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How to make ambiguous strategies ☆

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Abstract

Taking advantage of ambiguity in strategic situations is well documented in the literature. However, so far there are only few results on how to make ambiguous strategies.

In this paper we introduce a procedure which makes objective ambiguity, concretely it draws an element from a set of priors, defined by a belief function, in a way that it does not lead to any probability distribution over the priors. Moreover, we define the notion of ambiguous strategy, and by means of examples we show how to make ambiguous strategies in games.

JEL classification: C72; D81

Keywords: Ambiguity; Inner measure; Belief function; Game Theory; Ambiguous strategy

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1. Introduction

Ambiguity is a prevailing phenomenon in social and natural situations, and it is an old notion in decision theory and in economics. Ambiguity – more precisely the attitude towards it – can explain important paradoxes in human decision making, see e.g. the Ellsberg paradox (Ellsberg, 1961). The literature on ambiguity is rich including Schmeidler (1989); Gilboa and Schmeidler (1989); Ghirardato and Marinacci (2002); Klibanoff et al. (2005); Marinacci and Montucchio (2006); Maccheroni et al. (2006); Cerreia-Vioglio et al. (2011); Lehrer (2012); Gilboa and Marinacci (2016); Li et al. (2018), among others; for an overview of the notion and literature of ambiguity see Machina and Siniscalchi (2014).

It is quite common among the above mentioned papers that ambiguity does not appear as a primitive of the model, but it is encoded in the preferences of the decision maker. This situation is similar to the case of risk, where in von Neumann and Morgenstern (1944) the risk is a primitive of the model, hence it is objective – it does not depend on the decision maker, while in Savage (1954), risk is encoded in the preferences of the decision maker, hence it is subjective – it depends on the decision maker indeed (Anscombe and Aumann made this distinction – objective vs. subjective – explicit, in Anscombe and Aumann (1963) both types of risk (roulette vs. horse race) are considered). In other words, ambiguity is typically subjective in the models of the above mentioned papers. However, in order to achieve our goal of making ambiguous strategies, we need objective ambiguity.

In this paper we follow the multiple prior approach to model ambiguity. The subject of multiple priors was first proposed as maximin expected utility in Wald (1950). Subsequently it was also discussed in Hurwicz (1951), Arrow (1951) and Luce and Raiffa (1957). Gärdenfors and Sahlin (1982) made the connection of multiple priors to Ellsberg's paradox (Ellsberg, 1961). Gilboa and Schmeidler (1989) gave an axiomatization of this approach. Particularly, we consider the case when a set of priors is given by a belief function (Dempster, 1967; Shafer, 1976; Jaffray, 1992).

In several situations ambiguity can be used to gain strategic advantage, see e.g. Greenberg (2000), Binmore (2009), Bade (2011), Riedel and Sass (2014), Di Tillio et al. (2017), de Castro and Yannelis (2018) among others. However, in order to exploit the strategic advantage of ambiguity, players must be able to make ambiguous strategies. By making ambiguous strategies we mean a procedure which draws an element from a set of priors in a way that it does not give rise to any distribution over the priors. A player commits herself to the procedure, that is, whatever prior the procedure chooses, she applies this prior to play her action. Therefore, the procedure is a generalization of randomization which can be used to make mixed strategies.

In this paper, we introduce a mathematical construction (the skeleton of a device) with which one can make objective ambiguity, hence ambiguous strategies. The construction is based on the notion of inner measure. By Theorem 1 for any belief function one can take an appropriate probability space, where the inner measure of the probability distribution drives the same information as the belief function. Then, the decision maker can extend the probability distribution onto all subsets, however, the extension is typically not unique. The class of extensions are given by the inner measure of the probability distribution, hence an extension can be considered as choosing an element from a set of priors. In the steps of the extension the decision maker can apply Stecher et al. (2011)'s method to pick an extension in a way that it does not lead to any probability distribution over the possible extensions.

In other words, by Theorem 1 and Stecher et al. (2011)'s method, one can draw a prior from a class of priors given by a belief function, and the procedure does not lead to any probability

distribution over the priors, hence nobody can assign objective probability to the priors. Naturally, a decision maker can assign subjective probability to the priors, but this probability distribution is based on the decision maker's preferences, hence it is not objective.

Then as an application of the above described method, we introduce the notion of ambiguous strategy, which is a generalization of mixed strategy, and can be applied in game theory.

As in the case of mixed strategy, in the case of ambiguous strategy when a player evaluates a strategy profile, her attitude towards ambiguity is important. Objective ambiguity is not about attitudes, however, the Choquet integral (Choquet, 1953) applied by Schmeidler (1989), the maximin expected payoff applied by Gilboa and Schmeidler (1989) and the concave integral applied by Lehrer (2009, 2012) lead to the very same evaluation in our model (in the case of belief functions), hence at least one evaluation method of ambiguous strategies is already at hand.

To our knowledge the literature of applying objective ambiguity in strategic situations is not too rich. Binmore (2009) considers the strategic importance of ambiguity, and he introduces muddling boxes as source of objective ambiguity. However, he does not construct muddling boxes,[1] he takes them as given. Riedel and Sass (2014) consider games where the players can use a device to generate objective ambiguity in order to use ambiguity for strategic purposes. However, Riedel and Sass do not specify the details of the device, they assume that the proposed device is given. Di Tillio et al. (2017) apply ambiguous strategies based on objective ambiguity in mechanism design problems. However, Di Tillio et al. do not give any method to make ambiguous strategies, they take these strategies as given.

Battigalli et al. (2015); Epstein and Schneider (2007); Greenberg (2000); Bade (2011) and de Castro and Yannelis (2018) also use ambiguity in strategic situations (games), but these models work with subjective ambiguity.

We must elaborate on the connection between our main technical result (Theorem 1) on one side and Theorem 4 in Gul and Pesendorfer (2014) and Theorem 6 in Grant et al. (2022) on the other side. These papers are very different in their goals, Gul and Pesendorfer give a subjective foundation for the Dempster-Shafer theory of evidence, Grant et al. establish that there is nothing inherent in the Dempster-Shafer theory of evidence that necessitates the evaluation of an act via a linear expectation, while we propose a procedure to generate objective ambiguity. From a technical viewpoint, we can also see significant differences, since Gul and Pesendorfer's and Grant et al.'s mentioned results relate infinite state spaces to finite ones, but we relate a finite state space to a bigger, but finite one. However, it is common in these results that they all apply a partitioning approach; namely, the events of the smaller state space are corresponding to parts of a partition of the bigger state space. This partitioning approach allows for the transformation of a non-additive probability (on the small state space) into a probability distribution (on the bigger state space).

Stecher et al. (2011) is the closest to our model in its goal. Stecher et al. (2011) introduce a method to simulate ambiguous outcomes by applying composition of Cauchy random variables. Their method is suitable to simulate ambiguity in experiments, but it is less suitable for generating ambiguity in strategic situations. We use Stecher et al.'s method to get (real) numbers in a way that even the distribution of the numbers is not known. With these numbers we can draw an element from a set of priors given by a belief function (see above), hence we can make objective ambiguity.

[1] Following Binmore's terminology, in this paper we construct muddling boxes.

The outline of the paper is as follows. In Section 2 we introduce our procedure of generating objective ambiguity, and by considering the Ellsberg paradox (Ellsberg, 1961) we demonstrate how the procedure works. In Section 3 we introduce the notion of ambiguous strategy, and apply ambiguous strategies in Greenberg (2000)'s game and in one of the games in Riedel and Sass (2014). Finally, the last section briefly concludes. We relegate all proofs to the appendix.

2. Generating objective ambiguity

In this section we propose a procedure to generate objective ambiguity for strategic purposes. Our starting point is Stecher et al. (2011)'s method for simulating ambiguity.

Stecher et al. generate a sequence of (real) numbers for the purpose of simulating ambiguity in the lab. The generating process is based on the Cauchy distribution. The cdf of the Cauchy distribution $C(x_0, \gamma)$ is $F(x) = \frac{1}{\pi}\arctan(\frac{x-x_0}{\gamma}) + \frac{1}{2}$, where x_0 is the location, and $\gamma > 0$ is the scale parameter. The Cauchy distribution does not have any finite integer moments. Then Stecher et al. (2011) generate a sequence $(z_0, z_1, \ldots)$ in the following way:

1. Draw $Z_0 \sim C[0, 1]$;
2. Draw $Z_1 \sim C[z_0, 1]$;
3. Let $\phi, \psi \in [0, 1]$ with ϕ, ψ both small. For $n \geq 2$, draw $Z_n \sim C[z_{n-1}, \phi|z_{n-2}| + \psi]$.

In other words, Stecher et al.'s procedure can be written as a difference equation, $Z_n = Z_{n-1} + \psi X_n + \phi X_n |z_{n-2}|$, where $X_n \sim C[0, 1]$ are independent and identically distributed. Notice that Stecher et al.'s procedure draws from a non-ergodic process with inconsistent sample quantiles. Because the quantiles do not converge, an observer cannot learn probabilities from a sample. In other words, the result of the procedure is unpredictable in a strong sense, even the distribution of the generated numbers cannot be calculated.

Next we take a short mathematical detour to show how we apply the above procedure by Stecher et al. to generate objective ambiguity.

Consider a probability space $(X, \mathcal{M}, \mu)$, that is, X is a non-empty, finite set of the states of the world, $\mathcal{M}$ is a field on X, and μ is a probability distribution (measure) over $\mathcal{M}$. Then the inner measure μ_* of μ is a set function on $\mathcal{P}(X)$ defined as follows: for all $A \subseteq X$ it holds that

$$\mu_*(A) = \max_{\substack{B \subseteq A \\ B \in \mathcal{M}}} \mu(B).$$

By an inner measure for an event $A \subseteq X$ we define an interval $[\mu_*(A), 1 - \mu_*(A^{\complement})]$.[2] If for an event A we have that $\mu_*(A) \neq 1 - \mu_*(A^{\complement})$, then we say that event A is an ambiguous event. Notice that, if $\mu_*(A) \neq 1 - \mu_*(A^{\complement})$, then there are infinitely many extensions of μ onto $\mathcal{P}(X)$, particularly, for each $a \in [\mu_*(A), 1 - \mu_*(A^{\complement})]$ there exists an extension of μ onto $\mathcal{P}(X)$ such that its value at event A is a. Mathematically speaking ambiguity is the phenomenon that the extension is not unique.

[2] In terms of outer measure μ^*, $\mu^*(A) = \min_{\substack{B \supseteq A \\ B \in \mathcal{M}}} \mu(B)$, the interval $[\mu_*(A), 1 - \mu_*(A^{\complement})]$ can be written as $[\mu_*(A), \mu^*(A)]$.

In the following theorem (its proof can be found in Appendix A) we argue that taking an inner measure is equivalent to working with a belief function. We mean, one can take belief function instead of inner measure, and vice versa, one can take inner measure instead of belief function.

Theorem 1. *The following two assertions hold:*

1. *Take an arbitrary probability space $(X, \mathcal{M}, \mu)$. Then μ_* (the inner measure of μ) is a belief function.*
2. *Moreover, let ν be a belief function on a space $(\Omega, \mathcal{A})$. Then there exist a probability space $(X, \mathcal{M}, \mu)$ and a surjection $f: X \to \Omega$ such that $\nu = \mu_* \circ f^{-1}$.*

By Theorem 1 it is enough to consider a probability space $(X, \mathcal{M}, \mu)$, and the inner measure μ_* of μ on $\mathcal{P}(X)$. Then for each event $A \in \mathcal{P}(X)$ the interval $[\mu_*(A), 1 - \mu_*(A^{\complement})]$ is the "ambiguous probability" of the event. Differently, any extension of μ onto $\mathcal{P}(X)$ is a prior. What we have to do is pick a prior (an extension of μ) in a way it does not lead to any probability distribution over the extensions of μ. The proposed method is the following:

Method 2. *The method consists of the repetition of the following three steps:*

1. *Draw $A \in \mathcal{P}(X) \setminus \mathcal{M}$; if there does not exist such A, then stop.*
2. *If $\mu_*(A) = \alpha = 1 - \mu_*(A^{\complement})$, then let $\mu(A) = \alpha$, otherwise apply Stecher et al. (2011)'s method to get a number α from the interval $[\mu_*(A), 1 - \mu_*(A^{\complement})]$ and let $\mu(A) = \alpha$.*
3. *The probability distribution μ can be uniquely extended as a probability distribution onto the field generated by $\mathcal{M}$ and $\{A\}$ such that $\mu(A) = \alpha$; let $\mathcal{M}$ be the new larger field, μ be the extension onto the new $\mathcal{M}$; go to Point 1.*

The above method does exactly what we intended, that is, it extends μ from $\mathcal{M}$ onto larger and larger fields, finally onto $\mathcal{P}(X)$ as a probability distribution in a way it does not lead to any probability distribution over the extensions of μ. Therefore, this method generates objective ambiguity.

To illustrate how our method works let us consider the following example.

2.1. An example

Consider the (one urn) Ellsberg paradox, where $\Omega = \{\omega_B, \omega_Y, \omega_R\}$ is the set of the states of the world, $\mathcal{A} = \mathcal{P}(\Omega)$ is the class of events, and we have the belief function ν defined as follows, for each event $A \in \mathcal{A}$:

$$\nu(A) = \begin{cases} 0 & \text{if } A \in \{\emptyset, \{\omega_B\}, \{\omega_Y\}\}, \\ 1/3 & \text{if } A \in \{\{\omega_R\}, \{\omega_B, \omega_R\}, \{\omega_Y, \omega_R\}\}, \\ 2/3 & \text{if } A = \{\omega_B, \omega_Y\}, \\ 1 & \text{otherwise,} \end{cases}$$

that is, $\nu = \sum_{T \in \mathcal{P}(\Omega)\setminus\{\emptyset\}} \alpha_T u_T = \frac{1}{3} u_{\{\omega_R\}} + \frac{2}{3} u_{\{\omega_B, \omega_Y\}}$,[3] where u_T is the unanimity game on set T, that is,

[3] This representation of ν is unique. For the details see e.g. Grabisch (2016).

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

In words, ω_B and ω_Y stand for drawing a black and a yellow ball respectively, where the numbers of these balls in the urn are not known, and ω_R stands for drawing a red ball, where it is known that 30 of 90 balls are red.

Then let

$$\begin{aligned} X &= \{(\omega, A) \in \Omega \times \mathcal{P}(\Omega) \colon \omega \in A\} \\ &= \{(\omega_B, \{\omega_B\}), (\omega_B, \{\omega_B, \omega_Y\}), (\omega_B, \{\omega_B, \omega_R\}), (\omega_B, \Omega), \\ &\quad (\omega_Y, \{\omega_Y\}), (\omega_Y, \{\omega_B, \omega_Y\}), (\omega_Y, \{\omega_Y, \omega_R\}), (\omega_Y, \Omega), \\ &\quad (\omega_R, \{\omega_R\}), (\omega_R, \{\omega_B, \omega_R\}), (\omega_R, \{\omega_Y, \omega_R\}), (\omega_R, \Omega)\}, \end{aligned}$$

and let

$$f(\omega, A) = \begin{cases} \omega_B & \text{if } \omega = \omega_B, \\ \omega_Y & \text{if } \omega = \omega_Y, \\ \omega_R & \text{if } \omega = \omega_R, \end{cases}$$

that is, $f(x) = x|_\Omega$, $x \in X$, hence f is a surjection. Moreover, let $\mathcal{M}$ be the field generated by the sets $\{\{x \in X \colon x|_\mathcal{A} = A\},\ A \in \mathcal{A}\}$. Notice that

$$\begin{aligned} &\{\{x \in X \colon x|_\mathcal{A} = A\},\ A \in \mathcal{A}\} \\ &= \{\{(\omega_B, \{\omega_B\})\}, \{(\omega_Y, \{\omega_Y\})\}, \{(\omega_R, \{\omega_R\})\}, \\ &\{(\omega_B, \{\omega_B, \omega_Y\}), (\omega_Y, \{\omega_B, \omega_Y\})\}, \{(\omega_B, \{\omega_B, \omega_R\}), (\omega_R, \{\omega_B, \omega_R\})\}, \\ &\{(\omega_Y, \{\omega_Y, \omega_R\}), (\omega_R, \{\omega_Y, \omega_R\})\}, \{(\omega_B, \Omega), (\omega_Y, \Omega), (\omega_R, \Omega)\}\} \end{aligned}$$

is a partition of X. Finally, let

$$\mu(E) = \begin{cases} \alpha_{\{\omega_B\}} = 0 & \text{if } E = \{x \in X \colon x|_\mathcal{A} = \{\omega_B\}\}, \\ \alpha_{\{\omega_Y\}} = 0 & \text{if } E = \{x \in X \colon x|_\mathcal{A} = \{\omega_Y\}\}, \\ \alpha_{\{\omega_R\}} = \frac{1}{3} & \text{if } E = \{x \in X \colon x|_\mathcal{A} = \{\omega_R\}\}, \\ \alpha_{\{\omega_B, \omega_Y\}} = \frac{2}{3} & \text{if } E = \{x \in X \colon x|_\mathcal{A} = \{\omega_B, \omega_Y\}\}, \\ \alpha_{\{\omega_B, \omega_R\}} = 0 & \text{if } E = \{x \in X \colon x|_\mathcal{A} = \{\omega_B, \omega_R\}\}, \\ \alpha_{\{\omega_Y, \omega_R\}} = 0 & \text{if } E = \{x \in X \colon x|_\mathcal{A} = \{\omega_Y, \omega_R\}\}, \\ \alpha_\Omega = 0 & \text{if } E = \{x \in X \colon x|_\mathcal{A} = \Omega\}. \end{cases} \tag{1}$$

Then for every $T \subseteq \Omega$, $E \in \{\{x \in X \colon x|_\mathcal{A} = A\},\ A \in \mathcal{A}\}$ it holds that $E \subseteq f^{-1}(T)$ if and only if $E = \{x \in X \colon x|_\mathcal{A} = S\}$ for some $S \subseteq T$, hence $\nu = \mu_* \circ f^{-1}$.

Take a set from $\mathcal{P}(X) \setminus \mathcal{M}$; suppose that it is

$$\begin{aligned} f^{-1}(\{\omega_B, \omega_R\}) = \{&(\omega_B, \{\omega_B\}), (\omega_R, \{\omega_R\}), (\omega_B, \{\omega_B, \omega_Y\}), (\omega_B, \{\omega_B, \omega_R\}), \\ &(\omega_R, \{\omega_B, \omega_R\}), (\omega_R, \{\omega_Y, \omega_R\}), (\omega_B, \Omega), (\omega_R, \Omega)\}. \end{aligned}$$

Then $\mu_*(f^{-1}(\{\omega_B, \omega_R\})) = \alpha_{\{\omega_B\}} + \alpha_{\{\omega_R\}} + \alpha_{\{\omega_B, \omega_R\}} = \frac{1}{3}$, and $1 - \mu_*((f^{-1}(\{\omega_B, \omega_R\})^\complement) = 1$. Next, apply Stecher et al. (2011)'s method to get a number from the interval $[\frac{1}{3}, 1]$. Suppose that this number is $\frac{1}{2}$, that is, let $\bar{\mu}(f^{-1}(\{\omega_B, \omega_R\})) = \frac{1}{2}$, where $\bar{\mu}$ is the proposed extension of μ onto $\mathcal{P}(X)$.

Then from (1)

$$\begin{aligned}\bar{\mu}(\{(\omega_B, \{\omega_B\})\}) = \bar{\mu}(\{(\omega_Y, \{\omega_Y\})\}) &= \bar{\mu}(\{(\omega_B, \{\omega_B, \omega_R\})\}) \\ = \bar{\mu}(\{(\omega_R, \{\omega_B, \omega_R\})\}) &= \bar{\mu}(\{(\omega_Y, \{\omega_Y, \omega_R\})\}) \\ = \bar{\mu}(\{(\omega_R, \{\omega_Y, \omega_R\})\}) = \bar{\mu}(\{(\omega_B, \Omega)\}) &= \bar{\mu}(\{(\omega_Y, \Omega)\}) = \bar{\mu}(\{(\omega_R, \Omega)\}) = 0 \\ \bar{\mu}(\{(\omega_R, \{\omega_R\})\}) &= \frac{1}{3} \\ \bar{\mu}(\{(\omega_B, \{\omega_B, \omega_Y\})\}) + \bar{\mu}(\{(\omega_Y, \{\omega_B, \omega_Y\})\}) &= \frac{2}{3}\end{aligned}$$

From that $\bar{\mu}(f^{-1}(\{\omega_B, \omega_R\})) = \frac{1}{2}$ we have $\bar{\mu}(\{(\omega_R, \{\omega_R\}), (\omega_B, \{\omega_B, \omega_Y\})\}) = \frac{1}{2}$, hence $\bar{\mu}(\{(\omega_B, \{\omega_B, \omega_Y\})\}) = \frac{1}{6}$ and $\bar{\mu}(\{(\omega_Y, \{\omega_B, \omega_Y\})\}) = \frac{1}{2}$.

The construction above (and the proof of Theorem 1) clearly shows that point 2 of Theorem 1 can be stated in a stronger form, not only as an existence result, but the probability space and the mapping can also be constructed. Even more, the space $(X, \mathcal{M})$ depends only on the space $(\Omega, \mathcal{A})$, in other words in the probability space $(X, \mathcal{M}, \mu)$, $(X, \mathcal{M})$ depends only on $(\Omega, \mathcal{A})$, and only μ depends on the belief function ν.

It is also worth noticing that in this example it is enough to apply one round in Method 2, as we have seen above.

In point 2 of Theorem 1 the probability space $(X, \mathcal{M}, \mu)$ is not unique. Consider the above example, and alternatively let $X = \Omega$, $\mathcal{M} = \{\emptyset, X, \{\omega_R\}, \{\omega_B, \omega_Y\}\}$, μ be such that $\mu(\{\omega_R\}) = \frac{1}{3}$, and f be the identity. Then it is easy to see that $\nu = \mu_* = \mu_* \circ f^{-1}$.

The importance of the construction in the proof of Theorem 1 is that it is universal, it always works, therefore, with $(\Omega, \mathcal{A}, \nu)$ in hand we can give $(X, \mathcal{M}, \mu)$ explicitly.

3. Ambiguous strategies

In this section we introduce the notion of an ambiguous strategy, and show that it is a generalization of mixed strategies and therefore of pure strategies, as well. We also consider two examples for games with ambiguity from the literature and show how the players can make ambiguous strategies in these games.

Definition 3. Given a finite normal form game $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$, where N is the player set, S_i is player i's pure strategy set and u_i is player i's payoff function, $i \in N$. Then the ambiguous extension of the game is a tuple $(N, (\mathrm{b}(S_i)_{i\in N}), (\hat{u}_i)_{i\in N})$, where

- $\mathrm{b}(S_i)$ is the set of the belief functions over S_i, that is, it is the set of ambiguous strategies of player i, $i \in N$,
- $\hat{u}_i(\nu) = \min\limits_{\nu' \in \mathrm{core}(\nu)} \int u_i \, \mathrm{d}\nu'$, $\nu \in \prod_{j\in N} \mathrm{b}(S_j)$ is the payoff function of player i in the ambiguous extension, $i \in N$,

where $\mathrm{core}(\nu) = \{\nu' \in \prod_{j\in N} \Delta(S_j) \colon \nu'_j(E) \geq \nu_j(E),\ \forall E \subseteq S_j,\ \forall j \in N\}$.

Our definition of an ambiguous strategy is similar to Riedel and Sass (2014)'s notion. The two notions are different in two main points:

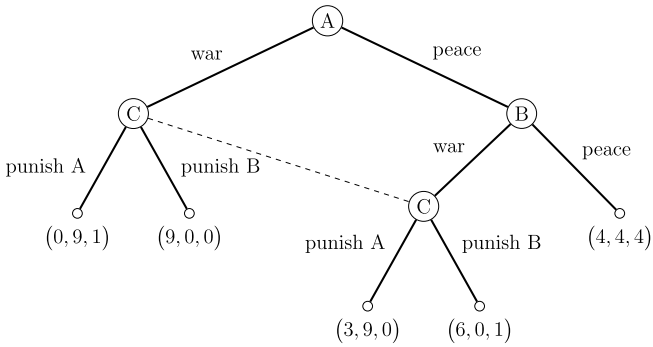

Fig. 1. Greenberg's example.

1. In Riedel and Sass (2014) the ambiguous strategy is defined as uncertainty over mixed strategies. Here, the ambiguous strategy is non-additive, more precisely, not necessarily additive mixing of pure strategies. In other words, while in Riedel and Sass (2014) the object of ambiguity is the mixed strategy, here the pure strategy is the object of ambiguity.
2. Riedel and Sass do not give explicitly any method to make ambiguous strategies, they refer to those as Ellsberg urns which are given exogenously. Here we give a method (see Method 2) which can generate ambiguous strategies.

According to Definition 3 every pure strategy is a mixed strategy, and every mixed strategy is an ambiguous strategy, but not each ambiguous strategy is a mixed strategy, as not each mixed strategy is a pure strategy.

Regarding the payoff functions in the ambiguous extension, in this paper we consider belief functions, in which case Schmeidler (1989)'s Choquet integral (Choquet, 1953), Gilboa and Schmeidler (1989)'s maximin expected payoff and Lehrer (2009, 2012)'s concave integral give the very same evaluation, hence when we use one of them, we use all.

Our method of generating objective ambiguity (Method 2) can be applied to make an ambiguous strategy in the following way:

1. The player chooses an ambiguous strategy (see Definition 3).
2. Method 2 assigns a probability to each pure strategy (Ω is the player's pure strategy set, and $\mathcal{A}$ is the class of all of its subsets).
3. The pure strategies are played according to the assigned probability distribution, meaning, the pure strategies are drawn randomly by the assigned probability distribution.

3.1. Two games from the literature

First we revisit Greenberg (2000)'s example and show how our method introduced above works.

Example 4. Consider the game in extensive form in Fig. 1.

There are three players, two of them – A and B – can choose between peace and war, and the third one – C – can punish any but only one of the two. If both A and B opt for peace, all three players obtain a payoff of 4. If one of A and B does otherwise, war breaks out, but C cannot

decide whose action started the war. Player C can punish one of A and B and support the other. The payoffs are for the players A, B and C respectively.

Greenberg showed that this game possesses a unique (mixed) Nash equilibrium where player A mixes with equal probabilities, and player B opts for war; player C has no clue who started the war given these actions. She is thus indifferent about whom to punish and mixes with equal probabilities as well. War happens with probability 1, and the resulting equilibrium (expected) payoff vector is (4.5, 4.5, 0.5).

Assume that player C can apply the following ambiguous strategy ν (over S_C), where

- $S_C = \{\text{punish A, punish B}\}$ is player C's strategy set,
- ν is a belief function such that $\nu(\{\text{punish A}\}) = \nu(\{\text{punish B}\}) = 0$.

Notice that the considered belief function assigns 0 to both pure strategies, therefore, this strategy is maximally ambiguous, and $\nu = u_{\{\text{punish A, punish B}\}}$.

If player A plays war his maximin expected payoff is 0, and if he opts for peace his payoff is higher than 0 independently from what player B plays. Therefore, player A's optimal strategy is to play peace. If player B opts for war his maximin expected payoff is 0, if he plays peace his payoff is 4. Therefore, for player B it is optimal to opt for peace. For player C applying the ambiguous strategy ν above gives 4, no other strategy can give higher payoff for her, meaning the (peace, peace, ν) is an equilibrium strategy profile.

If player C plays this strategy she commits herself to apply the following method: she takes the probability space $(X, \mathcal{M}, \mu)$, where

- $X = \{(\text{punish A}, \{\text{punish A}\}), (\text{punish B}, \{\text{punish B}\}), (\text{punish A}, \{\text{punish A, punish B}\}), (\text{punish B}, \{\text{punish A, punish B}\})\}$,
- The field $\mathcal{M}$ is generated by the partition $\{\{(\text{punish A}, \{\text{punish A}\})\}, \{(\text{punish B}, \{\text{punish B}\})\}, \{(\text{punish A}, \{\text{punish A, punish B}\}), (\text{punish B}, \{\text{punish A, punish B}\})\}\}$,
- $\mu(\emptyset) = \mu(\{(\text{punish A}, \{\text{punish A}\})\}) = \mu(\{(\text{punish B}, \{\text{punish B}\})\}) = 0$, and $\mu(\{(\text{punish A}, \{\text{punish A, punish B}\}), (\text{punish B}, \{\text{punish A, punish B}\})\}) = \mu(X) = 1$.

Then she considers the inner measure μ_* of μ, hence

$$\mu_*(E) = u_{\{x \in X:\, x|_{\mathcal{P}(S_C)} = \{\text{punish A,punish B}\}\}}$$
$$= \begin{cases} 1 & \text{if } \{x \in X:\, x|_{\mathcal{P}(S_C)} = \{\text{punish A, punish B}\}\} \subseteq E, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\nu = \mu_* \circ f^{-1}$, where $f\colon X \to S_C$ is defined as $f(x) = x|_{S_C}$, for all $x \in X$.

Next, she applies Stecher et al. (2011)'s method to assign a number from [0, 1] to the event $\{(\text{punish A}, \{\text{punish A, punish B}\})\}$. This gives rise to an extension of μ onto the class of all subsets of X; let $\bar{\mu}$ denote this extension. Finally, she plays the pure strategy "punish A" with probability $\bar{\mu}(f^{-1}(\{\text{punish A}\})) = \bar{\mu}(\{(\text{punish A}, \{\text{punish A}\}), (\text{punish A}, \{\text{punish A, punish B}\})\}) = \bar{\mu}(\{(\text{punish A}, \{\text{punish A, punish B}\})\})$. In words, player C will play the mixed strategy defined by $\bar{\mu}$, she plays the pure strategies "punish A" and "punish B" with probabilities $\bar{\mu}(f^{-1}(\{\text{punish A}\}))$ and $\bar{\mu}(f^{-1}(\{\text{punish B}\}))$ respectively.

Notice that in this case it is enough to apply one round in Method 2, there is no need for further rounds.

		Player 2		
		Rock	Scissors	Paper
Player 1	Rock	0,0	2,-1	-1,1
	Scissors	-1,1	0,0	1,-1
	Paper	1,-1	-1,1	0,0

Fig. 2. Modified Rock Scissors Paper.

Next we revisit the modified Rock Scissors Paper game from Riedel and Sass (2014). We have chosen this example to illustrate a case where more than one round in Method 2 is needed. Particularly, as we will see, a player has to apply three rounds in Method 2.

Example 5. Consider the modified Rock Scissors Paper game in Fig. 2.

Riedel and Sass (2014) give a non-trivial equilibrium of the modified Rock Scissors Paper game (Proposition 6). This equilibrium is the following: (ν_1, ν_2), where

- $$\nu_1(E) = \begin{cases} \frac{1}{3} & \text{if } E = \{\text{Rock}\}, \\ \frac{1}{3} & \text{if } E = \{\text{Scissors,Paper}\}, \\ \frac{1}{3} & \text{if } E = \{\text{Scissors}\}, \\ \frac{2}{3} & \text{if } E = \{\text{Rock,Paper}\}, \\ 0 & \text{if } E = \{\text{Paper}\}, \\ \frac{2}{3} & \text{if } E = \{\text{Rock,Scissors}\}, \end{cases}$$

and

- $$\nu_2(E) = \begin{cases} 0 & \text{if } E = \{\text{Rock}\}, \\ \frac{2}{3} & \text{if } E = \{\text{Scissors,Paper}\}, \\ \frac{1}{4} & \text{if } E = \{\text{Scissors}\}, \\ \frac{2}{3} & \text{if } E = \{\text{Rock,Paper}\}, \\ \frac{5}{12} & \text{if } E = \{\text{Paper}\}, \\ \frac{1}{4} & \text{if } E = \{\text{Rock,Scissors}\}. \end{cases}$$

In words

- Player 1 plays Rock at least with probability 1/3 but not with probability higher than 2/3, plays Paper with probability less than 1/3 and plays Scissors with probability exactly 1/3.
- Player 2 plays Rock at most with probability 1/3, plays Scissors at least with probability 1/4 but not with probability higher than 1/3, and plays Paper at least with probability 5/12 but not with probability higher than 3/4.

Let us see how player 2 can play the ambiguous strategy ν_2. Notice that $\nu_2 = \frac{1}{4}u_{\{\text{Scissors}\}} + \frac{5}{12}u_{\{\text{Paper}\}} + \frac{1}{4}u_{\{\text{Rock,Paper}\}} + \frac{1}{12}u_{\{\text{Rock,Scissors,Paper}\}}$.

First player 2 can take the probability space $(X, \mathcal{M}, \mu)$, where

- $X = \{(\text{Rock}, \{\text{Rock}\}), (\text{Scissors}, \{\text{Scissors}\}), (\text{Paper}, \{\text{Paper}\}), (\text{Rock}, \{\text{Rock,Scissors}\}), (\text{Scissors}, \{\text{Rock,Scissors}\}), (\text{Rock}, \{\text{Rock,Paper}\}), (\text{Paper}, \{\text{Rock,Paper}\}), (\text{Scissors}, \{\text{Scissors,Paper}\}), (\text{Paper}, \{\text{Scissors, Paper}\}), (\text{Rock}, \{\text{Rock,Scissors,Paper}\}), (\text{Scissors}, \{\text{Rock,Scissors,Paper}\}), (\text{Paper}, \{\text{Rock,Scissors,Paper}\})\}$.
- The field $\mathcal{M}$ is generated by the partition $\{\{(\text{Rock}, \{\text{Rock}\})\}, \{(\text{Scissors}, \{\text{Scissors}\})\}, \{(\text{Paper}, \{\text{Paper}\})\}, \{(\text{Rock}, \{\text{Rock,Scissors}\}), (\text{Scissors}, \{\text{Rock,Scissors}\})\}, \{(\text{Rock}, \{\text{Rock, Paper}\}), (\text{Paper}, \{\text{Rock,Paper}\})\}, \{(\text{Scissors}, \{\text{Scissors,Paper}\}), (\text{Paper}, \{\text{Scissors,Paper}\})\}, \{(\text{Rock}, \{\text{Rock, Scissors,Paper}\}), (\text{Scissors}, \{\text{Rock,Scissors,Paper}\}), (\text{Paper}, \{\text{Rock,Scissors, Paper}\})\}$, and let
- $$\mu(E) = \begin{cases} 0 & \text{if } E = \{(\text{Rock}, \{\text{Rock}\})\}, \\ \frac{1}{4} & \text{if } E = \{(\text{Scissors}, \{\text{Scissors}\})\}, \\ \frac{5}{12} & \text{if } E = \{(\text{Paper}, \{\text{Paper}\})\}, \\ 0 & \text{if } E = \{(\text{Rock}, \{\text{Rock,Scissors}\}), (\text{Scissors}, \{\text{Rock,Scissors}\})\}, \\ \frac{1}{4} & \text{if } E = \{(\text{Rock}, \{\text{Rock,Paper}\}), (\text{Paper}, \{\text{Rock,Paper}\})\}, \\ 0 & \text{if } E = \{(\text{Scissors}, \{\text{Scissors,Paper}\}), (\text{Paper}, \{\text{Scissors,Paper}\})\}, \\ \frac{1}{12} & \text{if } E = \{(\text{Rock}, \{\text{Rock,Scissors,Paper}\}), (\text{Scissors}, \{\text{Rock,Scissors,} \\ & \text{Paper}\}), (\text{Paper}, \{\text{Rock,Scissors,Paper}\})\}. \end{cases}$$

Next she considers the inner measure μ_* of μ, hence (we give μ_* only for the singleton sets)

$$\mu_*(E) = \begin{cases} \frac{1}{4} & \text{if } E = \{(\text{Scissors}, \{\text{Scissors}\})\}, \\ \frac{5}{12} & \text{if } E = \{(\text{Paper}, \{\text{Paper}\})\}, \\ 0 & \text{if } |E| = 1 \text{ and } E \neq \{(\text{Scissors}, \{\text{Scissors}\})\} \text{ and } E \neq \{(\text{Paper}, \\ & \{\text{Paper}\})\}. \end{cases}$$

Notice that $\nu = \mu_* \circ f^{-1}$, where $f\colon X \to \Omega$ is defined as $f(x) = x|_\Omega$, for all $x \in X$, where $\Omega = \{\text{Rock,Scissors,Paper}\}$.

Moreover, each of the sets $\{(\text{Rock}, \{\text{Rock,Scissors}\})\}$, $\{(\text{Scissors}, \{\text{Rock, Scissors}\})\}$, $\{(\text{Scissors}, \{\text{Scissors,Paper}\})\}$, $\{(\text{Paper}, \{\text{Scissors,Paper}\})\}$ is included by an event with μ-probability 0. Therefore, the probability of these events is 0 in any extension, hence w.l.o.g. we can assume that μ is defined on the field generated by the partition

$$\begin{aligned} \Pi = \{&\{(\text{Rock}, \{\text{Rock}\})\}, \{(\text{Scissors}, \{\text{Scissors}\})\}, \{(\text{Paper}, \{\text{Paper}\})\}, \\ &\{(\text{Rock}, \{\text{Rock,Scissors}\})\}, \{(\text{Scissors}, \{\text{Rock,Scissors}\})\}, \{(\text{Rock}, \\ &\{\text{Rock,Paper}\}), (\text{Paper}, \{\text{Rock,Paper}\})\}, \{(\text{Scissors}, \{\text{Scissors,Paper}\})\}, \\ &\{(\text{Paper}, \{\text{Scissors,Paper}\})\}, \{(\text{Rock}, \{\text{Rock, Scissors,Paper}\}), \\ &(\text{Scissors}, \{\text{Rock,Scissors,Paper}\}), (\text{Paper}, \{\text{Rock,Scissors,Paper}\})\}. \end{aligned}$$

Then, player 2 applies Stecher et al. (2011)'s method to assign a number from $[0, 1/4] = [\mu_*(\{(\text{Rock}, \{\text{Rock,Paper}\})\}), 1 - \mu_*(\{(\text{Rock}, \{\text{Rock,Paper}\})\}^{\complement})]$ to the event $\{(\text{Rock}, \{\text{Rock, Paper}\})\}$. This gives rise to an extension of μ onto the field generated by the partition

$$\Pi' = (\Pi \setminus \{(\text{Rock}, \{\text{Rock,Paper}\}), (\text{Paper}, \{\text{Rock,Paper}\})\})$$
$$\cup\{(\text{Rock}, \{\text{Rock,Paper}\})\} \cup \{(\text{Paper}, \{\text{Rock,Paper}\})\};$$

let μ' denote this extension.

Then, she applies Stecher et al. (2011)'s method again to assign a number from $[0, 1/12] = [\mu'_*(\{(\text{Rock}, \{\text{Rock,Scissors,Paper}\})\}), 1 - \mu'_*(\{(\text{Rock}, \{\text{Rock, Scissors,Paper}\})\}^{\complement})]$ to the event $\{(\text{Rock}, \{\text{Rock,Scissors,Paper}\})\}$. This gives rise to an extension of μ' onto the field generated by the partition

$$\hat{\Pi} = \big(\Pi' \setminus \{(\text{Rock}, \{\text{Rock,Scissors,Paper}\}), (\text{Scissors}, \{\text{Rock,Scissors,Paper}\}),$$
$$(\text{Paper}, \{\text{Rock,Scissors,Paper}\})\}) \cup \{(\text{Rock}, \{\text{Rock,Scissors,Paper}\})\}$$
$$\cup\{(\text{Scissors}, \{\text{Rock,Scissors,Paper}\}), (\text{Paper}, \{\text{Rock,Scissors,Paper}\})\};$$

let $\hat{\mu}$ denote this extension.

Then, she applies Stecher et al. (2011)'s method again to assign a number from $[0, 1/12 - \hat{\mu}(\{(\text{Rock}, \{\text{Rock,Scissors,Paper}\})\})]$ to the event $\{(\text{Scissor}, \{\text{Rock,Scissors,Paper}\})\}$. This gives rise to an extension of $\hat{\mu}$ onto the class of all subsets of X; let $\bar{\mu}$ denote this extension.

Finally, she plays the pure strategies "Rock", "Scissors" and "Paper" with probabilities $\bar{\mu}(f^{-1}(\{\text{Rock}\}))$, $\bar{\mu}(f^{-1}(\{\text{Scissors}\}))$ and $\bar{\mu}(f^{-1}(\{\text{Paper}\}))$ respectively.

4. Conclusion

In this paper we introduced a method to make ambiguous strategies. The proposed method applies Stecher et al. (2011)'s procedure in the extension of a probability distribution from a field on a set onto all subsets of the set. Our method draws a prior (an extension) from a set of priors (extensions) given by a belief function (inner measure) in a way that it does not induce a probability distribution over the priors (extensions), that is, the draw is not driven by any probability distribution.

We also consider two games to show how our method works in game theory applications.

Appendix A. The proof of Theorem 1

Proof. A belief function is a grounded, normalized, non-negative, totally monotone set function (for these notions see e.g. Grabisch (2016).

Point *1.*:

μ_* is grounded: Since $\emptyset \in \mathcal{M}$ it holds that $\mu_*(\emptyset) = \mu(\emptyset) = 0$.

μ_* is normalized: Since $X \in \mathcal{M}$ it holds that $\mu_*(X) = \mu(X) = 1$.

μ_* is non-negative: It is the direct corollary of that μ is non-negative.

μ_* is totally monotone: Take $A_1, \ldots, A_n \in \mathcal{P}(X)$ and let $B_1, \ldots, B_n \in \mathcal{M}$ be such that $B_m \subseteq A_m$ and $\mu_*(A_m) = \mu(B_m)$, $m = 1, \ldots, n$. Then

$$\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \mu_* \left(\bigcap_{m \in I} A_m \right) = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \mu \left(\bigcap_{m \in I} B_m \right),$$

moreover,

$$\mu_*(A_1 \cup \ldots \cup A_n) \geq \mu(B_1 \cup \ldots \cup B_n).$$

Therefore,

$$\mu_*(A_1 \cup \ldots \cup A_n) \geq \mu(B_1 \cup \ldots \cup B_n)$$
$$= \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \mu\left(\bigcap_{m \in I} B_m\right) = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \mu_*\left(\bigcap_{m \in I} A_m\right).$$

Point 2. Let $X = \{(\omega, A)) \in \Omega \times \mathcal{A} \colon \omega \in A\}$, and $f \colon X \to \Omega$ be defined as $f(x) = x|_\Omega$; it is clear that f is a surjection. Moreover, let $\mathcal{M}$ be the coarsest field on X which includes the following partition of X: $\{\{x \in X \colon x|_\mathcal{A} = A\},\ A \in \mathcal{A}\}$. Furthermore, for each $A \in \mathcal{A}$ let

$$\mu(\{x \in X \colon x|_\mathcal{A} = A\}) = \alpha_A,$$

where α_A is from $\nu = \sum_{A \in \mathcal{A}} \alpha_A u_A$ (see e.g. Grabisch (2016) Theorem 2.58 p. 79), and u_T is the unanimity game on set T, that is,

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that μ is a probability distribution on $\mathcal{M}$.

Let μ_* denote the inner measure of μ on $\mathcal{P}(X)$. Take a set $A \in \mathcal{A}$. Then it is easy to show that $\{x \in X \colon x|_\mathcal{A} = S\} \subseteq f^{-1}(A)$ if only if $S \subseteq A$. Therefore,

$$\nu(A) = \sum_{T \subseteq A} \alpha_T = \mu_*(f^{-1}(A)). \quad \square$$

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