4\epsilon})$ iteration complexity, where κ is the upper bound on the handicap of matrix M, μ^0 is the starting, average complementarity gap and ε is the final displacement from the complementarity gap, respectively. This is the first PC IPA for solving $P_*(\kappa)$-LCPs which uses the function $\varphi(t) = t^2$ in the new type of AET.

The paper is organized as follows. In Section 2 we give some basic concepts and useful results about the $P_*(\kappa)$-LCPs and $P_*(\kappa)$-matrices. Furthermore, in Section 2.2, depending on the representation of the nonlinear equation of the central path, a new way of applying the AET is discussed and compared to the earlier used AET technique. The usual, but important, scaling technique is discussed together with the unique solvability of the Newton-system, as well. In Section 3 we present a method for determining search directions in case of PC IPAs for $P_*(\kappa)$-LCPs by using the new type of AET approach. In Section 4, the new PC IPA is presented. While, Section 5 contains the complexity analysis of the introduced PC IPA with the new search directions. In Section 6 numerical computations are presented and compared to the computational performance of an earlier introduced PC IPA appeared in Darvay et al. (2020b) that used different function φ in the AET. In Section 7 some concluding remarks are enumerated.

2. Algebraic equivalent transformation technique of the central path equations

In this section we summarize important definitions and results related to $P_*(\kappa)$-LCPs. Furthermore, we introduce the AET of the central path equations. Following the steps of Darvay & Takács (2018), first we derive a known, equivalent description of the central path and then we apply the AET approach, see Section 2.2. An important novelty of the paper is that in this section we compare the two different AET techniques introduced in Darvay (2003) and Darvay & Takács (2018), respectively. An interesting observation is related to the fact that the same search directions can be obtained in different ways.

2.1. Central path of sufficient LCPs

The central path problem for (LCP) is:

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q}, \quad \mathbf{x},\mathbf{s}>\mathbf{0}, \quad \mathbf{x}\mathbf{s}=\mu\,\mathbf{e}, \tag{2}$$

where $\mathbf{e}$ denotes the n-dimensional vector of ones and $\mu>0$. Kojima et al. (1991) showed that the sequence $\{(\mathbf{x}(\mu),\mathbf{s}(\mu))\mid \mu>0\}$ of solutions lying on the central path parameterised by $\mu>0$ approaches a solution $(\mathbf{x},\mathbf{s})$ of the (LCP).

Illés, Roos, and Terlaky gave an elementary constructive proof for the existence and uniqueness of the central path for sufficient LCPs in an unpublished manuscript in 1997. The constructive proof of Illés et al. appears in Theorem 3.6 in the Ph.D. thesis of Nagy (2009).

Similarly to Darvay & Takács (2018), we use $\mathbf{x},\mathbf{s}>\mathbf{0}$ and $\mu>0$, hence we obtain:

$$\mathbf{x}\mathbf{s}=\mu\,e \Leftrightarrow \frac{\mathbf{x}\mathbf{s}}{\mu}=\mathbf{e} \Leftrightarrow \sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}=\mathbf{e} \Leftrightarrow \frac{\mathbf{x}\mathbf{s}}{\mu}=\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}.$$

Now the central path problem for (LCP) can be equivalently stated as

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q}, \quad \mathbf{x},\mathbf{s}>\mathbf{0}, \quad \frac{\mathbf{x}\mathbf{s}}{\mu}=\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}. \tag{3}$$

Different forms of the central path problem (2) and (3) will be used later in the AET context.

An important result was proved in Lemma 4.1 of Kojima et al. (1991), which plays important role in the solvability of the Newton system. An important corollary of Lemma 4.1 presented in Kojima et al. (1991) is the following.

Corollary 2.1. *Let $M\in\mathbb{R}^{n\times n}$ be a $P_*(\kappa)$-matrix, $\mathbf{x},\mathbf{s}\in\mathbb{R}^n_+$. Then, for all $\mathbf{a}_\varphi\in\mathbb{R}^n$ the system*

$$\begin{aligned} -M\Delta\mathbf{x}+\Delta\mathbf{s}&=\mathbf{0}\\ S\Delta\mathbf{x}+X\Delta\mathbf{s}&=\mathbf{a}_\varphi \end{aligned} \tag{4}$$

has a unique solution $(\Delta\mathbf{x},\Delta\mathbf{s})$, where X and S are the diagonal matrices obtained from the vectors $\mathbf{x}$ and $\mathbf{s}$.

2.2. Relationship between the two different types of AET approaches

The goal of the AET technique introduced by Darvay (2003) is to represent the central path in a different way and to derive Newton-system from these representations depending on the continuously differentiable, invertible, monotone increasing function $\bar{\varphi}:(\xi^2,\infty)\to\mathbb{R}$, where $0\le\xi<1$.

Now, we can apply the AET to the central path problem in the form (2) or (3). In case of applying the AET method to (2), we obtain the following form of the central path

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q}, \quad \mathbf{x},\mathbf{s}>\mathbf{0}, \quad \bar{\varphi}\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)=\bar{\varphi}(\mathbf{e}). \tag{5}$$

However, if the AET is applied to (3), using the continuously differentiable, invertible function $\varphi:(\xi^2,\infty)\to\mathbb{R}$, where $0\le\xi<1$, then using the idea presented in Darvay & Takács (2018), we get

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q}, \quad \mathbf{x},\mathbf{s}>\mathbf{0}, \quad \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)=\varphi\left(\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}\right). \tag{6}$$

The following interesting question arises: if we use different transformed forms of the central path (say (5) and (6)), is it necessary to use some extra criterion on functions φ? An answer will be given at the end of this subsection.

An interesting observation is the connection between systems (5) and (6). For this, let $\bar{\varphi}:(\xi^2,\infty)\to\mathbb{R}$

$$\bar{\varphi}(t)=\varphi(t)-\varphi(\sqrt{t}). \tag{7}$$

This leads to

$$\bar{\varphi}\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)=\varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)-\varphi\left(\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}\right). \tag{8}$$

Hence, we have

$$\begin{aligned}\bar{\varphi}\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)=\bar{\varphi}(\mathbf{e}) &\Leftrightarrow \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)-\varphi\left(\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}\right)=\varphi(\mathbf{e})-\varphi(\sqrt{\mathbf{e}})\\ &\Leftrightarrow \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)=\varphi\left(\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}\right).\end{aligned}$$

Majority of the published IPAs using the AET, derives the Newton-system from (5), while only few, like the ones proposed by Darvay & Takács (2018), and Zhang et al. (2020) applies the AET to (6). We follow the second approach to derive the corresponding Newton-system.

For an $(\mathbf{x},\mathbf{s})\in\mathcal{F}^+$ our aim is to find search directions $\Delta\mathbf{x}$ and $\Delta\mathbf{s}$ such that

$$\begin{aligned}&-M(\mathbf{x}+\Delta\mathbf{x})+(\mathbf{s}+\Delta\mathbf{s})=\mathbf{q},\\ &\varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}+\frac{\mathbf{x}\Delta\mathbf{s}+\mathbf{s}\Delta\mathbf{x}+\Delta\mathbf{x}\Delta\mathbf{s}}{\mu}\right)\\ &=\varphi\left(\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}+\frac{\mathbf{x}\Delta\mathbf{s}+\mathbf{s}\Delta\mathbf{x}+\Delta\mathbf{x}\Delta\mathbf{s}}{\mu}}\right),\end{aligned}$$

We neglect the quadratic terms and apply Taylor's theorem to the function $\bar{\varphi}(t)=\varphi(t)-\varphi(\sqrt{t})$. Hence, after some calculations we obtain (4) with

$$\mathbf{a}_\varphi=\mu\,\frac{-\varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)+\varphi\left(\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}\right)}{\varphi'\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)-\frac{1}{2\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}}\varphi'\left(\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}\right)}. \tag{9}$$

Now, from the denominator of the obtained fractional expression, it is clear that we need extra assumption on the function φ, namely

$$2t\,\varphi'(t^2) - \varphi'(t) > 0, \tag{10}$$

for all $t > \xi$, with $0 \le \xi < 1$.

Lemma 2.2. *Consider $\bar{\varphi} : (\xi^2, \infty) \to \mathbb{R}$ as given in (7). Then, $\bar{\varphi} : (\xi^2, \infty) \to \mathbb{R}$ is monotone increasing if and only if condition (10) is satisfied for the function φ.*

Proof. Using (7) we have $\bar{\varphi}'(t) = \varphi'(t) - \frac{1}{2\sqrt{t}}\varphi'(\sqrt{t})$. Hence,

$$\bar{\varphi}'(t) > 0,\ \forall t > \xi^2 \quad \text{if and only if} \quad \varphi'(t) - \frac{1}{2\sqrt{t}}\varphi'(\sqrt{t}) > 0, \quad \forall t > \xi^2. \tag{11}$$

Considering change of variable $u := \sqrt{t}$ in the second part of (11) we obtain condition (10). □

Depending on the used functions φ we can have different vectors $\mathbf{a}_\varphi$. In Darvay et al. (2020b) and Rigó (2020) the authors presented the functions $\bar{\varphi}$ already used in the literature in case of IPAs in order to derive complexity results for different class of optimization problems, including LO and sufficient LCPs, as well.

Now, if a function φ satisfying condition (10) is applied to (6), then using (7) and Lemma 2.2 we immediately obtain an IPA with $\bar{\varphi}$ applied to (5). However, if a function $\bar{\varphi}$ satisfying $\bar{\varphi}'(t) > 0$ is applied to (5) and we derive an IPA, we do not have guarantee that a correponding function φ exists, due to the fact that the connection between $\bar{\varphi}$ and φ is given as a functional equation given in Eq. (7). Thus, we do not have in this case immediately another description of the IPA. In other words, we should consider the following question: can we find a corresponding function $\varphi : (\xi^2, \infty) \to \mathbb{R}$ for a given $\bar{\varphi} : (\xi^2, \infty) \to \mathbb{R}$, $0 \le \xi < 1$? To answer this, we give counterexamples. Using the definition of the function $\bar{\varphi}$ given in (7), we have $\lim_{t\to 0}\bar{\varphi}(t) = \bar{\varphi}(1) = 0$. However, the functions $\bar{\varphi}$ are monotone increasing. Hence, all the functions $\bar{\varphi}$ that are defined on the whole interval $(0, \infty)$, i.e. $\xi = 0$, are counterexamples. It would be interesting to define a class of monotone increasing functions $\bar{\varphi}$ for which we can assign corresponding functions φ. For this, we should solve the functional equation $\varphi(t) - \varphi(\sqrt{t}) = \bar{\varphi}(t)$ for a given function $\bar{\varphi} : (\xi^2, \infty) \to \mathbb{R}$. This leads to further research topics.

3. Search directions in case of the new type of AET technique

In this section we present a method to determine search directions in case of IPAs for $P_*(\kappa)$-LCPs, by using the new type of AET approach presented in Section 2.2.

3.1. Scaling

Let us consider

$$\mathbf{v} = \sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}, \quad \mathbf{d} = \sqrt{\frac{\mathbf{x}}{\mathbf{s}}}, \quad \mathbf{d}_x = \frac{\mathbf{d}^{-1}\Delta\mathbf{x}}{\sqrt{\mu}} = \frac{\mathbf{v}\,\Delta\mathbf{x}}{\mathbf{x}}, \quad \mathbf{d}_s = \frac{\mathbf{d}\,\Delta\mathbf{s}}{\sqrt{\mu}} = \frac{\mathbf{v}\,\Delta\mathbf{s}}{\mathbf{s}}. \tag{12}$$

From (12) we obtain

$$\Delta\mathbf{x} = \frac{\mathbf{x}\,\mathbf{d}_x}{\mathbf{v}} \quad \text{and} \quad \Delta\mathbf{s} = \frac{\mathbf{s}\,\mathbf{d}_s}{\mathbf{v}}. \tag{13}$$

Hence, if we substitute these in the second equation of system (4) we get

$$\frac{\mathbf{x}\,\mathbf{s}\,\mathbf{d}_x}{\mathbf{v}} + \frac{\mathbf{x}\,\mathbf{s}\,\mathbf{d}_s}{\mathbf{v}} = \mu\frac{2\mathbf{v}\big(\varphi(\mathbf{v}) - \varphi(\mathbf{v}^2)\big)}{2\mathbf{v}\varphi'(\mathbf{v}^2) - \varphi'(\mathbf{v})}. \tag{14}$$

The transformed Newton system (4) with $\mathbf{a}_\varphi$ given in (9), obtained from (6) by applying the AET and then scaling it, leads to the following form of the scaled Newton-system:

$$\begin{aligned} -\bar{M}\mathbf{d}_x + \mathbf{d}_s &= \mathbf{0}, \\ \mathbf{d}_x + \mathbf{d}_s &= \mathbf{p}_\varphi, \end{aligned} \tag{15}$$

where $\bar{M} = DMD$, $D = diag(\mathbf{d})$ and

$$\mathbf{p}_\varphi = \frac{2\left(\varphi(\mathbf{v}) - \varphi(\mathbf{v}^2)\right)}{2\,\mathbf{v}\,\varphi'(\mathbf{v}^2) - \varphi'(\mathbf{v})}. \tag{16}$$

From Theorem 3.5 proposed in Kojima et al. (1991) and Corollary 2.1 it can be proved that system (15) has unique solution.

It should be mentioned that if we use the function $\varphi : \left(\frac{1}{2}, \infty\right) \to \mathbb{R}$, $\varphi(t) = t$, which satisfies condition (10), then we have

$$\mathbf{p}_\varphi = \frac{2\mathbf{v} - 2\mathbf{v}^2}{2\mathbf{v} - \mathbf{e}}. \tag{17}$$

Interestingly enough that exactly the same vector $\mathbf{p}_\varphi$ can be derived if the AET is applied to (5) with function $\bar{\varphi}(t) = t - \sqrt{t}$. For details see papers Darvay, Illés, Kheirfam, & Rigó (2020a); Darvay et al. (2016) for LO and Darvay, Illés, & Majoros (2021); Darvay et al. (2020b) for sufficient LCPs. This can be proved by using (7), because in this case we have $\bar{\varphi}(t) = \varphi(t) - \varphi(\sqrt{t}) = t - \sqrt{t}$. Furthermore, if we apply the AET to system (6) using the function $\varphi(t) = t^2$, then we obtain the same system as if we apply $\bar{\varphi}(t) = \varphi(t) - \varphi(\sqrt{t}) = t^2 - t$ to system (5). It should be mentioned, that this function has not been used in the literature in the AET technique. Hence, the function $\varphi(t) = t^2$ used in the AET approach and applied to (6) leads to novel search directions discussed in this paper.

In the following subsection we give a general method of determining the scaled predictor and scaled corrector systems in case of PC IPAs using this new type of AET.

3.2. Search directions in case of PC IPAs

Darvay et al. (2020b) gave a general framework to determine the scaled systems in case of PC IPAs for sufficient LCPs. Following the steps of their method, we give firstly the scaled corrector system, which coincides with system (15). This system has the unique solution: $\mathbf{d}_x^c = (I + \bar{M})^{-1}\mathbf{p}_\varphi$, $\mathbf{d}_s^c = \bar{M}(I + \bar{M})^{-1}\mathbf{p}_\varphi$. Analogous to the formula given in (13) we can define $\Delta^c\mathbf{x} = \frac{\mathbf{x}\,\mathbf{d}_x^c}{\mathbf{v}}$ and $\Delta^c\mathbf{s} = \frac{\mathbf{s}\,\mathbf{d}_s^c}{\mathbf{v}}$. The difference between this method and the one presented in Darvay et al. (2020b) is that we have different value of the vector $\mathbf{p}_\varphi$ due to the used function $\varphi(t) = t^2$ in the AET technique. In the transformed Newton system (4) we decompose $\mathbf{a}_\varphi$ given in (9) in the following way using the idea presented in Darvay et al. (2020b):

$$\mathbf{a}_\varphi = f(\mathbf{x}, \mathbf{s}, \mu) + g(\mathbf{x}, \mathbf{s}), \tag{18}$$

where $f : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}_\oplus \to \mathbb{R}^n$ with $f(\mathbf{x}, \mathbf{s}, 0) = \mathbf{0}$ and $g : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$. We set $\mu = 0$ in (18), because we would like to make as greedy predictor step as possible. From Darvay et al. (2020b) we obtain

$$\begin{aligned} -\bar{M}\mathbf{d}_x + \mathbf{d}_s &= \mathbf{0}, \\ \mathbf{d}_x + \mathbf{d}_s &= \frac{\mathbf{v}g(\mathbf{x}, \mathbf{s})}{\mathbf{x}\mathbf{s}}, \end{aligned} \tag{19}$$

where $\bar{M} = DMD$. The unique solution of system (19) is $\mathbf{d}_x^p = (I + \bar{M})^{-1}\frac{\mathbf{v}g(\mathbf{x},\mathbf{s})}{\mathbf{x}\mathbf{s}}$ and $\mathbf{d}_s^p = \bar{M}(I + \bar{M})^{-1}\frac{\mathbf{v}g(\mathbf{x},\mathbf{s})}{\mathbf{x}\mathbf{s}}$. The difference between this approach and the one given in Darvay et al. (2020b) lies in the different value of the vector $\mathbf{a}_\varphi$ and of $g(\mathbf{x}, \mathbf{s})$. Using (13) we can obtain the predictor search directions from $\Delta^p\mathbf{x} = \frac{\mathbf{x}\,\mathbf{d}_x^p}{\mathbf{v}}$ and $\Delta^p\mathbf{s} = \frac{\mathbf{s}\,\mathbf{d}_s^p}{\mathbf{v}}$. It should be mentioned that the decomposition (18) is not trivial and we have no guarantee that such decomposition exists for all functions φ suitable for AET.

4. New PC IPA for $P_*(\kappa)$-LCPs based on a new search direction

In this section we introduce a PC IPA using the AET technique presented in Section 2.2. We deal with the function $\varphi : \left(\frac{1}{2}, \infty\right) \to \mathbb{R}$, $\varphi(t) = t^2$, so we obtain

$$\mathbf{p}_\varphi = \frac{\mathbf{v} - \mathbf{v}^3}{2\mathbf{v}^2 - \mathbf{e}}. \tag{20}$$

It should be mentioned that the condition $2t\varphi'(t^2) - \varphi'(t) > 0, \forall t > \xi$ is satisfied in this case, where $\xi = \frac{\sqrt{2}}{2}$. Note that we can associate a corresponding kernel function to the search direction determined by the function φ in the new type of AET approach. In this way, we obtain a positive-asymptotic kernel function, see Darvay & Takács (2018); Rigó & Darvay (2018):

$$\psi : \left(\frac{\sqrt{2}}{2}, \infty\right) \to \mathbb{R}_\oplus, \quad \psi(t) = \frac{t^2 - 1}{4} - \frac{\log(2t^2 - 1)}{8}.$$

Let us define the centrality measure $\delta : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ as

$$\delta(\mathbf{x}, \mathbf{s}, \mu) := \delta(\mathbf{v}) := \frac{\|\mathbf{p}_\varphi\|}{2} = \frac{1}{2}\left\| \frac{\mathbf{v} - \mathbf{v}^3}{2\mathbf{v}^2 - \mathbf{e}} \right\|. \tag{21}$$

Beside this, we give the τ-neighbourhood of a fixed point of the central path as

$$\mathcal{N}_2(\tau, \mu) := \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \delta(\mathbf{x}, \mathbf{s}, \mu) \le \tau\}, \tag{22}$$

where $\delta(\mathbf{x}, \mathbf{s}, \mu)$ is given in (21), τ is a threshold parameter and $\mu > 0$ is fixed.

First, we need to find the decomposition of $\mathbf{a}_\varphi$ as it is given in (18):

$$\mathbf{a}_\varphi = \frac{\mu\, \mathbf{x}\mathbf{s}}{2\,(2\,\mathbf{x}\mathbf{s} - \mu\,\mathbf{e})} - \frac{\mathbf{x}\mathbf{s}}{2},$$

hence $f(\mathbf{x}, \mathbf{s}, \mu) = \frac{\mu\,\mathbf{x}\mathbf{s}}{2\,(2\,\mathbf{x}\mathbf{s} - \mu\,\mathbf{e})}$, which satisfies the condition $f(\mathbf{x}, \mathbf{s}, 0) = \mathbf{0}$ and $g(\mathbf{x}, \mathbf{s}) = -\frac{\mathbf{x}\mathbf{s}}{2}$. In this case, the transformed Newton system (4) with (9) is the following:

$$\begin{aligned} -M\Delta\mathbf{x} + \quad \Delta\mathbf{s} &= \mathbf{0}, \\ S\Delta\mathbf{x} + X\Delta\mathbf{s} &= \frac{\mu\,\mathbf{x}\mathbf{s}}{2\,(2\,\mathbf{x}\mathbf{s} - \mu\,\mathbf{e})} - \frac{\mathbf{x}\mathbf{s}}{2}. \end{aligned} \tag{23}$$

Note that some IPAs use firstly corrector steps and after that predictor step, see Potra (2008). Our algorithm also performs firstly a corrector step if the initial interior point is not well centered and after that a predictor one. The PC IPA starts with $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{N}_2(\tau, \mu)$ for which $\delta(\mathbf{x}^0, \mathbf{s}^0, \mu) \le \tau$. In a corrector step we obtain $\mathbf{d}_x^c$ and $\mathbf{d}_s^c$ by solving

$$\begin{aligned} -\bar{M}\mathbf{d}_x^c + \mathbf{d}_s^c &= \mathbf{0}, \\ \mathbf{d}_x^c + \mathbf{d}_s^c &= \frac{\mathbf{v} - \mathbf{v}^3}{2\mathbf{v}^2 - \mathbf{e}}, \end{aligned} \tag{24}$$

where we used the scaling notations considered in Section 3.1, $\bar{M} = DMD$ and $D = diag(\mathbf{d})$. From Theorem 3.5 given in Kojima et al. (1991) and Corollary 2.1 it can be proved that system (24) has unique solution:

$$\mathbf{d}_x^c = (I + \bar{M})^{-1}\frac{\mathbf{v} - \mathbf{v}^3}{2\mathbf{v}^2 - \mathbf{e}}, \quad \mathbf{d}_s^c = \bar{M}(I + \bar{M})^{-1}\frac{\mathbf{v} - \mathbf{v}^3}{2\mathbf{v}^2 - \mathbf{e}}.$$

From

$$\Delta^c\mathbf{x} = \frac{\mathbf{x}\,\mathbf{d}_x^c}{\mathbf{v}} \quad \text{and} \quad \Delta^c\mathbf{s} = \frac{\mathbf{s}\,\mathbf{d}_s^c}{\mathbf{v}} \tag{25}$$

the $\Delta^c\mathbf{x}$ and $\Delta^c\mathbf{s}$ search directions can be easily obtained. Let

$$\mathbf{x}^c = \mathbf{x} + \Delta^c\mathbf{x}, \quad \mathbf{s}^c = \mathbf{s} + \Delta^c\mathbf{s}.$$

Consider the following notations:

$$\mathbf{v}^c = \sqrt{\frac{\mathbf{x}^c\mathbf{s}^c}{\mu}}, \quad \mathbf{d}^c = \sqrt{\frac{\mathbf{x}^c}{\mathbf{s}^c}}, \quad D^+ = diag(\mathbf{d}^c), \quad \bar{M}^+ = D^+MD^+.$$

Then, the scaled predictor system is

$$\begin{aligned} -\bar{M}^+\mathbf{d}_x^p + \mathbf{d}_s^p &= \mathbf{0}, \\ \mathbf{d}_x^p + \mathbf{d}_s^p &= -\frac{\mathbf{v}^c}{2}, \end{aligned} \tag{26}$$

which has the solution

$$\mathbf{d}_x^p = -(I + \bar{M}^+)^{-1}\frac{\mathbf{v}^c}{2}, \quad \mathbf{d}_s^p = -\bar{M}^+(I + \bar{M}^+)^{-1}\frac{\mathbf{v}^c}{2}. \tag{27}$$

Then, using

$$\Delta^p\mathbf{x} = \frac{\mathbf{x}^c}{\mathbf{v}^c}\mathbf{d}_x^p \quad \text{and} \quad \Delta^p\mathbf{s} = \frac{\mathbf{s}^c}{\mathbf{v}^c}\mathbf{d}_s^p, \tag{28}$$

the search directions $\Delta^p\mathbf{x}$ and $\Delta^p\mathbf{s}$ can be easily calculated. The iterate after a predictor step is

$$\mathbf{x}^p = \mathbf{x}^c + \theta\Delta^p\mathbf{x}, \quad \mathbf{s}^p = \mathbf{s}^c + \theta\Delta^p\mathbf{s}, \quad \mu^p = \left(1 - \frac{\theta}{2}\right)\mu,$$

where $\theta \in (0, 1)$ is the update parameter.

5. Analysis of the PC IPA

In the first part of the analysis we deal with the corrector step.

5.1. The corrector step

In the corrector part of the proposed PC IPA we use the classical small-update step of IPAs. Therefore, the results of Zhang et al. (2020) can be applied to analyse the corrector steps of the proposed PC IPA. It should be mentioned that the default value $\tau = \frac{1}{16(1+4\kappa)}$ given in Algorithm 4.1 is smaller than the upper bounds

Algorithm 4.1 PC IPA for sufficient LCPs based on a new type of AET.

Let $\epsilon > 0$ be the accuracy parameter, $0 < \theta < 1$ the update parameter (default value $\theta = \frac{1}{4(1+4\kappa)\sqrt{n}}$) and $\tau$the proximity parameter (default value $\tau = \frac{1}{16(1+4\kappa)}$). Furthermore, a known upper bound κ of the handicap $\hat{\kappa}(M)$ is given. Assume that for $(\mathbf{x}^0, \mathbf{s}^0)$ the $\left(\mathbf{x}^0\right)^T\mathbf{s}^0 = n\mu^0$, $\mu^0 > 0$ holds such that $\delta(\mathbf{x}^0, \mathbf{s}^0, \mu^0) \le \tau$ and $\frac{\mathbf{x}^0\mathbf{s}^0}{\mu^0} > \frac{1}{2}\mathbf{e}$.

begin
    $k := 0$;
    while $\left(\mathbf{x}^k\right)^T\mathbf{s}^k > \epsilon$ **do**
    begin

        (corrector step)
        compute $(\Delta^c x^k, \Delta^c s^k)$ from system (24) using (25);
        let $\left(\mathbf{x}^c\right)^k := \mathbf{x}^k + \Delta^c x^k$ and $\left(\mathbf{s}^c\right)^k := \mathbf{s}^k + \Delta^c s^k$;

        (predictor step)
        compute $(\Delta^p x^k, \Delta^p s^k)$ from system (26) using (28);
        let $\left(\mathbf{x}^p\right)^k := \left(\mathbf{x}^c\right)^k + \theta\Delta^p\mathbf{x}^k$ and $\left(\mathbf{s}^p\right)^k := \left(\mathbf{s}^c\right)^k + \theta\Delta^p\mathbf{s}^k$;

        (update of the parameters and the iterates)
        $\mathbf{x}^{k+1} := \left(\mathbf{x}^p\right)^k, \quad \mathbf{s}^{k+1} := \left(\mathbf{s}^p\right)^k, \quad \mu^{k+1} := \left(1 - \frac{\theta}{2}\right)\mu^k$;
        k:=k+1;

    end
end

of centrality measures given in the following theorem and lemma, hence we can use these results in the analysis of the corrector step. Furthermore, a detailed description of how the default values of the parameters have been chosen is given in Section 5.4. In the next theorem the strict feasibility of the full-Newton IPA is proved, where $\mathbf{v}^c = \sqrt{\frac{\mathbf{x}^c\mathbf{s}^c}{\mu}}$.

Theorem 5.1 (Theorem 1 in Darvay & Takács (2018), and Lemma 3 in Zhang et al. (2020)). *Let $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu) < \frac{1}{\sqrt{1+4\kappa}}$ and $\mathbf{v} > \frac{\sqrt{2}}{2}\mathbf{e}$. Then, we have $(\mathbf{x}^c, \mathbf{s}^c) \in \mathcal{F}^+$ and $\mathbf{v}^c \geq \sqrt{1-(1+4\kappa)\delta^2}\,\mathbf{e}$. Moreover, if we choose $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu) < \frac{1}{\sqrt{2(1+4\kappa)}}$, then we have $\mathbf{v}^c > \frac{\sqrt{2}}{2}\mathbf{e}$.*

The next lemma shows the quadratic convergence of the corrector step.

Lemma 5.2 (Theorem 2 in Zhang et al. (2020)). *Let $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu) < \frac{1}{\sqrt{2(1+4\kappa)}}$ and $\mathbf{v} > \frac{\sqrt{2}}{2}\mathbf{e}$. Then,*

$$\delta^c := \delta(\mathbf{x}^c, \mathbf{s}^c, \mu) \leq \frac{5(1+4\kappa)\delta^2}{1-2(1+4\kappa)\delta^2}\sqrt{1-(1+4\kappa)\delta^2}.$$

Corollary 5.3. *Let $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu) \leq \frac{1}{2\sqrt{1+4\kappa}}$ and $\mathbf{v} > \frac{\sqrt{2}}{2}\mathbf{e}$. Then, $\delta^c \leq 10(1+4\kappa)\delta^2$.*

Proof. From $\delta(\mathbf{x}, \mathbf{s}, \mu) < \frac{1}{2\sqrt{1+4\kappa}}$ we have

$$1-2(1+4\kappa)\delta^2 \geq \frac{1}{2}.$$

Using this, Lemma 5.2 and $\sqrt{1-(1+4\kappa)\delta^2} \leq 1$ we obtain

$$\delta(\mathbf{x}^c, \mathbf{s}^c, \mu) \leq \frac{5(1+4\kappa)\delta^2}{1-2(1+4\kappa)\delta^2} \leq 10(1+4\kappa)\delta^2,$$

which yields the result. □

Next lemma provides an upper bound for the duality gap after a full-Newton step.

Lemma 5.4 (Lemma 4 in Zhang et al. (2020)). *Let $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu)$ given as in (21). Then,*

$$(\mathbf{x}^c)^T\mathbf{s}^c < \mu(n+9\delta^2).$$

5.2. Technical lemmas

In this subsection we present important results that will be used in the next part of the analysis. We assume that M is a $P_*(\kappa)$-matrix for a given $\kappa \geq \hat{\kappa}(M) \geq 0$. From $-M\Delta^p\mathbf{x} + \Delta^p\mathbf{s} = \mathbf{0}$, we have

$$(1+4\kappa)\sum_{i\in I_+}\Delta^p x_i\Delta^p s_i + \sum_{i\in I_-}\Delta^p x_i\Delta^p s_i \geq 0, \tag{29}$$

where $I_+ = \{i : \Delta^p x_i\Delta^p s_i > 0\}$ and $I_- = \{i : \Delta^p x_i\Delta^p s_i < 0\}$. Using (12) we obtain $\mathbf{d}_x^p\mathbf{d}_s^p = \frac{\Delta^p\mathbf{x}\Delta^p\mathbf{s}}{\mu}$. Hence, (29) can be written as

$$(1+4\kappa)\sum_{i\in I_+} d_{x_i}^p d_{s_i}^p + \sum_{i\in I_-} d_{x_i}^p d_{s_i}^p \geq 0. \tag{30}$$

The following lemma is similar to that of Lemma 1 in the paper of Kheirfam (2014) and Lemma 5.3 in Darvay et al. (2020b). However, we use another type of AET transformation and different function φ.

Lemma 5.5. *Let $\delta^c = \delta(\mathbf{x}^c, \mathbf{s}^c, \mu) = \frac{1}{2}\left\|\frac{\mathbf{v}^c-(\mathbf{v}^c)^3}{2(\mathbf{v}^c)^2-\mathbf{e}}\right\|$. Then, the following inequality holds*

$$\|\mathbf{d}_x^p\mathbf{d}_s^p\| < \frac{n(2+\kappa)(1+4\delta^c)^2}{4}$$

Proof. Using the second equation of the scaled predictor system (26) we obtain

$$\sum_{i\in I_+} d_{x_i}^p d_{s_i}^p \leq \frac{1}{4}\|\mathbf{d}_x^p + \mathbf{d}_s^p\|^2 = \frac{\|\mathbf{v}^c\|^2}{16}.$$

Using the proof of Lemma 5.3 given in Darvay et al. (2020b) and from the relation (30) we have

$$\frac{\|\mathbf{v}^c\|^2}{4} \geq \|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2 - 8\kappa\sum_{i\in I_+} d_{x_i}^p d_{s_i}^p \geq \|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2 - \frac{1}{2}\kappa\|\mathbf{v}^c\|^2. \tag{31}$$

Hence, $\|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2 \leq \left(\frac{1}{4}+\frac{1}{2}\kappa\right)\|\mathbf{v}^c\|^2 < \left(1+\frac{1}{2}\kappa\right)\|\mathbf{v}^c\|^2$. Similar to the proof of Lemma 5.3 of Darvay et al. (2020b), we give an upper bound for $\|\mathbf{v}^c\|$. Consider the notation $\sigma^c = \|\mathbf{e}-\mathbf{v}^c\|$, which is the centrality measure used in Darvay (2003); Kheirfam (2014). Using the relation (5.6) given in Darvay et al. (2020b) we have

$$\|\mathbf{v}^c\| \leq \sqrt{n}(\sigma^c + 1). \tag{32}$$

Moreover,

$$\delta^c = \frac{1}{2}\left\|\frac{\mathbf{v}^c-(\mathbf{v}^c)^3}{2(\mathbf{v}^c)^2-\mathbf{e}}\right\| = \frac{1}{2}\left\|\frac{\mathbf{v}^c(\mathbf{e}+\mathbf{v}^c)}{2(\mathbf{v}^c)^2-\mathbf{e}}(\mathbf{e}-\mathbf{v}^c)\right\| > \frac{1}{4}\|\mathbf{e}-\mathbf{v}^c\| = \frac{\sigma^c}{4}, \tag{33}$$

where we used that the function $\bar{h}(t) = \frac{t^2+t}{2t^2-1} > \frac{1}{2}$, for $t > \frac{\sqrt{2}}{2}$. Hence, we have $\sigma^c < 4\delta^c$. Using (32) and (33) we get

$$\|\mathbf{v}^c\| < \sqrt{n}(1+4\delta^c). \tag{34}$$

Thus,

$$\|\mathbf{d}_x^p\mathbf{d}_s^p\| \leq \|\mathbf{d}_x^p\|\|\mathbf{d}_s^p\| \leq \frac{1}{2}\left(\|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2\right) \leq \frac{1}{2}\left(1+\frac{1}{2}\kappa\right)\|\mathbf{v}^c\|^2 < \frac{n(2+\kappa)(1+4\delta^c)^2}{4},$$

which proves the lemma. □

Consider

$$\mathbf{q}_v = \mathbf{d}_x^c - \mathbf{d}_s^c. \tag{35}$$

Then, we have

$$\mathbf{d}_x^c = \frac{\mathbf{p}_\varphi + \mathbf{q}_v}{2}, \quad \mathbf{d}_s^c = \frac{\mathbf{p}_\varphi - \mathbf{q}_v}{2} \quad \text{and} \quad \mathbf{d}_x^c\mathbf{d}_s^c = \frac{\mathbf{p}_\varphi^2 - \mathbf{q}_v^2}{4}. \tag{36}$$

We give an upper bound for the norm of $\mathbf{q}_v$ depending on the centrality measure. The proof technique is similar to the one given in Asadi, Mahdavi-Amiri, Darvay, & Rigó (2020) for $P_*(\kappa)$-LCPs over Cartesian product of symmetric cones.

Lemma 5.6 (c.f. Lemma 5.4 in Darvay et al. (2020b) and Lemma 5.1 in Asadi et al. (2020)). *The following inequality holds:*

$$\|\mathbf{q}_v\| \leq 2\sqrt{1+4\kappa}\,\delta,$$

where $\delta = \delta(\mathbf{x}, \mathbf{s}, \mu)$ is the proximity measure given in (21) and it is related to the iterates before the corrector step.

Proof. The proof is similar to Lemma 5.4 given in Darvay et al. (2020b) and Lemma 5.1 appeared in Asadi et al. (2020). However, we consider a different search direction. In the proof we use only the $\mathbf{d}_x^c + \mathbf{d}_s^c = \mathbf{p}_v$ equation, which is valid in our case as well, independently on the used search direction. □

The next subsection contains the analysis of the predictor step.

5.3. The predictor step

Lemma 5.7 gives a sufficient condition for the strict feasibility of the predictor step.

Lemma 5.7. *Let $(\mathbf{x}^c, \mathbf{s}^c) > \mathbf{0}$, $0 < \theta < 1$ and $\mu > 0$ such that $\delta^c := \delta(\mathbf{x}^c, \mathbf{s}^c, \mu) < \frac{1}{4}$. Consider $\mathbf{x}^p = \mathbf{x}^c + \theta\Delta^p\mathbf{x}$ and $\mathbf{s}^p = \mathbf{s}^c + \theta\Delta^p\mathbf{s}$. Let*

$$z(\delta^c, \theta, n) := (1-4\delta^c)^2 - \frac{n(2+\kappa)\theta^2(1+4\delta^c)^2}{2(2-\theta)}.$$

If $z(\delta^c,\theta,n)>0$, then $\mathbf{x}^p>\mathbf{0}$ and $\mathbf{s}^p>\mathbf{0}$.

Proof. Let us consider $\mathbf{x}^p(\alpha)=\mathbf{x}^c+\alpha\,\theta\,\Delta^p\mathbf{x}$ and $\mathbf{s}^p(\alpha)=\mathbf{s}^c+\alpha\,\theta\,\Delta^p\mathbf{s}$, for $0\le\alpha\le1$. Then, $\mathbf{x}^p(\alpha)=\frac{\mathbf{x}^c}{\mathbf{v}^c}(\mathbf{v}^c+\alpha\,\theta\,\mathbf{d}_x^p)$ and $\mathbf{s}^p(\alpha)=\frac{\mathbf{s}^c}{\mathbf{v}^c}(\mathbf{v}^c+\alpha\,\theta\,\mathbf{d}_s^p)$. Using relation (5.17) given in Darvay et al. (2020b) and from the second equation of system (26) we obtain:

$$\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha)=\mu\Big((\mathbf{v}^c)^2+\alpha\theta\mathbf{v}^c(\mathbf{d}_x^p+\mathbf{d}_s^p)+\alpha^2\,\theta^2\,\mathbf{d}_x^p\mathbf{d}_s^p\Big)$$
$$=\mu\left(\left(1-\frac{1}{2}\alpha\theta\right)(\mathbf{v}^c)^2+\alpha^2\,\theta^2\,\mathbf{d}_x^p\mathbf{d}_s^p\right). \tag{37}$$

Hence, we obtain

$$\min\left(\frac{\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha)}{\mu\left(1-\frac{\alpha\theta}{2}\right)}\right)=\min\left((\mathbf{v}^c)^2+\frac{\alpha^2\theta^2}{1-\frac{\alpha\theta}{2}}\mathbf{d}_x^p\mathbf{d}_s^p\right)\ge\min\left((\mathbf{v}^c)^2\right)-\frac{2\alpha^2\theta^2}{2-\alpha\theta}\|\mathbf{d}_x^p\mathbf{d}_s^p\|_\infty.$$

We have $1-\sigma^c\le v_i^c\le1+\sigma^c,\ \forall i=1,\dots,n$. Using these bounds, (33) and $\delta^c<\frac14$ we have

$$\min\,(\mathbf{v}^c)^2\ge(1-\sigma^c)^2\ge(1-4\delta^c)^2. \tag{38}$$

We will use that the real valued function $f(\alpha)=\frac{2\alpha^2\theta^2}{2-\alpha\theta}$ is strictly increasing for $0\le\alpha\le1$ and each fixed $0<\theta<1$. Moreover, from Lemma 5.5 and (38) we obtain

$$\min\left(\frac{\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha)}{\mu\left(1-\frac{\alpha\theta}{2}\right)}\right)\ge(1-4\delta^c)^2-\frac{2n(2+\kappa)\theta^2(1+4\delta^c)^2}{4(2-\theta)}$$
$$=z(\delta^c,\theta,n)>0. \tag{39}$$

Hence, we have $\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha)>0$ for $0\le\alpha\le1$. Therefore, $\mathbf{x}^p(\alpha)$ and $\mathbf{s}^p(\alpha)$ do not change sign on $0\le\alpha\le1$. Using $\mathbf{x}^p(0)=\mathbf{x}^c>\mathbf{0}$ and $\mathbf{s}^p(0)=\mathbf{s}^c>\mathbf{0}$, we obtain $\mathbf{x}^p(1)=\mathbf{x}^p>\mathbf{0}$ and $\mathbf{s}^p(1)=\mathbf{s}^p>\mathbf{0}$, which yields the result. □

Let us introduce

$$\mathbf{v}^p=\sqrt{\frac{\mathbf{x}^p\mathbf{s}^p}{\mu^p}},$$

where $\mu^p=\left(1-\frac{\theta}{2}\right)\mu$. If we substitute $\alpha=1$ in (37) and (39) we have

$$(\mathbf{v}^p)^2=(\mathbf{v}^c)^2+\frac{2\theta^2}{2-\theta}\mathbf{d}_x^p\mathbf{d}_s^p\quad\text{and}\quad\min\,(\mathbf{v}^p)^2\ge z(\delta^c,\theta,n)>0. \tag{40}$$

The next lemma analyses the effect of a predictor step and the update of μ on the proximity measure.

Lemma 5.8. *Let $\delta^c:=\delta(\mathbf{x}^c,\mathbf{s}^c,\mu)<\frac14$, $\mu^p=\left(1-\frac{\theta}{2}\right)\mu$, where $0<\theta<1$, $z(\delta^c,\theta,n)>\frac12$ and consider $\delta:=\delta(\mathbf{x},\mathbf{s},\mu)$ given in (21). The iterates after a predictor step are denoted as $\mathbf{x}^p$ and $\mathbf{s}^p$. Then, we have $\mathbf{v}^p>\frac{\sqrt2}{2}\mathbf{e}$ and*

$$\delta^p:=\delta(\mathbf{x}^p,\mathbf{s}^p,\mu^p)\le\frac{\sqrt{z(\delta^c,\theta,n)}\big(10(1+4\kappa)\delta^2+(1-4\delta^c)^2-z(\delta^c,\theta,n)\big)}{4z(\delta^c,\theta,n)-2}.$$

Proof. Using $z(\delta^c,\theta,n)>\frac12>0$, from Lemma 5.7 we get $\mathbf{x}^p>\mathbf{0}$ and $\mathbf{s}^p>\mathbf{0}$, thus the predictor step is strictly feasible. From (40) we obtain

$$\min\,(\mathbf{v}^p)\ge\sqrt{z(\delta^c,\theta,n)}>\frac{\sqrt2}{2},$$

which yields the first part of the result. Beside this,

$$\delta^p:=\frac12\left\|\frac{\mathbf{v}^p-(\mathbf{v}^p)^3}{2(\mathbf{v}^p)^2-\mathbf{e}}\right\|=\frac12\left\|\frac{\mathbf{v}^p\big(\mathbf{e}-(\mathbf{v}^p)^2\big)}{2(\mathbf{v}^p)^2-\mathbf{e}}\right\|. \tag{41}$$

Consider $h:\left(\frac{\sqrt2}{2},\infty\right)\to\mathbb{R},\ h(t)=\frac{t}{2t^2-1}$, which is a decreasing function with respect to t. Using this, (40) and (41) we get

$$\delta^p\le\frac{\min\,(\mathbf{v}^p)}{4\min\,(\mathbf{v}^p)^2-2}\left\|\mathbf{e}-(\mathbf{v}^p)^2\right\|$$
$$\le\frac{\sqrt{z(\delta^c,\theta,n)}}{4z(\delta^c,\theta,n)-2}\left\|\mathbf{e}-(\mathbf{v}^c)^2-\frac{2\theta^2}{2-\theta}\mathbf{d}_x^p\mathbf{d}_s^p\right\|$$
$$\le\frac{\sqrt{z(\delta^c,\theta,n)}}{4z(\delta^c,\theta,n)-2}\left(\left\|\mathbf{e}-(\mathbf{v}^c)^2\right\|+\frac{2\theta^2}{2-\theta}\left\|\mathbf{d}_x^p\mathbf{d}_s^p\right\|\right). \tag{42}$$

Using the proof of Lemma 2 in Darvay & Takács (2018) we obtain the following upper bound for $\left\|\mathbf{e}-(\mathbf{v}^c)^2\right\|$:

$$\left\|\mathbf{e}-(\mathbf{v}^c)^2\right\|\le\left\|\frac{\mathbf{q}_v^2}{4}\right\|+\left\|\frac{9\mathbf{v}^2-4\mathbf{e}}{\mathbf{v}^2}\cdot\frac{\mathbf{p}_\varphi^2}{4}\right\| \tag{43}$$

Hence, using (43) and Lemma 5.6 we may write

$$\left\|\mathbf{e}-(\mathbf{v}^c)^2\right\|\le\left\|\frac{\mathbf{q}_v^2}{4}\right\|+\left\|\frac{9\mathbf{v}^2-4\mathbf{e}}{\mathbf{v}^2}\cdot\frac{\mathbf{p}_\varphi^2}{4}\right\|$$
$$<\frac{\|\mathbf{q}_v\|^2}{4}+9\frac{\|\mathbf{p}_\varphi\|^2}{4}\le10(1+4\kappa)\delta^2. \tag{44}$$

We used that $\mathbf{0}<\frac{9\mathbf{v}^2-4\mathbf{e}}{\mathbf{v}^2}<9\mathbf{e}$ for $\mathbf{v}>\frac{\sqrt2}{2}\mathbf{e}$. From (42), (44), Lemma 5.5 and the definition of the function z we get:

$$\delta^p\le\frac{\sqrt{z(\delta^c,\theta,n)}}{4z(\delta^c,\theta,n)-2}\left(\left\|\mathbf{e}-(\mathbf{v}^c)^2\right\|+\frac{2\theta^2}{2-\theta}\left\|\mathbf{d}_x^p\mathbf{d}_s^p\right\|\right)$$
$$\le\frac{\sqrt{z(\delta^c,\theta,n)}\big(10(1+4\kappa)\delta^2+(1-4\delta^c)^2-z(\delta^c,\theta,n)\big)}{4z(\delta^c,\theta,n)-2}, \tag{45}$$

which proves the second statement of the lemma. □

It should be mentioned that in Lemma 5.8 the condition $z(\delta^c,\theta,n)>\frac12$ should hold, because due to the used function $\varphi(t)=t^2$ in the new type of AET technique for the determination of the search directions, we have to ensure that in each iteration of the algorithm, the components of the vector $\mathbf{v}$ are greater than $\frac{\sqrt2}{2}$.

In the following lemma we give an upper bound for the duality gap after a main iteration.

Lemma 5.9. *Let $0<\theta<1$. If $\delta\le\frac{1}{16(1+4\kappa)}$, $\mathbf{x}^p$ and $\mathbf{s}^p$ are the iterates obtained after the predictor step of the algorithm, then*

$$(\mathbf{x}^p)^T\mathbf{s}^p\le\left(1-\frac{\theta}{2}+\frac{\theta^2}{8}\right)(\mathbf{x}^c)^T\mathbf{s}^c<\frac{3n\mu^p}{2(2-\theta)}.$$

Proof. Using (37) with $\alpha=1$ and the definition of $\mathbf{v}^p$ we have

$$(\mathbf{x}^p)^T\mathbf{s}^p=\mu^p\mathbf{e}^T(\mathbf{v}^p)^2=\mu\mathbf{e}^T\left(\left(1-\frac{\theta}{2}\right)(\mathbf{v}^c)^2+\theta^2\mathbf{d}_x^p\mathbf{d}_s^p\right)$$
$$=\left(1-\frac{\theta}{2}\right)(\mathbf{x}^c)^T\mathbf{s}^c+\mu\theta^2\left(\mathbf{d}_x^p\right)^T\mathbf{d}_s^p. \tag{46}$$

We multiply the second equation of (26) by $\left(\mathbf{d}_x^p\right)^T$ and by $\left(\mathbf{d}_s^p\right)^T$, respectively. After that, we sum the obtained two equations, hence

$$\left(\mathbf{d}_x^p\right)^T\mathbf{d}_s^p=\frac{(\mathbf{x}^c)^T\mathbf{s}^c}{8\mu}-\frac{\|\mathbf{d}_x^p\|^2+\|\mathbf{d}_s^p\|^2}{2}\le\frac{(\mathbf{x}^c)^T\mathbf{s}^c}{8\mu}. \tag{47}$$

Using (46) and (47) we get

$$(\mathbf{x}^p)^T\mathbf{s}^p\le\left(1-\frac{\theta}{2}+\frac{\theta^2}{8}\right)(\mathbf{x}^c)^T\mathbf{s}^c.$$

If $0 < \theta < 1$, then

$$1 - \frac{\theta}{2} + \frac{\theta^2}{8} < 1. \tag{48}$$

Furhermore, if $\delta \le \frac{1}{16(1+4\kappa)}$ and $n \ge 1$, then

$$\delta^2 \le \frac{n}{256(1+4\kappa)^2}.$$

Using this, $\mu^p = \left(1-\frac{\theta}{2}\right)\mu$, (48) and Lemma 5.4 we have

$$\begin{aligned}(\mathbf{x}^p)^T\mathbf{s}^p &\le \left(1-\frac{\theta}{2}+\frac{\theta^2}{8}\right)(\mathbf{x}^c)^T\mathbf{s}^c < (\mathbf{x}^c)^T\mathbf{s}^c < \mu(n+9\delta^2)\\ &< \frac{\mu^p}{1-\frac{\theta}{2}}\left(n+\frac{9n}{256(1+4\kappa)^2}\right) < \frac{2\mu^p n}{2-\theta}\left(1+\frac{9}{256}\right)\\ &= \frac{265 n\mu^p}{256(2-\theta)} < \frac{3n\mu^p}{2(2-\theta)},\end{aligned}$$

which yields the result. □

5.4. Determination of the values of the proximity and update parameters

We choose the values of the parameters τ and θ in such a way that after a corrector and a predictor step, the proximity measure will not exceed the proximity parameter. The following lemma is a technical one.

Lemma 5.10. *Let $\delta \le \frac{1}{16(1+4\kappa)}$ be the centraltiy measure related to the iterates before the corrector step. Then, we have $\delta^c \le \frac{10}{256(1+4\kappa)} < \frac{1}{4}$.*

Proof. Using $\frac{1}{16(1+4\kappa)} \le \frac{1}{2\sqrt{1+4\kappa}}$, by applying Corollary 5.3 and from $\kappa \ge 0$ we have

$$\delta^c \le 10(1+4\kappa)\delta^2 \le \frac{10}{256(1+4\kappa)} < \frac{1}{4},$$

which proves the lemma. □

Let $(\mathbf{x},\mathbf{s}) \in \mathcal{N}_2(\tau,\mu)$. Using Lemma 5.2, after a corrector step we have

$$\delta^c := \delta(\mathbf{x}^c,\mathbf{s}^c,\mu) \le \frac{5(1+4\kappa)\delta^2}{1-2(1+4\kappa)\delta^2}\sqrt{1-(1+4\kappa)\delta^2},$$

which is monotonically increasing with respect to δ, where $\delta < \frac{1}{\sqrt{2(1+4\kappa)}}$. In this way,

$$\delta^c \le \frac{5(1+4\kappa)\tau^2}{1-2(1+4\kappa)\tau^2}\sqrt{1-(1+4\kappa)\tau^2} =: \omega(\tau).$$

From $\delta \le \frac{1}{16(1+4\kappa)}$ and using Lemma 5.10 we have $\delta^c < \frac{1}{4}$. Using Lemma 5.8, after a predictor step and a μ-update we have

$$\begin{aligned}\delta^p &:= \delta(\mathbf{x}^p,\mathbf{s}^p,\mu^p)\\ &\le \frac{\sqrt{z(\delta^c,\theta,n)}\left(10(1+4\kappa)\delta^2+(1-4\delta^c)^2-z(\delta^c,\theta,n)\right)}{4z(\delta^c,\theta,n)-2},\end{aligned}$$

where $\delta := \delta(\mathbf{x},\mathbf{s},\mu)$ is the proximity measure given in (21). The function $z(\delta^c,\theta,n)$ is decreasing with respect to δ^c. Thus, $z(\delta^c,\theta,n) \ge z(\omega(\tau),\theta,n)$. In Lemma 5.8 we have seen that the function $h(t) = \frac{t}{2t^2-1}$, $t > \frac{\sqrt{2}}{2}$ is decreasing with respect to t, hence

$$h(\sqrt{z(\delta^c,\theta,n)}) \le h(\sqrt{z(\omega(\tau),\theta,n)}).$$

Note that $(1-4\delta^c)^2 - z(\delta^c,\theta,n) = \frac{2n(2+\kappa)\,\theta^2\,(1+4\delta^c)^2}{4(2-\theta)}$ is increasing with respect to δ^c. Using this and $\delta < \tau$, $\delta^c < \omega(\tau)$, we obtain

$$\begin{aligned}&\frac{\sqrt{z(\delta^c,\theta,n)}\left(10(1+4\kappa)\delta^2+(1-4\delta^c)^2-z(\delta^c,\theta,n)\right)}{4z(\delta^c,\theta,n)-2}\\ &\le \frac{\sqrt{z(\omega(\tau)),\theta,n)}\left(10(1+4\kappa)\tau^2+(1-4\omega(\tau))^2-z(\omega(\tau),\theta,n)\right)}{4z(\omega(\tau),\theta,n)-2}.\end{aligned} \tag{49}$$

Our aim is to keep $\delta^p \le \tau$. For this, it suffices that

$$\frac{\sqrt{z(\omega(\tau)),\theta,n)}\left(10(1+4\kappa)\tau^2+(1-4\omega(\tau))^2-z(\omega(\tau),\theta,n)\right)}{4z(\omega(\tau),\theta,n)-2} \le \tau.$$

Setting $\tau = \frac{1}{16(1+4\kappa)}$ and $\theta = \frac{1}{4(1+4\kappa)\sqrt{n}}$, the above inequality holds. Thus, $\mathbf{x},\mathbf{s} > \mathbf{0}$ and $\delta(\mathbf{x},\mathbf{s},\mu) \le \frac{1}{16(1+4\kappa)} < \frac{1}{\sqrt{2(1+4\kappa)}}$ are maintained during the algorithm. This means that the proposed IPA is well defined. Furthermore, we have

$$\begin{aligned}z(\delta^c,\theta,n) &= (1-4\delta^c)^2 - \frac{2n(2+\kappa)\theta^2(1+4\delta^c)^2}{4(2-\theta)}\\ &\ge (1-4\omega(\tau))^2 - \frac{2n(2+\kappa)\theta^2(1+4\omega(\tau))^2}{4(2-\theta)} > \frac{1}{2},\end{aligned}$$

hence the predictor step is strictly feasible. The way we have chosen the neighbourhood parameter shows that $(\mathbf{x}^p,\mathbf{s}^p) \in \mathcal{N}_2(\tau,\mu^p)$.

5.5. Complexity bound

The next lemma gives an upper bound for the number of iterations produced by the PC IPA.

Lemma 5.11. *Let $\mathbf{x}^0$ and $\mathbf{s}^0$ be strictly feasible, $\theta = \frac{1}{4(1+4\kappa)\sqrt{n}}$, $\mu^0 = \frac{(\mathbf{x}^0)^T\mathbf{s}^0}{n}$ and $\delta(\mathbf{x}^0,\mathbf{s}^0,\mu^0) \le \tau = \frac{1}{16(1+4\kappa)}$. Moreover, let $\mathbf{x}^k$ and $\mathbf{s}^k$ be the iterates obtained after k iterations. Then, $\left(\mathbf{x}^k\right)^T\mathbf{s}^k \le \epsilon$ for*

$$k \ge 1 + \left\lceil \frac{2}{\theta}\log\frac{3\left(\mathbf{x}^0\right)^T\mathbf{s}^0}{4\epsilon}\right\rceil.$$

Proof. Using Lemma 5.9 we have

$$\left(\mathbf{x}^k\right)^T\mathbf{s}^k < \frac{3n\mu^k}{4\left(1-\frac{\theta}{2}\right)} = \frac{3n\left(1-\frac{\theta}{2}\right)^{k-1}\mu^0}{4} = \frac{3\left(1-\frac{\theta}{2}\right)^{k-1}\left(\mathbf{x}^0\right)^T\mathbf{s}^0}{4}.$$

The inequality $\left(\mathbf{x}^k\right)^T\mathbf{s}^k \le \epsilon$ holds if $\frac{3\left(1-\frac{\theta}{2}\right)^{k-1}(\mathbf{x}^0)^T\mathbf{s}^0}{4} \le \epsilon$. We take logarithms, hence

$$(k-1)\log\left(1-\frac{\theta}{2}\right) + \log\frac{3\left(\mathbf{x}^0\right)^T\mathbf{s}^0}{4} \le \log\epsilon.$$

From $\log(1+\theta) \le \theta$, $\theta \ge -1$, it follows that the above inequality holds if

$$-\frac{\theta}{2}(k-1) + \log\frac{3\left(\mathbf{x}^0\right)^T\mathbf{s}^0}{4} \le \log\epsilon.$$

This yields the desired result. □

Theorem 5.12. *Let $\tau = \frac{1}{16(1+4\kappa)}$ and $\theta = \frac{1}{4(1+4\kappa)\sqrt{n}}$. Then, Algorithm 4.1 is well defined and requires at most*

$$O\left((1+4\kappa)\sqrt{n}\log\frac{3n\mu^0}{4\epsilon}\right)$$

iterations. The output is a pair $(\mathbf{x},\mathbf{s})$ satisfying $\mathbf{x}^T\mathbf{s} \le \epsilon$.

6. Numerical results

We implemented a variant of the proposed PC IPA in the C++ programming language using the code presented in Darvay & Takó (2012). We did all computations on a desktop computer with Intel quad-core 2.6 GHz processor and 8 GB RAM. It should be mentioned that the value of the parameter κ can be very large, which leads to a very small value of the parameter θ, see Theorem 5.12. This motivated us to make some modifications in the implementation of the proposed PC IPA.

Algorithm 6.1 illustrates the computational version of the theoretical PC IPA given in Algorithm 4.1. In the predictor step we calculated the maximal step size α_x^p and α_s^p to the boundary of non-negative orthant by using minimal ratio test. We considered the minimum value of these step sizes and we determined the vectors $\mathbf{x}^p$ and $\mathbf{s}^p$ without modifying the actual points $\mathbf{x}^k$ and $\mathbf{s}^k$. The value of ρ in our case was 0.5. Note that the vectors $\mathbf{x}^p$ and $\mathbf{s}^p$ were used in the computation of step lengths α_x^c, α_s^c in the corrector step.

Algorithm 6.1 PC IPA from the implementation point of view.

Let $\epsilon = 10^{-5}$, $\mathbf{x}^0 = \mathbf{s}^0 = \mathbf{e}$, $\mu^0 = 1$, $0 < \rho < 1$, $0 < \sigma < 1$ and $lb = \frac{1}{2}$.
begin
  $k := 0$;
  while $\left(\mathbf{x}^k\right)^T \mathbf{s}^k > \epsilon$ **do**
  begin
  predictor step
    compute $(\Delta^p x^k, \Delta^p s^k)$ from system (26) using (28);
    $\alpha_x^p = \min\left\{-\frac{x_i^k}{\Delta^p x_i^k} \,|\, \Delta^p x_i^k < 0, 1 \le i \le n\right\}$;
    $\alpha_s^p = \min\left\{-\frac{s_i^k}{\Delta^p s_i^k} \,|\, \Delta^p s_i^k < 0, 1 \le i \le n\right\}$;
    $\alpha^p = \min\{\alpha_x^p, \alpha_s^p\}$;
    $(\mathbf{x}^p)^k := \mathbf{x}^k + \rho\alpha^p \Delta^p \mathbf{x}^k$; $(\mathbf{s}^p)^k := \mathbf{s}^k + \rho\alpha^p \Delta^p \mathbf{s}^k$;
  corrector step
    $\mu_c^k = \sigma \frac{\min\{(\mathbf{x}_i^p)^k (\mathbf{s}_i^p)^k :\, 1 \le i \le n\}}{lb}$;
    compute $(\Delta^c x^k, \Delta^c s^k)$ from system (24) using (25);
    $\Delta x^k = \Delta^p x^k + \Delta^c x^k$; $\Delta s^k = \Delta^p s^k + \Delta^c s^k$;
    $\alpha_x^c = \min\left\{-\frac{(x_i^p)^k}{\Delta x_i^k} \,|\, \Delta x_i^k < 0, 1 \le i \le n\right\}$;
    $\alpha_s^c = \min\left\{-\frac{(s_i^p)^k}{\Delta s_i^k} \,|\, \Delta s_i^k < 0, 1 \le i \le n\right\}$;
    $\alpha^c = \min\{\alpha_x^c, \alpha_s^c\}$;
    $(\mathbf{x}^c)^k := \mathbf{x}^k + \rho\alpha^c \Delta x^k$; $(\mathbf{s}^c)^k := \mathbf{s}^k + \rho\alpha^c \Delta s^k$;
    $\mathbf{x}^{k+1} := (\mathbf{x}^c)^k$, $\mathbf{s}^{k+1} := (\mathbf{s}^c)^k$; $k := k+1$;
  end
end

The value of the parameter μ in the corrector step was calculated as $\mu_c^k = \sigma \frac{\min\{(\mathbf{x}_i^p)^k (\mathbf{s}_i^p)^k :\, 1 \le i \le n\}}{lb}$, where $0 < \sigma < 1$, lb denotes a given lower bound, which in our case is $\frac{1}{2}$. In our case the value of σ was 0.1. The way of determining the value of the parameter μ_c^k ensures that the components of the vector $\mathbf{v}$ are greater than a positive constant, which is important in our case due to the used search direction. It should be mentioned that we considered the search directions obtained by the sum of the predictor and the corrector directions. In the determination of the step length in case of the corrector step we used the same strategy as in case of the predictor step.

We tested the PC IPA on LCPs with sufficient matrices having positive κ parameters generated by Illés & Morapitiye (2018). We generated the test problems in the following way: $\mathbf{q} := -M\mathbf{e} + \mathbf{e}$. We considered $\mathbf{x}^0 = \mathbf{e}$ and $\mathbf{s}^0 = \mathbf{e}$ as starting points for our PC IPA.

We have tested the PC IPA for all 61 $P_*(\kappa)$-LCPs from the selection given in Illés & Morapitiye (2018). We could easily obtain results for variants of the PC IPA using different functions φ in this new type of AET technique by changing the right hand side of the Newton-system. In our computational study we compared our PC IPA using the function $\varphi(t) = t^2$ in system (6) with the variant of the IPA which uses the $\varphi(t) = t$ in the new type of AET technique characterized by system (6). Note that in the case when $\varphi(t) = t$ is used, then the value of lb is $\frac{1}{4}$, $g(\mathbf{x}, \mathbf{s}) = -\mathbf{xs}$ and $\mathbf{a}_\varphi = \frac{\sqrt{\mu}\mathbf{xs}}{2\sqrt{\mathbf{xs}} - \sqrt{\mu}\mathbf{e}}$. This yields the same direction as the one used in Darvay et al. (2020b), where system (5) was considered with $\bar{\varphi}(t) = t - \sqrt{t}$. Table 1 contains the average of iteration numbers and CPU times (in seconds) for 10 given LCPs for each size n listed in the table. We can observe that the results are similar for both variants of the PC IPA using the different search directions.

Table 1
Numerical results for $P_*(\kappa)$-LCPs from Illés & Morapitiye (2018) having positive handicap.

n	$\varphi(t)=t^2$; $\bar{\varphi}(t)=t^2-t$		$\varphi(t)=t$; $\bar{\varphi}(t)=t-\sqrt{t}$	
	Avg. Iter.	CPU (s)	Avg. Iter.	CPU (s)
10	19	0.003	18.9	0.0016
20	20.5	0.041	20.2	0.0405
50	18.1	0.2798	17.9	0.2741
100	18.4	1.563	18.1	1.5241
200	19	10.3192	18.5	10.0423
500	19.2	146.905	19.2	147.1175

Table 2
Numerical results for $P_*(\kappa)$-LCPs with matrix given in (50).

n	$\varphi(t)=t^2$; $\bar{\varphi}(t)=t^2-t$		$\varphi(t)=t$; $\bar{\varphi}(t)=t-\sqrt{t}$	
	Nr. of Iter.	CPU (s)	Nr. of Iter.	CPU (s)
20	29	0.058	30	0.06
50	45	0.67	46	0.688
100	72	6.184	73	6.151
300	181	307.276	181	307.081
400	235	964.821	236	1016.09

de Klerk & Nagy (2011) proved that the handicap of the matrix can be exponential in the size of the problem. They considered the following matrix which was proposed by Csizmadia:

$$M = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 1 \end{pmatrix}, \tag{50}$$

and they proved that $\hat{\kappa}(M) \ge 2^{2n-8} - 0.25$. However, in our computational study we obtained promising results for the two variants of PC IPAs. The results are summarized in Table 2.

The obtained results can be further analysed, because it seems that the practical iteration complexity is significantly better than the theoretical (worst case) guarantee for the special class of LCPs with the lower triangular P-matrix M, introduced by Zs. Csizmadia.

7. Conclusions and further research

In this paper we proposed a new PC IPA for solving $P_*(\kappa)$-LCPs which uses the new type of AET given in Darvay & Takács (2018) for LO. The presented IPA applies the function $\varphi(t) = t^2$ on the nonlinear equation $\mathbf{v}^2 = \mathbf{v}$ in order to determine the new search directions. The corresponding kernel function is a positive-asymptotic kernel function. Furthermore, similar to Darvay et al. (2020b), we presented the method for determining the Newton systems and scaled systems in case of PC IPAs using this new type of AET. Due to the used search direction we had to ensure during the whole process of the IPA that the components of the vector $\mathbf{v}$ were greater than $\frac{\sqrt{2}}{2}$. In spite of this fact, we proved that the PC IPA retains polynomial iteration complexity in the handicap of the problem's matrix, the size of the problem and the deviation from the complementarity gap. This is the first

PC IPA for solving $P_*(\kappa)$-LCPs which uses the function $\varphi(t) = t^2$ in the new type of AET. Moreover, we also provided numerical results where we compared our PC IPA to another variant of this algorithm using $\varphi(t) = t$ in the new type of AET technique. As further research, it would be interesting to find a class of monotone increasing functions $\bar{\varphi}$ for which we can assign corresponding functions φ. This would lead to a case where we can establish equivalence between the two approaches of the AET presented in this paper. Furthermore, it would be interesting to define a PC IPA using this new type of AET approach, where the central path parameter update is adaptive, for example as it is in Potra & Wright (2000).

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