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Mixed-strategy equilibrium of the symmetric production in advance game: the missing case*

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Abstract

The mixed-strategy equilibrium of the symmetric production-in-advance type capacity-constrained Bertrand-Edgeworth duopoly game has not been derived analytically over the entire range of intermediate capacities in the literature. Tasnádi (2020) constructed a symmetric mixed-strategy equilibrium for the production-in-advance game for a large range of intermediate capacities. In this paper we derive for the missing region a symmetric mixed-strategy equilibrium analytically.

Keywords: Price-quantity games; Bertrand-Edgeworth competition; inventories.

JEL Classification Number: D43, L13.

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1 Introduction

We investigate a homogeneous good duopoly model introduced by Shubik (1955) in which the firms set both their prices and quantities simultaneously. For this game Shubik (1955) already found that it may not have an equilibrium in pure strategies. The existence of a mixed-strategy equilibrium was established by Maskin (1986) who called this game the production-in-advance game in which production takes place before sales are realized. In contrast, in the case of production to order, production takes place after prices are known.

From one point of view it is the most natural case that the firms have the freedom to set their own prices and quantities though not necessarily simultaneously. Markets of perishable goods are usually mentioned as examples of advance production in a market. Spot markets in general can also be regarded as production-in-advance markets. Phillips et al. (2001) emphasized that there are also goods that can be traded both in a production-in-advance and in a production-to-order environment. For example, coal and electricity are sold in both types of environments. Based on Italian industry data Casaburi and Minerva (2011) investigated the endogenous mode of production, that is whether firms choose to produce to order or in advance. They observed that production in advance occurs more frequently in homogenous industries than in differentiated ones, while production to order is more prevalent as product differentiation increases. In an experimental setting production in advance under the assumption of a non-atomistic buyer side has been investigated by Davis (1999), Muren (2000), and Orland and Selten (2016).

In an earlier work Tasnádi (2004) demonstrated for the case of identical capacities and constant unit costs that in equilibrium production-in-advance profits are equal to production-to-order profits, while prices are higher in the former case. Montez and Schutz (2021) considered quantity as an unobservable inventory, hence though in their context the quantity decision precedes the price decision their game is equivalent to the production-in-advance game. Somogyi and Vergote (2020) introduced capacity uncertainty into the model in order to explain the empirical observation that large firms set lower prices. Among others Hirata and Matsumura (2010) analyzed the standard Bertrand price-setting game without capacity constraints.

Turning to the results on the mixed-strategy equilibrium in closed form of the production-in-advance game, Levitan and Shubik (1978) computed the mixed-strategy equilibrium for the case of production in advance under linear demand and unlimited capacities. In the same framework Gertner (1986) determined the mixed-strategy equilibrium under more general conditions.

Montez and Schutz (2021) resolved limitations and corrected flaws of previous works. They calculated the mixed-strategy equilibrium of the production-in-advance game for the case of large capacities in their work on unsold inventories.

Recently, Tasnádi (2020) calculated a symmetric mixed-strategy equilibrium for a large range of intermediate capacities. In this paper we address the missing region of intermediate capacities on which the mixed-strategy equilibrium is far more complex and has to be determined successively in a finite number of steps.

From a broader perspective in basic models of duopoly price and quantity are the most frequently employed strategic variables. Friedman (1988) discussed their roles and the effect of possible orderings of these two variables. Kreps and Scheinkman (1983) gave a game-theoretic foundation to the Cournot game through a two-stage capacity-then-price-setting game. We would like to emphasize that in the last section of their paper they also considered a third separate quantity-setting stage, which can be considered as a capacity-setting stage followed by a production-to-order game. To relate our model to theirs we take capacities as exogenously given, while we merge their second and third stages into a simultaneous-move price-quantity game, which results in the same game as described in the first paragraph.

The remainder of the paper is organized as follows: Section 2 presents the framework, Section 3 determines a symmetric mixed-strategy equilibrium for the missing case, and Section 4 concludes.

2 Preliminaries

This section contains the necessary assumptions, notations, and the required available results in the literature.

Assumption 1. The demand curve $D : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly decreasing on $[0, b]$, identically zero on $[b, \infty)$, continuous at b and twice continuously differentiable on $(0, b)$. Furthermore, the revenue function $pD(p)$ is strictly concave on $[0, b]$.

Let $a = D(0)$ and P be the inverse demand function.

We consider the duopoly model in which both firms set their prices and quantities simultaneously.

Assumption 2. Firms 1 and 2 have identical positive unit costs $c \in (0, b)$ up to the same positive capacity constraint k. Each of them sets its price $p_1, p_2 \in [0, b]$ and production quantity $q_1, q_2 \in [0, k]$.

When referring to firms with A and B, our convention is that $A, B \in \{1,2\}$ and $A \neq B$.

Assumption 3. Incorporating the efficient rationing rule,[1] the demand faced by firm A is given by

$$\Delta_A(p_1, q_1, p_2, q_2) = \begin{cases} D(p_A) & \text{if } p_A < p_B, \\ \frac{q_A}{q_A+q_B} D(p_A) & \text{if } p_A = p_B, \\ (D(p_A) - q_B)^+ & \text{if } p_A > p_B, \end{cases}$$

where, as usual, $f^+(x)$ stands for $\max\{f(x), 0\}$ for an arbitrary function $f : \mathbb{R} \to \mathbb{R}$. The interpretation of Assumption 3 is as follows: the low-price firm faces the entire demand, in case of ties firms split the demand in proportion to the firms' quantity decisions[2] and the high-price firm faces the demand minus the quantity produced by the low-price firm. Then the firms' profits are given by

$$\pi_A((p_1, q_1), (p_2, q_2)) = p_A \min\{\Delta_A(p_1, q_1, p_2, q_2), q_A\} - cq_A$$

for both $A \in \{1,2\}$.

From Dasgupta and Maskin (1986, Theorem 6*) it follows that the symmetric production-in-advance game possesses a symmetric equilibrium in mixed strategies. In the following, a mixed strategy μ_A is a probability measure defined on the σ-algebra of Borel measurable sets on $[0,b] \times [0,k]$, which can be restricted without loss of generality to $S = [c,b] \times [0,k]$. In equilibrium, each firm optimally chooses μ_A conditional on μ_B, $A \neq B$. Such an equilibrium is denoted by (μ_1^*, μ_2^*). A mixed-strategy equilibrium (μ_1^*, μ_2^*) can be calculated by the following two conditions:

$$\pi_1((p_1, q_1), \mu_2^*) \leq \pi_1^*, \qquad \pi_2(\mu_1^*, (p_2, q_2)) \leq \pi_2^* \tag{1}$$

holds true for all $(p_1, q_1), (p_2, q_2) \in S$,[3] and

$$\pi_1((p_1^*, q_1^*), \mu_2^*) = \pi_1^*, \qquad \pi_2(\mu_1^*, (p_2^*, q_2^*)) = \pi_2^* \tag{2}$$

holds true μ_1^*-almost everywhere and μ_2^*-almost everywhere, where π_1^*, π_2^* stand for the equilibrium profits corresponding to (μ_1^*, μ_2^*).

[1] For more details in which markets the efficient rationing rule can be applied or the description of the other frequently applied so-called proportional rationing rule we refer to Vives (1998) and Wolfstetter (2001).

[2] Our analysis remains valid for a large class of tie-breaking rules satisfying that a firm's demand is strictly increasing in its own quantity.

[3] For notational convenience we do not introduce a separate notation for the expected profits that have to be determined in (1) and (2). If a mixed-strategy appears in the argument of the profit function π_A, we mean expected profits.

We define market-clearing price p^* by

$$p^* = \begin{cases} D^{-1}(2k) & \text{if } \ D(0) > 2k \\ 0 & \text{if } \ D(0) \leq 2k. \end{cases}$$

The function $\pi^r(p) = (p-c)(D(p)-k)$ equals a firm's residual profit whenever its opponent sells k and $D(p) \geq k$. Let $\overline{p} = \arg\max_{p\in[c,b]} \pi^r(p)$ and $\overline{\pi} = \pi^r(\overline{p})$. Assumptions 1 and 2 assure that p^* and $\overline{p}$ are well defined. Furthermore, let $\underline{p}$ the price at which a firm is indifferent between selling its entire capacity and maximizing profits on the residual demand curve, i.e. $\underline{p} = c + \overline{\pi}/k$.

For the case of small capacities, i.e. $p^* \geq \overline{p}$, the game has a unique equilibrium in pure strategies in which the firms produce at their capacity limits and set the market-clearing price (e.g. Tasnádi, 2004, Proposition 2). The mixed-strategy equilibrium for the case of large capacities, i.e. $D(c) \leq k$, has been determined recently by Montez and Schutz (2021) in which the firms charge prices above their common unit costs. Recently, we have determined a symmetric mixed-strategy equilibrium on a subregion of intermediate capacities (i.e. $\overline{p} > \max\{p^*, c\}$).

Before recalling our recent proposition, we need to introduce several further notations. Let $F(p) = \mu^*\left(\left[\underline{p}, p\right] \times [0,k]\right)$ denote the cumulative distribution of equilibrium prices. We shall denote by $\widehat{p} = \inf\{p \in [c,b] \mid \mu((p,b] \times [0,k]) = 0\}$ the highest possible price set by a firm when playing an arbitrary strategy μ.

In the symmetric mixed-strategy equilibrium at prices $p \in [c, \overline{p}) \subset [c,b]$ firms set at most one quantity $s(p) \in [0,k]$ by Tasnádi (2004, Proposition 7). At least in that price region the associated quantity is proven to be unique and equals k. Furthermore, for any $p \in [\overline{p}, \widehat{p}] \subset [c,b]$ there is a symmetric mixed-strategy equilibrium in which the firms set at most one quantity $s(p) \in [0,k]$ at price p by Tasnádi (2020, Proposition 2). Note that prices and quantities are chosen simultaneously by the firms, but nevertheless the chosen price-quantity pairs lie on a curve. Therefore, a symmetric mixed-strategy equilibrium can be given by the triple $(\widehat{p}, s, F)$.

Proposition 1 (Tasnádi, 2020, Proposition 2)**.** *Let Assumptions 1-3 hold. If $\overline{p} > \max\{p^*, c\}$, then a symmetric mixed-strategy equilibrium (μ^*, μ^*) of the production-in-advance game is given by the following equilibrium price distribution*

$$F(p) = \begin{cases} 0 & \text{if } 0 \leq p < \underline{p}, \\ \frac{(p-c)k-\overline{\pi}}{p(2k-D(p))} & \text{if } \underline{p} \leq p < \overline{p}, \\ 1 - \frac{c}{p} & \text{if } \overline{p} \leq p < \widehat{p}, \text{ and} \\ 1 & \text{if } \widehat{p} \leq p \leq b \end{cases} \tag{3}$$

and by the 'supply' function $s(p)$ *given by* $s(p) = k$ *for all* $p \in [\underline{p}, \overline{p})$ *and determined by*

$$s(p) = D'(p)\left(\frac{p^2}{c} - p\right) + D(p) + \frac{\overline{\pi}}{c} \tag{4}$$

for all $p \in [\overline{p}, \widehat{p}]$ *if*

$$\widehat{p} \leq P(k), \tag{5}$$

where $\widehat{p}$ *is the unique solution of* $s(r) = D(r)/2$.

To illustrate the region of intermediate capacities covered by Proposition 1 we consider the demand curve $D(p) = 1 - p$. Hence, without loss of generality we can restrict ourselves to $c, k \in [0, 1]$. Figure 1 shows the four different cases we can have. The triangle labelled 'Large' depicts the case of large capacities ($c \geq 1 - k$), the triangle labelled 'Small' ($c \leq 1 - 3k$) depicts the case of small capacities, the shaded area labelled 'Int1' in the middle ($s(1 - k) \leq k/2$) depicts the case of intermediate capacities covered by Proposition 1, and the white area labelled 'Int2' depicts the region of intermediate capacities for which this paper determines an equilibrium in mixed strategies.

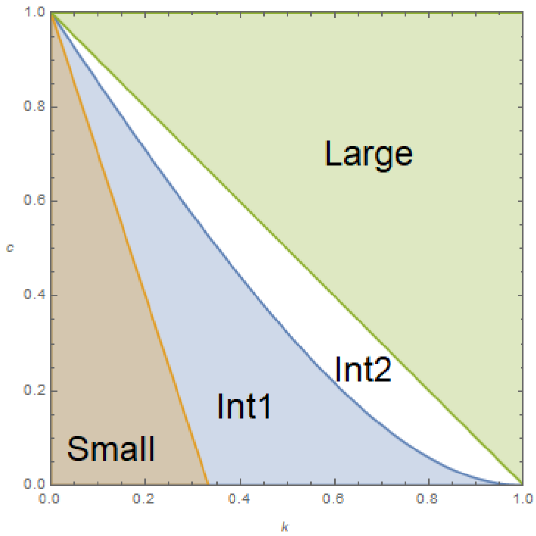

Figure 1: Four different cases

3 Mixed-strategy equilibrium

In this section we calculate a symmetric mixed-strategy equilibrium for the missing range of intermediate capacities on which $s(p)$ is only piecewise strictly decreasing. Though the number of pieces on which $s(p)$ is defined by different expressions is finite the number of pieces cannot be bounded from above as we are getting closer and closer to the large-capacity region.

Nevertheless, the price distribution F specified in Proposition 1 remains still the equilibrium price distribution in the upper range of intermediate capacities. Furthermore, the expression on the right-hand side of (4) still specifies $s(p)$ on the interval $[\overline{p}, P(k)]$ since in this case in the proof of Proposition 1 $D(p) - s(r)$ is non-negative for any $p \in [\overline{p}, P(k)]$ and any $r \in [\overline{p}, p]$. Since s will be defined piecewise on a finite set of disjoint and consecutive intervals, we shall denote by s_1 the expression on the right-hand side of (4). Since we determine s iteratively and at the same time the respective intervals with the boundary points (i.e. prices) too for notational convenience we let $p_0 = \overline{p}$, $p_1 = P(k)$ and $s_0(p) = k$ for any $p \in \left[\underline{p}, \widehat{p}\right]$, which can be regarded as a kind of initialization.[4]

When extending function s to prices above $\overline{p}$ one needs to integrate $D(p) - s_1(r)$ only above prices r on which the integrand is non-negative. To determine the lowest price from which the integration of $D(p) - s_1(r)$ should start for a given p we define $t_1(p) = s_1^{-1}(D(p)) = r$. The strategy for constructing the mixed-strategy equilibrium is to determine the next piece of s denoted by s_2. Then we arrive either to a solution delivering an r^* satisfying $s_2(r^*) = D(r^*)/2$ and $r^* \leq p_2 = P(s_2(p_1))$ or we define $t_2(p) = s_2^{-1}(D(p)) = r$ and continue with determining the next piece of s denoted by s_3. We repeat the whole process until we obtain an r^* satisfying $s_n(r^*) = D(r^*)/2$ and $r^* \leq p_n = P(s_n(p_{n-1}))$, where n stands for the required number of steps. We shall denote by r_i the value of r^* obtained at the ith step, that is $s_i(r_i) = D(r_i)/2$.

The next proposition contains the results of the described procedure and the proof of their correctness.

Proposition 2. *Let Assumptions 1-3 hold. If $\overline{p} > \max\{p^*, c\}$ and $P(k) = p_1 < r_1$, then there exists an $n \in \{1, \ldots\}$ such that a symmetric mixed-strategy equilibrium (μ^*, μ^*) of the production-in-advance game is given by the equilibrium price distribution (3) and by the 'supply' function $s(p)$ given by $s(p) = s_0(p) = k$ for all $p \in \left[\underline{p}, \overline{p}\right]$, given by (4) for all $p \in (\overline{p}, p_1]$, and given by*

$$s_{i+1}(p) = D'(p)\left(\frac{p^2}{t_i(p)} - p + \sum_{j=l(p)}^{i-1}\left(\frac{p^2}{t_j(p)} - \frac{p^2}{p_j}\right)\right) + D(p) + \frac{\overline{\pi}}{c} \qquad (6)$$

[4]From here on the subscripts of p stand for indexing the steps of the iterative process and not for the labeling of firms, which we highlight by using i and j as indexes instead of A and B.

for all $p \in (p_i, \min\{p_{i+1}, r^*\}]$ *and all* $i \in \{1, \dots, n\}$[5] *if*

$$p_1 < r_1, \dots, p_n < r_n, r_{n+1} \le p_{n+1}, \tag{7}$$

where r_i *is the unique solution of* $s_i(r_i) = D(r_i)/2$, $t_i(p) = s_i^{-1}(D(p))$, $p_i = P(s_i(p_{i-1}))$ *for all* $i \in \{1, \dots, n+1\}$, *and* $l(p) \in \{1, \dots, i\}$ *is increasing in* p. *Then* $\widehat{p} = r_{n+1}$. *Furthermore, the numerical sequences* $(s_i(p_{i-1}))_{i=1}^{n+1}$ *and* $(s_i(p_i))_{i=1}^{n}$ *are strictly decreasing,* $s_i(p_i) \le s_{i+1}(p_i)$ *and the functions* $s_i(p)$ *are strictly decreasing in* p *on* $[p_{i-1}, \min\{p_i, r^*\}]$ *for all* $i \in \{1, 2, \dots, n+1\}$.

Before giving a proof of Proposition 2 we illustrate the supply functions given by (6) and shed light on why the equilibrium supply function is discontinuous and has kinks. We depict a possible supply function in Figure 2. We would like to emphasize that there is definitely a discontinuity at p_1, otherwise, we have either a discontinuity or kink at p_i, where the former case occurs if $l(p_i)$ is different for s_i and s_{i+1} in equation (6) and otherwise the latter case occurs. A concrete numerical example for the case of linear demand will be provided after the proof of Proposition 2. First, observe that

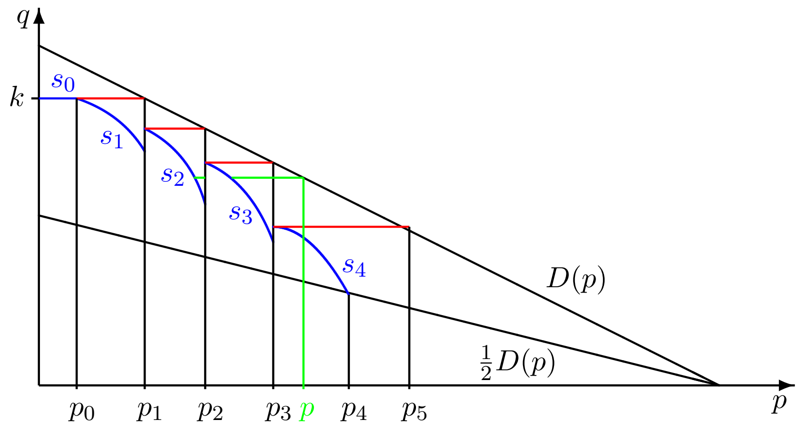

Figure 2: Supply function

if a firm sets price $p \in (p_0, p_1)$ and its opponent sets a price $r \in [\underline{p}, p)$ (i.e. it is the high-price firm), it can always sell a positive amount, and therefore $D(p) - s(r)$ is positive. However, if $p \in (p_1, p_2)$ and $r \in [\underline{p}, p_0)$, its residual demand equals zero, while $D(p) - s(r)$ becomes negative. Second, let us move a bit further to the right and pick a price $p \in (p_3, p_4)$ as indicated with a green p in Figure 2. If the firm's opponent sets a price such that $s_3(r) < D(p)$ and $p \le p_3$, then the firm faces a positive residual demand, but this is also

[5] As usually, if $l(p) > i - 1$ the sum equals zero since there is no summand.

true if the firm's opponent sets a price such that $s_2(r) < D(p)$ and $p \leq p_2$. Therefore, in this case we have $i(p) = 2$. A kink in s arises at a price satisfying $s_3(p_3) = D(p)$.

Now we turn to the proof of Proposition 2.

Proof. Proposition 1 can be considered as the initialization step of our recursive procedure, i.e. the statement of our proposition holds for $n = 1$. Then we assume that we have already obtained the sequence of prices $p_1, \ldots, p_i$, the sequence of supply functions $s_1, \ldots, s_i$, and the sequence of functions $t_1, \ldots, t_i$ in the way as stated in the proposition recursively.

Since s and F are known for all $p \in \left[\underline{p}, p_i\right)$ in what follows we consider only prices such that $p \geq p_i$.[6] When determining the next piece of s, we shall denote by $r^*_{i+1} \in [p_i, b]$ the price at which $s_{i+1}(r^*_{i+1}) = D(r^*_{i+1})/2$ and assume that such a price exists uniquely.[7]

Given that we are looking for a symmetric equilibrium we denote the rival firm 2's strategy simply by μ (and thus omitting its subscript). Then firm 1's profit equals

$$\begin{aligned}\pi_1\left((p,q),\mu\right) &= pq\left(1-F(p)\right) + p\int_{p_i}^{p} \min\left\{(D(p)-s_{i+1}(r))^+, q\right\} dF(r) \\ &\quad + \sum_{j=l(p)}^{i} p\int_{t_j(p)}^{p_j} \min\left\{D(p)-s_j(r), q\right\} dF(r) - cq \end{aligned} \tag{8}$$

for any $p \in (p_i, p_{i+1}]$ and any $q \in [0, D(p)]$, where we have already taken into account that $D(p) < s_{i+1}(p) = q$ does not make sense since then the firms produce a superfluous amount for sure and $l(p) \geq 1$ is the smallest index for which $D(p) > s_{l(p)}(p_{l(p)})$. Note that $l(p) \geq 1$ since for any $p \in (p_i, p_{i+1}]$ we have $D(p) < D(p_0) = k$. (8) simplifies to

$$\begin{aligned}\pi_1\left((p,q),\mu\right) &= pq\left(1-F(p)\right) + p\int_{p_i}^{p} \min\left\{D(p)-s_{i+1}(r), q\right\} dF(r) \\ &\quad + \sum_{j=l(p)}^{i} p\int_{t_j(p)}^{p_j} \min\left\{D(p)-s_j(r), q\right\} dF(r) - cq, \end{aligned} \tag{9}$$

[6] We would like to emphasize that in line with the statement of Proposition 2 we are not showing the uniqueness of the symmetric mixed-strategy equilibrium. Nevertheless, we have to deal with the successive construction of the supply function of the symmetric mixed-strategy equilibrium. However, we do not derive the cumulative distribution function F given in Proposition 2, we just verify its correctness.

[7] We will verify in the proof that the s_{i+1} given by (6) is continuous and strictly decreasing on $p \in [p_i, p_{i+1}]$ and that r^*_{i+1} is uniquely determined by the properties of D and s_{i+1}.

where we could drop the non-negativity operation in the first integral of (8) because we will speak only about the next piece of a solution if $p \leq p_{i+1}$ and $p \leq r^*_{i+1}$. In addition, if $i = n$, then (7) will hold.

Since the equilibrium price distribution is given by $F(p) = 1 - c/p$ on $(\overline{p}, \widehat{p})$ (9) takes the following form

$$\begin{aligned}
\pi_1\left((p,q),\mu\right) &= pq\frac{c}{p} + p\int_{p_i}^{p} \min\left\{D(p) - s_{i+1}(r), q\right\} dF(r) \\
&\quad + \sum_{j=l(p)}^{i} p\int_{t_j(p)}^{p_j} \min\left\{D(p) - s_j(r), q\right\} dF(r) - cq \\
&= p\int_{p_i}^{p} \min\left\{D(p) - s_{i+1}(r), q\right\} dF(r) \\
&\quad + \sum_{j=l(p)}^{i} p\int_{t_j(p)}^{p_j} \min\left\{D(p) - s_j(r), q\right\} dF(r) \qquad (10)
\end{aligned}$$

from which we can see that (10) is strictly increasing in q on $\left[0, \max_{j=l(p),l(p)+1,\ldots,i+1} D(p) - s_j(p)\right]$ and constant on $\left[\max_{j=l(p),l(p)+1,\ldots,i+1} D(p) - s_j(p), D(p)\right]$ since $F(p) = 1 - c/p$, and therefore it follows that we can derive s_{i+1} on the respective interval by solving $\pi_1\left((p,q),\mu\right) =$

$$\begin{aligned}
\overline{\pi} &= p\int_{p_i}^{p} \left(D(p) - s_{i+1}(r)\right)\frac{c}{r^2}dr \\
&\quad + \sum_{j=l(p)}^{i} p\int_{t_j(p)}^{p_j} \left(D(p) - s_j(r)\right)\frac{c}{r^2}dr \\
&= pD(p)\left(\frac{c}{t_i(p)} - \frac{c}{p} + \sum_{j=l(p)}^{i-1}\left(\frac{c}{t_j(p)} - \frac{c}{p_j}\right)\right) \\
&\quad -p\sum_{j=l(p)}^{i}\left(S_j(p_j) - S_j(t_j(p))\right) - p\int_{p_i}^{p} s_{i+1}(r)\frac{c}{r^2}dr, \qquad (11)
\end{aligned}$$

where

$$S_j(p) = \int_{p_{j-1}}^{p} s_j(r)\frac{c}{r^2}dr \qquad (12)$$

for any $p \in [p_{j-1}, p_j)$. By simple rearrangements we get (13)

$$\begin{aligned} S_{i+1}(p) &= D(p)\left(\frac{c}{t_i(p)} - \frac{c}{p} + \sum_{j=l(p)}^{i-1}\left(\frac{c}{t_j(p)} - \frac{c}{p_j}\right)\right) \\ &\quad - \sum_{j=l(p)}^{i}\left(S_j(p_j) - S_j(t_j(p))\right) - \frac{\overline{\pi}}{p} \end{aligned} \tag{13}$$

from which by differentiation we obtain

$$\begin{aligned} S'_{i+1}(p) &= D'(p)\left(\frac{c}{t_i(p)} - \frac{c}{p} + \sum_{j=l(p)}^{i-1}\left(\frac{c}{t_j(p)} - \frac{c}{p_j}\right)\right) \\ &\quad + D(p)\left(-\frac{ct'_i(p)}{t_i^2(p)} + \frac{c}{p^2} + \sum_{j=l(p)}^{i-1} - \frac{ct'_j(p)}{t_j^2(p)}\right) \\ &\quad + \sum_{j=l(p)}^{i}\left(s_j(t_j(p))\frac{c}{t_j^2(p)}t'_j(p)\right) + \frac{\overline{\pi}}{p^2} \\ &= D'(p)\left(\frac{c}{t_i(p)} - \frac{c}{p} + \sum_{j=l(p)}^{i-1}\left(\frac{c}{t_j(p)} - \frac{c}{p_j}\right)\right) \\ &\quad + D(p)\left(-\frac{ct'_i(p)}{t_i^2(p)} + \frac{c}{p^2} + \sum_{j=l(p)}^{i-1} - \frac{ct'_j(p)}{t_j^2(p)}\right) \\ &\quad + \sum_{j=l(p)}^{i} D(p)\frac{c}{t_j^2(p)}t'_j(p) + \frac{\overline{\pi}}{p^2} \\ &= D'(p)\left(\frac{c}{t_i(p)} - \frac{c}{p} + \sum_{j=l(p)}^{i-1}\left(\frac{c}{t_j(p)} - \frac{c}{p_j}\right)\right) \\ &\quad + D(p)\frac{c}{p^2} + \frac{\overline{\pi}}{p^2}, \end{aligned} \tag{14}$$

where the fact that $l(p)$ is an increasing step function of p implies that S_{i+1} in not differentiable at at most i points. Since F does not have an atom at these points the value of s can be set arbitrarily there. Rearranging (14), we get

$$s_{i+1}(p) = D'(p)\left(\frac{p^2}{t_i(p)} - p + \sum_{j=l(p)}^{i-1}\left(\frac{p^2}{t_j(p)} - \frac{p^2}{p_j}\right)\right) + D(p) + \frac{\overline{\pi}}{c} \tag{15}$$

It can be verified that $s'_{i+1}(p) < D'(p)$ holds for prices higher than p_i.

The process of constructing the next piece of s has to be repeated if $p_{i+1} < r^*_{i+1}$. After a finite number of steps, we have to arrive at an n such that $r^*_{n+1} \leq p_{n+1}$ since equilibrium profits are positive. Clearly, both S_{n+1} and s_{n+1} can be extended through equations (13) and (15) for prices higher than r^*_{n+1}, respectively, where for $p \geq r^*_{n+1}$ equation (11) takes the following form

$$\begin{aligned} \overline{\pi} &= p\int_{p_n}^{p} s_{n+1}(r)\frac{c}{r^2}dr \\ &\quad + \sum_{j=l(p)}^{n} p\int_{t_j(p)}^{p_j} \left(D(p) - s_j(r)\right)\frac{c}{r^2}dr \\ &= pD(p)\left(1 - \frac{c}{r^*} + \sum_{j=l(p)}^{n-1}\left(\frac{c}{t_j(p)} - \frac{c}{p_j}\right)\right) \\ &\quad - p\sum_{j=l(p)}^{n}\left(S_j(p_j) - S_j(t_j(p))\right) - p\int_{p_i}^{p} s_{n+1}(r)\frac{c}{r^2}dr, \qquad (16) \end{aligned}$$

since $s_{n+1}(p) < D(p) - s_{n+1}(p)$ for any $p > r^*_{n+1}$.

For any $p \geq r^*_{n+1}$ let

$$Q(p) = \int_{r^*_{n+1}}^{p} s_{n+1}(r)\frac{c}{r^2}dr. \qquad (17)$$

Then we have

$$Q(r^*_{n+1}) = 0 \text{ and } Q'(p) = s_{n+1}(p)\frac{c}{p^2} \qquad (18)$$

for any $p \in \left[r^*_{n+1}, r'\right)$, where r' is uniquely defined by the implicit equation $s(r') = D(r') - k$. Clearly, setting prices above r' does no make sense, since playing these pure strategies against mixed-strategy $\mu_{s,F}$ will result in less profits than pure-strategy $(\overline{p}, D(\overline{p}) - k)$. From (16) we get

$$\begin{aligned} Q(p) &= D(p)\left(1 - \frac{c}{r^*_{n+1}} + \sum_{j=l(p)}^{n-1}\left(\frac{c}{t_j(p)} - \frac{c}{p_j}\right)\right) \\ &\quad - \sum_{j=l(p)}^{n}\left(S_j(p_j) - S_j(t_j(p))\right) - \frac{\overline{\pi}}{p} \qquad (19) \end{aligned}$$

for any $p \in \left[r^*_{n+1}, r',\right)$ from which by differentiation we obtain Q' and finally by simple rearrangements $s_{n+1}(p)$. With a slight abuse of notation we will still denote the obtained function by $s_{n+1}(p)$ on $p \in \left(r^*_{n+1}, r'\right)$ though, as it will turn out, the firms will not produce at prices above r^*_{n+1}. These extensions will be helpful for us in the price interval $[r^*_{n+1}, r']$.

Now we will verify that having an atom at price r^*_{n+1} of mass $c/r^*_{n+1} = 1 - F(r^*_{n+1})$ completes a symmetric mixed-strategy equilibrium. Assume that firm 2 plays the same mixed strategy. Then we already know that for any $p \in \left[\underline{p}, r^*_{n+1}\right)$ producing an amount of $q = s(p)$ results in $\overline{\pi}$ profit. Furthermore, for any $p \in \left[\underline{p}, r^*_{n+1}\right)$ and any quantity $[D(p) - s(p), k]$ profits equal $\overline{\pi}$, while they are strictly less for quantities less than $D(p) - s(p)$ by (10).

We claim that in the derived symmetric mixed-strategy equilibrium firms produce at price r^*_{n+1} an amount of $s(r^*_{n+1}) = D(r^*_{n+1})/2$. Suppose that they would produce more than $D(r^*_{n+1})/2$. Then there will be superfluous production at price r^*_{n+1}, and therefore by the continuity of profits for prices below r^*_{n+1} profits at price r^*_{n+1} would be less than at prices $r^*_{n+1} - \varepsilon$ if ε is sufficiently small. Suppose that they would produce an amount of q^* less than $D(r^*_{n+1})/2$. Then $\pi_1\left((p,q), \mu_{s,F}\right)$ is continuous at (r^*_{n+1}, q^*), and therefore $\pi_1\left((r^*_{n+1}, q^*), \mu_{s,F}\right) < \overline{\pi}$; a contradiction. Thus, we must have indeed $s(r^*_{n+1}) = D(r^*_{n+1})/2$. By the left continuity at price r^*_{n+1} it follows that $\pi_1\left((r^*_{n+1}, D(r^*_{n+1})/2), \mu_{s,F}\right) = \overline{\pi}$.

To verify that the triple $(\widehat{p}, s, F)$ specified in the previous paragraphs specifies a strategy of a symmetric mixed-equilibrium it remains to be shown that prices above r^*_{n+1} combined with any quantity $q \in [0, k]$ result in less profits than $\overline{\pi}$.

The profit function of firm 1 in response to firm 2 playing the mixed strategy associated with $(\widehat{p}, s, F)$ for prices $p \geq r^*_{n+1}$ equals

$$\begin{aligned}
\pi_1\left((p,q), \mu_{s,F}\right) &= p \min\left\{D(p) - \frac{D(r^*_{n+1})}{2}, q\right\} \frac{c}{r^*_{n+1}} \\
&\quad + p \int_{p_n}^{r^*_{n+1}} \left(D(p) - s_{n+1}(r)\right) \frac{c}{r^2} dr \\
&\quad + \sum_{j=l(p)}^{n} p \int_{t_j(p)}^{p_j} \left(D(p) - s_j(r)\right) \frac{c}{r^2} dr - cq, \qquad (20)
\end{aligned}$$

from which we get

$$\frac{\partial \pi_1}{\partial q}\left((p,q), \mu\right) = \begin{cases} -c & \text{if } D(p) - \frac{D(r^*_{n+1})}{2} < q, \\ p\frac{c}{r^*_{n+1}} - c & \text{if } D(p) - \frac{D(r^*_{n+1})}{2} > q \geq D(p) - s(p) \end{cases} \qquad (21)$$

for any $p > \widehat{p} = r^*_{n+1}$. Since $pc/r^*_{n+1} - c > 0$ we get that quantity $q = D(p) - \frac{D(r^*_{n+1})}{2}$ results in the highest profit in (20) for any price $p > \widehat{p} = r^*_{n+1}$.

Hence, we define the profit function of firm 1 at the best quantities for prices $p \geq r^*_{n+1}$ by

$$\begin{aligned}\pi^*(p) &= p\left(D(p) - \frac{D(r^*_{n+1})}{2}\right)\frac{c}{r^*_{n+1}} \\ &+p\int_{p_n}^{r^*_{n+1}}\left(D(p) - s_{n+1}(r)\right)\frac{c}{r^2}dr \\ &+\sum_{j=l(p)}^{n} p\int_{t_j(p)}^{p_j}\left(D(p) - s_j(r)\right)\frac{c}{r^2}dr \\ &-c\left(D(p) - \frac{D(r^*_{n+1})}{2}\right) \end{aligned} \tag{22}$$

It can be verified that $\pi^*(p)$ is strictly concave, and it would be straightforward to check that the derivative $\pi^*(p)$ is non-positive at r^*_{n+1}, which unfortunately does not result in a manageable inequality. Therefore, we consider the equality in (16) defining s and let us denote by

$$\begin{aligned}\pi^s(p) &= p\int_{r^*_{n+1}}^{p} s(r)\frac{c}{r^2}dr + p\int_{p_n}^{r^*_{n+1}}\left(D(p) - s_{n+1}(r)\right)\frac{c}{r^2}dr \\ &+\sum_{j=l(p)}^{n} p\int_{t_j(p)}^{p_j}\left(D(p) - s_j(r)\right)\frac{c}{r^2}dr = \overline{\pi} \end{aligned} \tag{23}$$

for prices $p \in \left[r^*_{n+1}, r'\right]$. Clearly, $d\pi^s(p)/dp = 0$ for any $p \in \left[r^*_{n+1}, r'\right]$ by the definition of s, which we will utilize by considering $\Delta(p) = \pi^*(p) - \pi^s(p) =$

$$\begin{aligned} &= p\left(D(p) - \frac{D(r^*_{n+1})}{2}\right)\frac{c}{r^*_{n+1}} - c\left(D(p) - \frac{D(r^*_{n+1})}{2}\right) - p\int_{r^*_{n+1}}^{p} s(r)\frac{c}{r^2}dr \\ &= \left(D(p) - \frac{D(r^*_{n+1})}{2}\right)\left(p\frac{c}{r^*_{n+1}} - c\right) - p\int_{r^*_{n+1}}^{p} s(r)\frac{c}{r^2}dr. \end{aligned} \tag{24}$$

Then

$$\begin{aligned}\Delta'(p) &= D'(p)\left(p\frac{c}{r^*_{n+1}} - c\right) + \left(D(p) - \frac{D(r^*_{n+1})}{2}\right)\frac{c}{r^*_{n+1}} - \\ &\int_{r^*_{n+1}}^{p} s(r)\frac{c}{r^2}dr - ps(p)\frac{c}{p^2}. \end{aligned} \tag{25}$$

By substituting r^*_{n+1} for p in (25) and taking $s(r^*_{n+1}) = D(r^*_{n+1})/2$ into consideration we get $\Delta'(r^*_{n+1}) = 0$, which implies $d\pi^*(p)/dp = 0$, which completes the proof. $\square$

Now we consider a numerical example and provide its numerical solution based on Proposition 2.

Example 1. Let $D(p) = 1 - p$, $c = 0.38$ and $k = 0.46$.

It can be verified that the cost and capacity pair given in Example 1 is very close to the Int1 region, but within the Int2 region shown in Figure 1. Since the derivation of the cumulative distribution function is straightforward we only present the supply function $s(p)$ in Figure 3, which is drawn in green and has four pieces s_0, s_1, s_2 and s_3. We can see that the termination condition is satisfied at $r^* = r_3$ since there s_3 crosses $D(p)/2$ left to p_3. r^* determines also the position of the atom of the cumulative distribution function F of the symmetric mixed-strategy equilibrium prices. We can see that at p_2 the

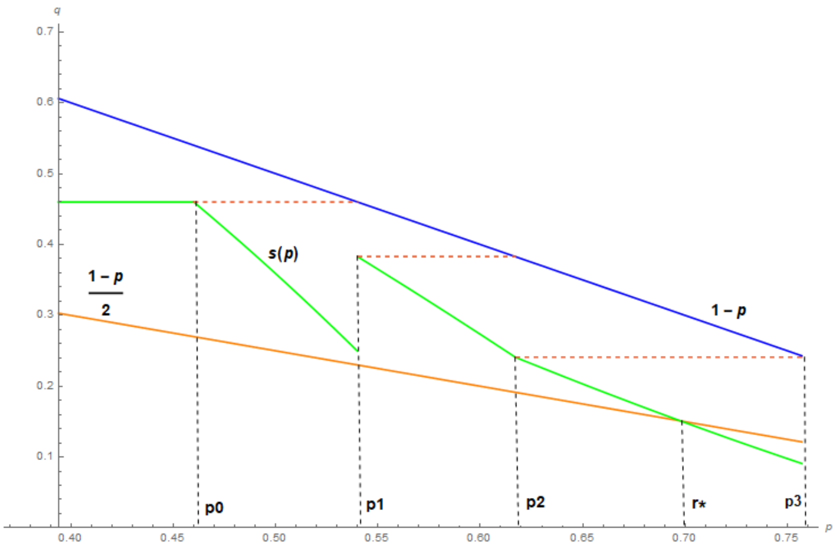

Figure 3: Four different cases

supply curve $s(p)$ just has a kink and no discontinuity.

It is worthwhile to note that in the case of linear demand there is no symmetric mixed-strategy equilibrium just requiring two steps. Though s_1, s_2 and s_3 look linear in Figure 3 they are highly nonlinear. To determine s_3, we need to find the appropriate root of a polynomial of degree 4. Thus, considering an example requiring an additional step seems to be intractable since the degree of the polynomial to be solved could be only approximated numerically and we would even need to determe t_3, which is an inverse function of s_3. For the same reasons we did not draw an extended version of Figure 1 containing the area on which in case of linear demand we would have an equilibrium in three steps.

4 Concluding remarks

In this paper we have derived analytically a symmetric mixed-strategy equilibrium of the production-in-advance game for the missing part of intermediate capacities. Though the cumulative distribution function of prices remains simple the construction of the supply function required a recursive procedure, which resulted in only a piecewise continuous supply function with kinks, where kinks emerge in case of $l(p) \leq i$ in (6).

From an economic point of view the discontinuities in s imply that certain unsold amounts are more likely than others. This may have implications on optimal store sizes or disposal units, but requires a richer model and further analysis.

It is straightforward to see that the closer we are coming to the large capacity region the lowest price in the support of the equilibrium price distribution tends to c, the equilibrium profits tend to zero, r^* to b and the price distribution tends the Montez and Schutz (2021) equilibrium price distribution in distribution. Furthermore, it can be verified that $s(p)$ approximates $D(p)$, and therefore the solution approaches to the solution obtained by Montez and Schutz (2021) for the case of large capacities.

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