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Which Social Choice Rule is More Dictatorial?

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Abstract *Social choice rules (SCRs) aggregate individual preferences to social preferences. By Arrow's (1951) impossibility theorem there does not exist a non-dictatorial SCR satisfying three desirable properties. Considering this negative axiomatic result, in this paper we determine distances of SCRs from the dictatorial rules to rank common SCRs. In particular, we apply the Kendall τ, the Spearman rank correlation and the Spearman footrule metrics. We find that from the investigated SCRs the Borda, the Copeland and the Kemény-Young SCRs stand out. Furthermore, we show that anonymous SCRs approach the constant rule when the number of alternatives is fixed and the number of voters tends to infinity.*

Keywords: Simulation, Asymptotic behavior, Dictatorship, Kendall τ, Spearman rank correlation, Spearman footrule.
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1 Introduction

The debate between Borda and Condorcet at the end of the 18th century demonstrated well the challenge in choosing a widely accepted voting rule. Arrow (1951) settled the problem by terminating the search for an 'ideal' voting rule or even for generally appealing social choice rules (SCRs). In his famous impossibility theorem he showed that if there are at least three alternatives (or candidates), there does not exist an SCR that fulfills all of the following four natural requirements.

- It is defined on the so-called universal domain, i.e. every combination of ranking alternatives is admitted.

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- It satisfies the Pareto property, i.e. if an alternative is preferred by everybody to another alternative, then the former one should be preferred to the latter one by the SCR.
- It guarantees the independence of irrelevant alternatives, i.e. if in two combinations of preferences each voter individually orders two alternatives in the same way, then the SCR should rank these two alternatives in both combinations of preferences in the same way.
- It is non-dictatorial, i.e. it cannot happen that in all combinations of preferences the choice of a distinguished voter and the SCR are the same.

For social choice functions (SCFs), also called voting rules, which select an alternative instead of a social preference, Gibbard (1973) and Satterthwaite (1975) independently proved an impossibility theorem by showing in case of at least three alternatives that for SCFs, which are onto (i.e. for each alternative there exists a combination of preferences such that the SCF selects it, or put it otherwise the SCF is surjective), the properties of non-manipulability (none of the voters can benefit from misreporting its ranking) and non-dictatorship are incompatible.

Clearly, this was not the end of the story and by relaxing some of the properties researchers could find possibility results. For instance, by restricting the universal domain to the set of single-peaked preferences, Moulin (1981) proved a possibility result. Furthermore, following the axiomatic route many voting rules could be characterized by a set of properties. For instance, the Borda rule (Smith, 1973; Young, 1974; Saari, 2000), the Copeland method (Henriet, 1985) and the Kemény-Young method (Young and Levenglick, 1978) were successfully characterized. These and further axiomatic results highlighted the benefits and shortcomings of certain voting rules, and therefore, moved the problem of selecting directly the 'right voting rule' to the question of which properties are more adequate under certain circumstances.

In this paper we follow an alternative route to the axiomatic one, the 'operations research approach', which strives for selecting voting rules as solutions to appropriately defined distance minimization problems. When treating each voter equally it makes sense to minimize the sum of distances to the dictators, which leads to frequently applied voting rules. The later approach is either equivalent to the definitions of certain rules, like the Kemény-Young method (Kemény, 1959), or results in known voting rules, like the Borda count (Dwork et al., 2002). Other voting rules defined directly as solutions of optimization problems are Slater's and Dogson's rules both using the Kendall τ distance (see Eckert and Klammer, 2011).

There is a quite extensive literature on deriving voting rules as solutions of optimization problems related to desired properties. Farkas and Nitzan (1979) derived the Borda count as the solution of an optimization problem on the set of SCFs by minimizing the distance from the unanimity principle. Taking other metrics, Nitzan (1981) obtained the plurality rule among other rules. However, the obtained voting rules are functions of the respective distance function and principle(s). Lehrer and Nitzan (1985) and Campbell and Nitzan (1986) showed that basically any voting rule can be distance rationalized. The approach of minimizing the distance from a set of profiles with a clear winner such as the unanimous winner, the majority winner, or the Condorcet winner has been developed further by Elkind et al. (2015), Andiga et al. (2014) and Mahajne et al. (2015), and Zwicker (2014) among others. Bednay et al. (2017) offered a 'dual' approach based on distance maximization from the closest dictator in which the dictatorial rules are taken as the benchmark and the distances of SCFs from them are determined. When looking at voting rules being farthest away from the closest dictatorial rule, very undesirable voting rules were obtained, which would lead intuitively to the largest level of dissimilarity. Continuing this approach, Bednay et al. (2019) introduced a non-dictatorship index and compared SCFs based on it. In addition, in Bednay et al. (2022) the limiting behavior of the non-dictatorship index has been determined. Above the set of alternatives there is less structure, and therefore in this paper we consider the richer framework of SCRs, which allows for a set of reasonable distance functions on the set of preferences. However, in contrast to the axiomatic approach we have to choose between distance functions instead of properties.

In this paper, when ordering SCRs, we investigate three prominent distance functions used in operations research, which are meaningful in the social choice context. These distance functions are well-know in mathematics and statistics. Diaconis and Graham (1977) considered them as distances on permutations and in statistics they are used as measures of association for ordinal data (for more on historical notes we refer to Monjardet, 1997). For two of them (the Kendall τ[1] and the Spearman rank correlation rule) the optimal voting rule is known. For the third one (the Spearman footrule) the MedRank algorithm is usually mentioned as its solution,[2] however as it will be clear from our results this is only true on a small set of possible combinations of rankings. Furthermore, we would like to find out how close other voting rules are to the optimal one. It is also interesting to know whether in all our investigated cases the same rules are the second or third closest to the optimal one. This question is also of importance because some rules are difficult to calculate, like the Kemény-Young method, and therefore the best approximating polynomial time rule can serve as a replacement. We find for the three selected distance functions in our paper that the Borda, the Copeland and the Kemény-Young SCRs are the best performing ones, while the plurality rule performs the worst. Burka et al. (2022) obtained a similar ordering of these voting rules through employing neural networks. In addition, we show that for a given number of alternatives when the number of voters tends to infinity all rules tend to the constant rule. Qualitatively, it also implies that for a large number of voters any SCR performs almost equally well. We can interpret this result in favor of the very simple plurality rule, which is the most frequently employed one in elections, while based on its axiomatic properties it is the most widely criticized rule.

[1]According to Knuth (1973) the basic idea behind the Kendall τ distance was already used by Cramer in 1750.

[2]This only holds true if the median ranks of the alternatives form a permutation.

We would like to emphasize that SCRs are used besides voting situations in the collective selection of rankings in many areas. In general, aggregating rankings can be also regarded as combining inputs from multiple sources like in automated decision making, machine learning (e.g. Volkovs and Zemel, 2014) or database middleware (e.g. Masthoff, 2004), or determining the results in sport competitions (e.g. Csató, 2022). The problem also arises in coding theory since the alternatives can be regarded as letters and the rankings as strings, and the distance function can be utilized in error detection (Bortolussi et al., 2012). Nowadays, the problem of aggregating rankings also emerges in the link analysis in networks like the world wide web, which lie at the heart of web search algorithms (Borodin et al., 2005). Therefore, a partially parallel literature emerged, which sometimes uses different names for the procedures known in the theory of voting and more importantly it pursues frequently a different goal so that, for instance, the appropriate choice of a distance can be determined in the framework of a given problem. From another point of view in the social choice context the number of voters is usually significantly larger than the number of candidates, while in the applications in computer science and operations research the number of alternatives is frequently larger than the number of experts (or voters).

The structure of the paper is as follows. Section 2 introduces the basic notations, the employed metrics on the set of SCRs and the indices to measure the degree of dictatorship of SCRs. Section 3 presents the SCRs under study. Section 4 explains the computational scheme, presents and discusses the results obtained by simulations. Section 5 proves the limiting results for the introduced indices. Finally, Section 6 concludes. Additional figures are relegated to the Appendix.

2 The Framework

Let $A = \{a_1, \ldots, a_m\}$ be the set of alternatives, where $m \geq 2$, and $N = \{1, \ldots, n\}$ be the set of voters. We shall denote by $\mathcal{P}$ the set of all linear (or preference) orderings (irreflexive, transitive and total binary relations) on A and by $\mathcal{P}^n$ the set of all preference profiles. If $\succ \in \mathcal{P}^n$ and $i \in N$, then $\succ_i$ is the preference ordering of voter i over A. We write $\succ_{-i}$ if we drop voter i's preference ordering from profile $\succ$. Then $\succ = (\succ_i, \succ_{-i})$. Moreover, let $rk[a, \succ]$ denote the *rank* of alternative a in the ordering $\succ \in \mathcal{P}$ (i.e. $rk[a, \succ] = 1$ if a is the top alternative in the ranking $\succ$, $rk[a, \succ] = 2$ if a is second-best, and so on).

Definition 1. A mapping $f : \mathcal{P}^n \to \mathcal{P}$ that selects the linear ordering is called a *social choice rule*, henceforth, SCR.

As our definition of an SCR does not allow for possible ties, while the formulas defining the well-known voting rules just determine weak orderings, we employ fixed anonymous[3] tie-

[3]The linear ordering selected by an anonymous tie-breaking is invariant to the ordering of voters' preferences.

breaking rules to break ties. A tie-breaking rule $\tau : \mathcal{P}^n \to \mathcal{P}$ maps preference profiles to linear orderings on A, which will be only employed when a formula leaves alternatives indifferent. If there are more tied alternatives at a certain position based on a formula 'almost' specifying an SCR, then the tied alternatives follow the ordering given by the tie-breaking rule. When determining our non-dictatorship indices for the SCRs under study, we employ the fixed (also called the lexicographic) tie-breaking rule $a_1\,\tau(\succ)\,a_2\,\tau(\succ)\ldots\tau(\succ)\,a_m$ for each profile $\succ\in\mathcal{P}^n$, which is anonymous and does not depend on the actual preference profile $\succ\in\mathcal{P}^n$. Henceforth, we will briefly write $a_1\,\tau\,a_2\,\tau\ldots\tau\,a_m$ for the ordering of the alternatives by this tie-breaking rule.

Let $\mathcal{F} = \mathcal{P}^{\mathcal{P}^n}$ be the set of SCRs and $\mathcal{F}^{an} \subset \mathcal{F}$ be the set of anonymous SCRs. The subset of $\mathcal{F}$ consisting of the dictatorial rules will be denoted by $\mathcal{D} = \{D_1, \ldots, D_n\}$, where D_i is the dictatorial rule with voter i as the dictator. We will consider three metrics on $\mathcal{F}$ which for any $F, G \in \mathcal{F}$ are defined by

$$\rho_1(F,G) = \sum_{\succ\in\mathcal{P}^n} \sum_{i=1}^{m} |rk[a_i, F(\succ)] - rk[a_i, G(\succ)]| \tag{2.1}$$

$$\rho_2(F,G) = \sum_{\succ\in\mathcal{P}^n} \sum_{i=1}^{m} (rk[a_i, F(\succ)] - rk[a_i, G(\succ)])^2 \tag{2.2}$$

$$\rho_K(F,G) = \sum_{\succ\in\mathcal{P}^n} \#\{(a_i, a_j) \in A^2 \mid i < j,\ a_i\,F(\succ)\,a_j \text{ and } a_j\,G(\succ)\,a_i\} \tag{2.3}$$

Note that usually these types of metrics are defined for profiles, and thus the outer sum is missing in the above equations when the metrics are defined for preferences or permutations. Since we are interested in the comparison of SCRs we added the outer sums. For two voters of a given profile (that is considering the distance of two linear orderings) the metrics defined by the inner sums of (2.1), (2.2) and (2.3) are known in the literature as the Spearman footrule, the Spearman rank correlation and the Kendall τ (or Kemény) distances. Clearly, other weighting schemes are possible, however here we give equal weight to each profile if we estimate the distances by random sampling as in our simulations in Section 4.

It will be helpful for us that Diaconis and Graham (1977) provide the maximum values for the Kendall τ, the Spearman footrule and the Spearman rank correlation distances between two linear orderings, which equal $(m-1)m/2$, $\lfloor m^2/2 \rfloor$ and $(m^3-m)/3$, respectively. Therefore, in the forthcoming Definition 2 for normalization purposes we let

$$C_K = (m-1)m/2, C_1 = \lfloor m^2/2 \rfloor \text{ and } C_2 = (m^3 - m)/3.$$

In defining our first set of normalized indices we consider the distance to all dictatorial rules by treating each of them equally, which means that we take the sum of the distances to obtain the indices.

Definition 2. The *balancedness indices* (BI_I) are given by

$$BI_I(F) = \frac{1}{n}\frac{\sum_{i\in N}\rho_I(F, D_i)}{(m!)^n C_I},$$

where $I \in \{1, 2, K\}$.

In defining the set of balanced SCRs we try to get close to all dictatorial rules by minimizing the sum of the distances to obtain the respective optimal SCR, where we restrict ourselves in our analysis to anonymous SCRs.

Definition 3. We define the set of (anonymous) *balanced rules* by

$$\mathcal{F}_b^{an} = \arg\min_{F\in\mathcal{F}^{an}} \sum_{i\in N} \rho(F, D_i) = \arg\min_{F\in\mathcal{F}^{an}} BI(F)$$

over the set of anonymous SCRs.

Some of the famous voting rules are defined in this way, like the Kemény-Young method and the Spearman footrule. The former one uses the Kendall-τ distance, which is also called the Kemény distance. Others like the Spearman's rank correlation can be shown to result in the Borda count (see Dwork et al., 2002).[4]

We define our non-dictatorship index (NDI) by taking the distance to the closest dictator.

Definition 4. The *non-dictatorship indices* (NDI_I) are given by

$$NDI_I(F) = \frac{\min_{i\in N}\rho_I(F, D_i)}{(m!)^n C_I},$$

where $I \in \{1, 2, K\}$.

As an alternative approach we specify the set of least dictatorial rules as those ones which are the furthest away from the closest dictatorial rule, which means that we are maximizing the minimum of the distances to the dictatorial rules.

Definition 5. We define the set of (anonymous) *least dictatorial rules* by

$$\mathcal{F}_{ld}^{an} = \arg\max_{F\in\mathcal{F}^{an}} \min_{i\in N} \rho(F, D_i) = \arg\max_{F\in\mathcal{F}^{an}} NDI(F)$$

over the set of anonymous SCRs.

[4]More precisely, they show this for those profiles for which the Borda count determines a linear ordering. As we show in Section 5 this does not pose a severe restriction since for large n the set of profiles on which the Borda count has tied alternatives becomes negligible.

From another point of view the least dictatorial rules are getting far away from each individual, and therefore not surprisingly result in undesirable rules as it will become clear from our simulations in Section 4.

For any anonymous SCR F we have $\rho_I(F, D_i) = \rho_I(F, D_j)$ for any $i, j = 1, \ldots, n$ and any $I \in \{1, 2, K\}$, and therefore it follows that

$$\min_{i \in N} \rho(F, D_i) = \frac{1}{n} \sum_{i=1}^{n} \rho(F, D_i),$$

which in turn implies that for anonymous SCRs the BI_I and NDI_I indices equal each other.

3 Social Choice Rules

We consider the following seven common voting rules, a trivial rule and four derived SCRs. To introduce these SCRs we pick a profile with 5 alternatives and 7 voters given in Table 1. Since

Rank	$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$
1	e	d	a	e	d	c	c
2	a	a	d	b	a	e	e
3	d	b	c	d	e	d	d
4	b	e	b	c	b	b	a
5	c	c	e	a	c	a	b

Table 1: A profile with 5 alternatives and 7 voters

they can be tied alternatives based on the formula partially defining an SCR we use the fixed tie-breaking rule $a \,\tau\, b \,\tau\, c \,\tau\, d \,\tau\, e$ to resolve ties in order to arrive to the linear ordering chosen by the respective SCR.

1. The *plurality rule* counts the number of top positions for each alternative and orders them accordingly. Looking at the profile in Table 1, we see that the resulting ordering of the alternatives are $c \succ d \succ e \succ a \succ b$, where we used the above mentioned tie-breaking rule τ to resolve ties. The SCR PL_τ is the *plurality rule* if for all $(\succ_i)_{i=1}^n \in \mathcal{P}^n$ and all pairs of distinct alternatives

$$\begin{aligned} a\ PL_\tau\left((\succ_i)_{i=1}^n\right)\ b \quad \Leftrightarrow \quad & \#\left\{i \in N \mid rk[a, \succ_i] = 1\right\} > \#\left\{i \in N \mid rk[b, \succ_i] = 1\right\} \text{ or} \\ & \#\left\{i \in N \mid rk[a, \succ_i] = 1\right\} = \#\left\{i \in N \mid rk[b, \succ_i] = 1\right\} \text{ and } a\tau b. \end{aligned}$$

2. The *Borda count*, briefly denoted by BC, orders the alternatives based on the sum of their ranks. In particular, an alternative with a lower sum of ranks is preferred over an alternative with a higher sum of ranks. In case of the profile in Table 1 the sum of ranks of alternatives a,

b, c, d and e are 21, 26, 24, 16 and 18, respectively. Therefore, the social ordering determined by BC is $d \succ e \succ a \succ c \succ b$. The SCR BC_τ is the *Borda count* if for all $(\succ_i)_{i=1}^n \in \mathcal{P}^n$ and all pairs of distinct alternatives a and b we have

$$a\ BC_\tau\left((\succ_i)_{i=1}^n\right)\ b \quad\Leftrightarrow\quad \begin{array}{l} \sum_{i=1}^n rk[a,\succ_i] < \sum_{i=1}^n rk[b,\succ_i] \text{ or} \\ \sum_{i=1}^n rk[a,\succ_i] = \sum_{i=1}^n rk[b,\succ_i] \text{ and } a\tau b. \end{array}$$

3. The *k-approval rule* counts how many times an alternative is ranked among the top k alternatives in a profile and orders them accordingly. We shall denote this rule by kAV. In case of the profile given in Table 1 alternatives a, b, c, d and e are 4, 1, 2, 3 and 4 times top two alternatives, and therefore the respective ranking by $2AV_\tau$ is $a \succ e \succ d \succ c \succ b$. In an analogous way, for the 3-approval rule for the same profile alternatives a, b, c, d and e are 4, 2, 3, 7 and 5 times top three alternatives, which implies the ranking $d \succ e \succ a \succ c \succ b$ by $3AV_\tau$. Formally, the kAV_τ rule is the *k-approval rule* if for all $(\succ_i)_{i=1}^n \in \mathcal{P}^n$ and all pairs of distinct alternatives a and b we have

$$a\ kAV_\tau\left((\succ_i)_{i=1}^n\right)\ b \quad\Leftrightarrow\quad \begin{array}{l} \#\left\{i \in N \mid rk[a,\succ_i] \leq k\right\} > \#\left\{i \in N \mid rk[b,\succ_i] \leq k\right\} \text{ or} \\ \#\left\{i \in N \mid rk[a,\succ_i] \leq k\right\} = \#\left\{i \in N \mid rk[b,\succ_i] \leq k\right\} \text{ and } a\tau b. \end{array}$$

In our computations we will just use the $2AV$ and $3AV$ rules. Note that $1AV$ equals PL.

4. The *Copeland method* carries out for all voters pairwise comparisons of two alternatives, where an alternative beats the other one if it is ranked higher by more voters than the other alternative. In this case the former alternative wins while the other looses their competition. This procedure is carried out for any pair of distinct alternatives. Thereafter, the alternatives are ranked by the Copeland method following the ordering based on their numbers of wins. This is in fact the usual way how round-robin tournaments are organized. Of course, possible ties have to be broken by a tie-breaking rule. Again we consider the profile given in Table 1 and employ the same tie-breaking rule as in case of the previously introduced SCRs. We can see that a beats alternatives b and c, b beats alternative c, c does not beat another alternative, d beats alternatives a, b and c, and finally e beats all alternatives. Therefore, the Copeland method arrives to the linear ordering $e \succ d \succ a \succ b \succ c$. We shall denote by CM the Copeland method, which we define now formally. For a given profile $(\succ_i)_{i=1}^n \in \mathcal{P}^n$ we say that alternative $a \in A$ *beats* alternative $x \in A$ if $\#\{i \in N \mid a \succ_i x\} > \#\{i \in N \mid x \succ_i a\}$, i.e. a wins over x by pairwise comparison. Furthermore, alternative $a \in A$ is *tied with* alternative $x \in A$ if $\#\{i \in N \mid a \succ_i x\} = \#\{i \in N \mid x \succ_i a\}$, which can only happen if n is even. We shall denote by $l[a,(\succ_i)_{i=1}^n]$ the number of alternatives beaten by alternative $a \in A$ plus half of the number

of alternatives a is tied with for a given profile $(\succ_i)_{i=1}^n$. Then an SCR CM_τ is the *Copeland method* if for all $(\succ_i)_{i=1}^n \in \mathcal{P}^n$ and all pairs of distinct alternatives a and b we have

$$\begin{aligned} a\ CM_\tau\left((\succ_i)_{i=1}^n\right)\ b \quad \Leftrightarrow \quad & l[a,(\succ_i)_{i=1}^n] > l[b,(\succ_i)_{i=1}^n] \text{ or} \\ & l[a,(\succ_i)_{i=1}^n] = l[b,(\succ_i)_{i=1}^n] \text{ and } a\tau b. \end{aligned}$$

5. The *Bucklin rule* determines for each alternative $a \in A$ the highest rank h_a that has still to be taken into account so that considering the first h_a ranked alternatives it appears at least $n/2$ times in a given profile. Looking at Table 1, we see that no alternative receives a majority (i.e. 4 votes) when counting only the numbers of top ranked alternatives. Now taking also the second ranked alternatives into consideration we see that both a and e appear four times, where the tie-breaking rule τ gives priority to a. Thus, $h_a = h_e = 2$. Admitting also third ranked alternatives, d appears 6 times in the first three rows and we have $h_c = 3$. If we also take the fourth ranked alternatives into account b and c appear 6 and 4 times, respectively. Hence, $h_b = h_c = 4$. In case of tied values h_b and h_c priority is given to the alternative with the higher number of occurrences and if even these ones are the same the tie breaking rule τ is applied. Then the Bucklin rule gives the ranking $a \succ e \succ d \succ b \succ c$. For each alternative a the rank h_a is determined by the median voter in the increasingly reordered sequence of voters by ranks based on how they rank alternative a. We shall denote the Bucklin rule by BR. Then the SCR BR_τ is the *Bucklin rule* if for all $\succ \in \mathcal{P}^n$ and all pairs of distinct alternatives we have

$$\begin{aligned} & aBR_\tau\,(\succ)\,b \Leftrightarrow \\ & \qquad h_a < h_b \text{ or} \\ & \qquad \left(h_a = h_b \text{ and } \#\left\{i \in N \mid rk[a,\succ_i] \leq h_a\right\} > \#\left\{i \in N \mid rk[b,\succ_i] \leq h_b\right\}\right) \text{ or} \\ & \qquad \left(h_a = h_b \text{ and } \#\left\{i \in N \mid rk[a,\succ_i] \leq h_a\right\} = \#\left\{i \in N \mid rk[b,\succ_i] \leq h_b\right\} \text{ and } a\tau b\right). \end{aligned}$$

6. The Bucklin rule has itself several variants, but there is a related one, which is called the *MedRank rule* in the computer science literature as an abbreviation for the aggregation rule based on the median ranks of the alternatives, henceforth also briefly MR. The modification lies in the fact that only the medium ranks of the alternatives matter, that is only the above defined h_{a_1},...,h_{a_n} matter (see for instance, Dwork et al., 2002). In our framework of SCRs for two distinct alternatives a and b for which $h_a = h_b$ the tie-breaking rules have to be employed immediately without taking into account how many times these two alternatives appear in the top h_a positions of all voters. Therefore, the MedRank algorithm is less decisive than the Bucklin rule and in fact has tied alternatives far more frequently. Dwork et al. (2002) showed that at least for those profiles in which all median ranks of the alternatives are pairwise distinct (that is, they form a permutation), then the ranking obtained by the MedRank rule equals the Spearman footrule optimal ranking.

7. The *Kemény-Young* method, henceforth also KY, selects for a given profile the KY solution by minimizing the sum of Kendall τ distances over the set of voters. For the profile given in Table 1 the brute-force method would require the evaluation of $5! = 120$ linear orderings and there is no polynomial time algorithm for determining an optimal KY ranking. Therefore, we just state for our profile that $e \succ d \succ a \succ b \succ c$ is an optimal KY ranking for which the sum of Kendall τ distances equals $1+3+7+3+2+5+4 = 25$. However, even verifying that this is an optimal solution is not easy. Therefore, we continue with the formal definition of KY. The preference relation chosen by the *Kemény-Young method* for profile $(\succ_1, ..., \succ_n) \in \mathcal{P}^n$ selects a preference relation $\succ^* \in \mathcal{P}$ minimizing

$$\sum_{k=1}^{n} \#\{(a_i, a_j) \in A^2 \mid i < j,\ a_i \succ^* a_j \text{ and } a_j \succ_k a_i\}, \tag{3.4}$$

which is not necessarily unique, however in this case we can pick $\succ^*$ arbitrarily from the set of (3.4) minimizing preference relations.

8. We also include the trivial *constant rule*, denoted by CR, which assigns to each profile the same fixed preference relation. Note that considering the properties in Arrow's impossibility theorem the constant rule is defined on the universal domain, is non-dictatorial and satisfies the independence of irrelevant alternatives, but violates the Pareto property. Formally, let $\succ^* \in \mathcal{P}$ be a fixed linear ordering and we define the *constant rule* by $CR(\succ) = \succ^*$ for all $\succ \in \mathcal{P}^n$.

9. The *reverse plurality* rule, denoted by RPL, assigns to each preference profile essentially the opposite ordering as determined by the plurality rule with the slight exception that for tied alternatives we use the tie-breaking rule τ^{-1} (i.e. $a_m \tau^{-1} a_{m-1} \tau^{-1} \dots \tau^{-1} a_1$) to resolve ties. In an analogous way we define the *reverse Borda count*, the *reverse Copeland method* and the *reverse Kemény-Young* method, which we shall denote by RBC, RCM and RKY, respectively.

We introduce the notion of a Condorcet winner since some of the above introduced SCRs satisfy the Condorcet criterium. Let μ be a majority relation for a given profile $(\succ_i)_{i=1}^n \in \mathcal{P}^n$, then $a \mu x$ if $\#\{i \in N \mid a \succ_i x\} > \#\{i \in N \mid x \succ_i a\}$. The Condorcet winner CW of a profile $(\succ_i)_{i=1}^n$ is an element beating any other alternative based on the majority relation μ (constructed according to the profile), i.e.

$$CW((\succ_i)_{i=1}^n) = \{a \in A \mid \text{for all } x \in A \setminus \{a\} : a \mu x\}\,.$$

An SCR is *Condorcet consistent* or satisfies the Condorcet criterium if it selects the Condorcet winner as its top alternative if such one exists. From the SCRs defined in this section CM and KY are Condorcet consistent.

4 Computation Scheme and Results

For three alternatives and a smaller number of voters (up to 7) we have determined the exact values of the balancedness indices with the Kendall τ distance by the brute-force algorithm. We present these results in Tables 2-4.[5] For a larger number of alternatives and voters we performed the estimation of the indices for 3, 4 and 5 alternatives and up to 100 voters. In our simulations we have generated 2500 random preference profiles, where each profile is chosen with equal probability. The graphs for 3 and 4 alternatives can be found in the Appendix. To be focused in this section we present only the graphs for 5 alternatives. Qualitatively, there are no relevant differences for 3 and 4 alternatives. The exact values in the tables are close to the results obtained by our simulations. Since the speed of convergence is slow it is not obvious that the graphs converge to a common limit. We will show that this is the case in Section 5.

Because of the slow rate of convergence in the figures we show the graphs only up to 50 voters. To each case we present two types of figures: the first type contains the graphs of the BI_I minimizing, the BI_I maximizing, the PL, the RPL and the CR SCRs for the number of voters ranging from 3 to 50 and the second type is a zoomed in version of the first one in which we omit the strange reverse type rules, but include all discussed usual SCRs. In the first type of figures we had to be selective because the graphs of many SCRs lie very close to each other, while the second type of figures contain only the parts ranging from 15 to 50 voters because the graphs are steeply increasing for a smaller number of voters. Regarding the first type of figures, if shown, then the graphs of the BI_I of the usual SCRs would lie between that of the BI_I minimizing SCR and that of the PL, while concerning the reverse type of $SCRs$ they would lie between that of the BI_I maximizing SCR and that of the RPL.

We proceed sequentially and start with the Kendall τ distance. From Table 2 we see that KY has the lowest BI_K values with the exception of $n = 4$, where it is tied with BC and CM. Since for 3 alternatives the $3AV$ rule is identical with the constant rule it is not surprising that they have identical BI_K values. There is also a strong connection between PL and $2AV$ since the former one just singles out the top alternatives, while the latter one the bottom alternatives, which also leads to identical BI_K values. We observe that among the common rules PL and $MedRank$ perform worst. We also find that the reverse kind rules perform worst. They have at the same time highest BI_K and NDI_K indices. The latter indicates that maximizing the NDI_K, that is searching for the rule which is the furthest away from the closest dictatorial rule, results in strange rules. By symmetry RKY has the highest NDI_K.

Figures 1 and 2 show the results of our simulation for the case of 5 alternatives. They confirm the results observed in Table 2. On the vertical axis we can see the values of the indices for the 2500 generated profiles, while on the horizontal axis we can see the number of voters.

[5] Our Python code is downloadable at `http://www.uni-corvinus.hu/~tasnadi/SCRsRank_and_Kemeny_distances_optimized2b.py`.

SCR \ n	3	4	5	6	7
Plurality	0.314815	0.339506	0.355967	0.368313	0.377915
Bucklin	0.268519	0.321759	0.331019	0.356610	0.361325
MedRank	0.277778	0.326389	0.347222	0.367541	0.383402
2-approval	0.314815	0.339506	0.355967	0.368313	0.377915
3-approval	0.500000	0.500000	0.500000	0.500000	0.500000
Borda	0.268519	0.312500	0.327932	0.345465	0.356181
Copeland	0.259259	0.312500	0.320987	0.344393	0.351323
Kemény-Young	0.256173	0.312500	0.317130	0.344179	0.347322
Reverse plurality	0.685185	0.660494	0.644033	0.631688	0.622085
Reverse Borda	0.731481	0.687500	0.672068	0.654535	0.643819
Reverse Copeland	0.740741	0.687500	0.679012	0.655607	0.648677
Reverse Kemény	0.743827	0.687500	0.682870	0.655821	0.652678
Constant	0.500000	0.500000	0.500000	0.500000	0.500000

Table 2: Exact values of BI_K for $m = 3$

The graphs in Figure 1 show also the values for the obscure reverse Kemény-Young method and reverse-plurality rule, and strengthen our observation that looking for least dictatorial rules (as we have defined them) result in strange SCRs since they are getting far away from the preference relations of all voters. In the middle we see the graph for the constant rule which always chooses the same fixed preference relation independently from the composition of a given preference profile. We will show in the next section that the graphs of all anonymous SCRs tend to the value of the constant rule. Since in Figure 1 the graphs of the meaningful SCRs would lie close to each other we highlight these ones in Figure 2 so that we can see the differences between them clearly. We can also see that for slightly more than twenty voters the MedRank algorithm performs even worse than the plurality rule.

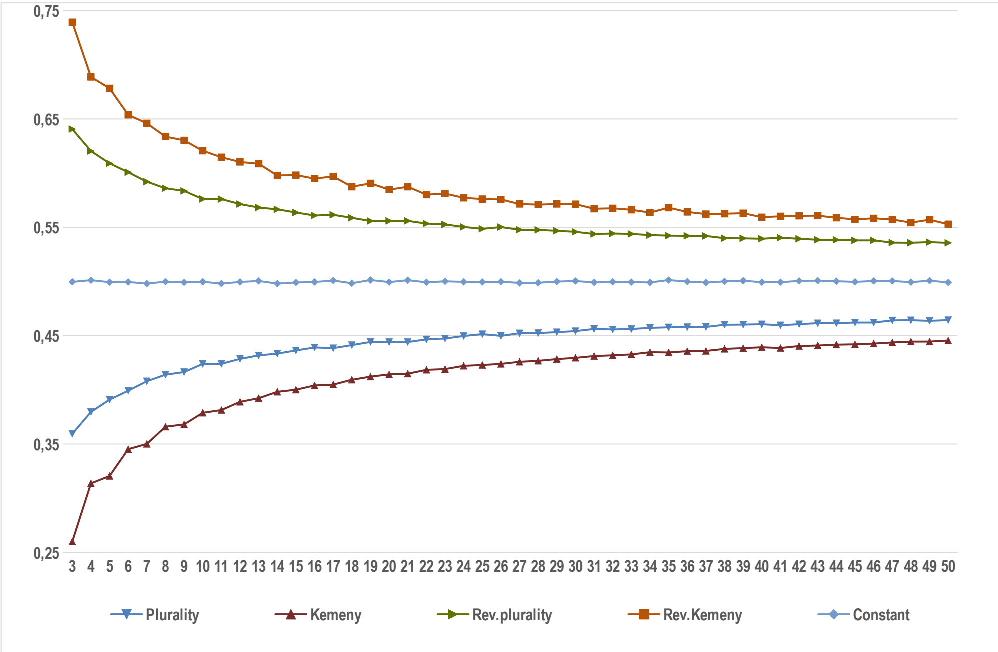

Figure 1: Balancedness in case of ρ_K and $m = 5$

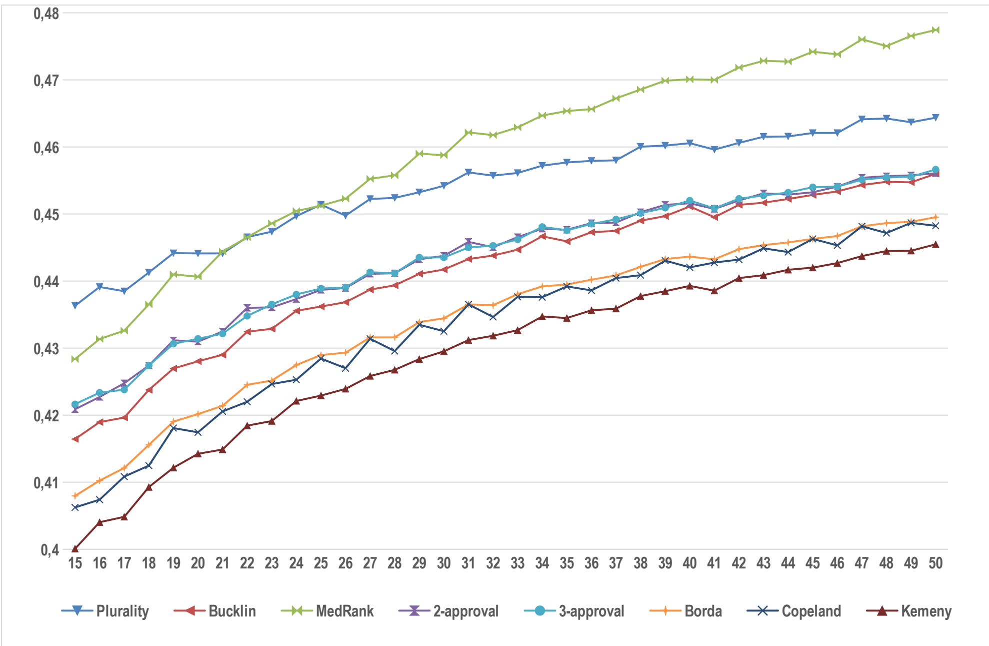

Figure 2: Balancedness in case of ρ_K and $m = 5$

Since the Kemény-Young rule is defined by selecting for each profile a preference relation by minimizing the inner sum of ρ_k it is obvious that no anonymous rule can be more balanced than KY. We see that the other Condorcet-consistent rule, the Copeland method is the second one concerning balancedness. The Borda count preforms just slightly worse than CM. The Bucklin method is more further away, and thereafter the 2-approval and 3-approval methods perform almost identically. However, this is due to the fact that we are considering 5 alternatives, since then $2AV$ means two votes for and three votes against an alternative, while for $3AV$ we have the opposite case. Indeed, looking at the figures in the Appendix. we can observe that $2AV$ and $3AV$ perform differently for 3 and 4 alternatives. Finally, the plurality rule and the MedRank algorithm are the least balanced ones.

Considering the fact that computing the Kemény-Young rule is NP-hard (Bartoldi et al., 1989), our simulations indicate that on average the CM gives results close to KY, which is interesting since considering only given profiles other rules like the Bucklin rule (more precisely the rule minimizing the Spearman footrule distance) are mentioned as possible approximations of KY by Dwork et al. (2002) based on a result by Diaconis and Graham (1977).

Turning to the distance ρ_2, we see in Table 3 the exact values for $m = 3$ and in Figures 3-4 the results of our simulations for $m = 5$. In all of these we find that the Borda count is the most balanced one. In fact it has been proven by Dwork et al. (2002) that minimizing the inner sum of (2.2) results in the Borda count for any profile for which BC gives a linear ordering, and therefore our simulations are basically in line with this theoretical finding and show that possible ties (needing the application of a tie-breaking) does not spoil this result for the investigated SCRs, when taking the averages. Again on the vertical axis we can see the values of indices for the 2500 generated profiles. In Figure 3 we see the reverse-Borda rule, which is the least dictatorial rule because of the previously mentioned theoretical finding and symmetry. The graphs of the most common SCRs are shown in more detail in Figure 4.

SCR \ n	3	4	5	6	7
Plurality	0.291667	0.319444	0.337963	0.351852	0.362654
Bucklin	0.256944	0.305556	0.318287	0.342303	0.349055
MedRank	0.263889	0.310764	0.334491	0.354681	0.372471
2-approval	0.291667	0.319444	0.337963	0.351852	0.362654
3-approval	0.500000	0.500000	0.500000	0.500000	0.500000
Borda	0.250000	0.288194	0.309414	0.326710	0.339463
Copeland	0.250000	0.291667	0.312500	0.330247	0.343750
Kemény-Young	0.250000	0.306134	0.311343	0.338552	0.342089
Reverse plurality	0.708333	0.680556	0.662037	0.648148	0.637346
Reverse Borda	0.750000	0.711806	0.690586	0.673290	0.660537
Reverse Copeland	0.750000	0.708333	0.687500	0.669753	0.656250
Reverse Kemény	0.750000	0.693866	0.688657	0.661448	0.657911
Constant	0.500000	0.500000	0.500000	0.500000	0.500000

Table 3: Exact values of BI_2 for $m = 3$

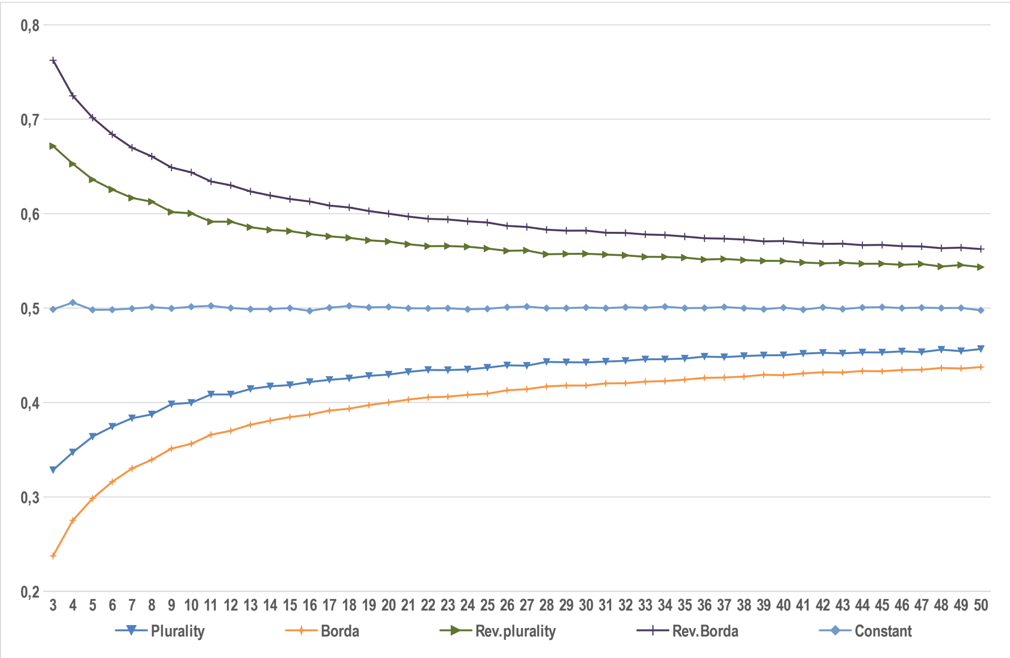

Figure 3: Balancedness in case of ρ_2 and $m = 5$

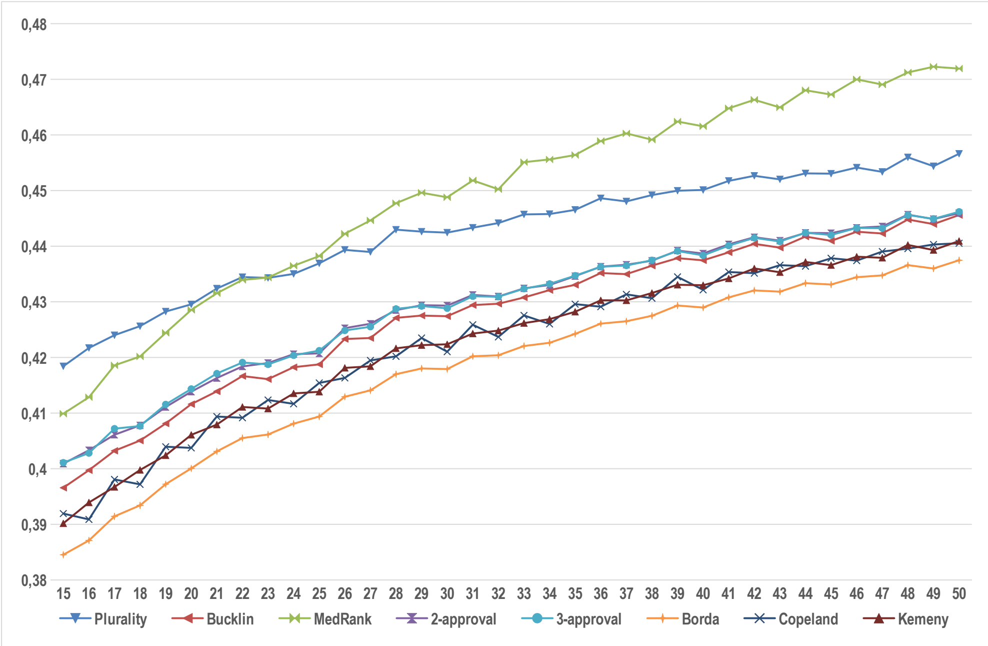

Figure 4: Balancedness in case of ρ_2 and $m = 5$

We observe that the two Condorcet-consistent rules (CM and KY) are the closest ones to the Borda count in terms of balancedness. CM is more balanced in case of even numbers of voters, while KY in case of odd numbers of voters; however, the difference between them is minor. The Bucklin method is clearly less balanced, while the next ones in terms of balancedness are the 2-approval and 3-approval rules, which again perform almost identically. Once again the plurality rule and the MedRank algorithm are the least balanced ones.

If we look at Table 4 and Figures 5-6 determining BI_1, we can make similar observations as based on Figures 3-4. In both the ordering with respect to balancedness is BC, CM or KY, BR, $2AV$ or $3AV$, and PL or $MedRank$. Nevertheless, the result in Figure 6 is surprising since the so called $MedRank$ algorithm, which is quite similar to BR, minimizes the Spearman footrule distance (see Dwork et al., 2002). However, to be precise the statement only holds true for those profiles for which the median ranks of the alternatives form a permutation of the ranks. It easy to see that a large set of such profiles does not satisfy this property. For instance, rank 1 as the median rank of an alternative emerges in a profile if and only if it has a strict majority of the top positions. m alternatives satisfy this property with probability

$$m \sum_{k=\lceil n/2 \rceil}^{n} \binom{n}{k} \left(\frac{1}{m}\right)^k \left(\frac{m-1}{m}\right)^{n-k}.$$

For example, in case of $m = 3$ and $n = 10$ the above probability approximately equals 0.229691.

SCR \ n	3	4	5	6	7
Plurality	0.453704	0.481481	0.503086	0.516461	0.527092
Bucklin	0.370370	0.449074	0.453704	0.494663	0.493377
MedRank	0.370370	0.449074	0.464506	0.496914	0.513489
2-approval	0.453704	0.481481	0.503086	0.516461	0.527092
3-approval	0.666667	0.666667	0.666667	0.666667	0.666667
Borda	0.384259	0.439815	0.460648	0.482017	0.495842
Copeland	0.370370	0.439815	0.453704	0.481696	0.492913
Kemény-Young	0.370370	0.445602	0.452160	0.486518	0.490805
Reverse plurality	0.870370	0.842593	0.827160	0.812757	0.801783
Reverse Borda	0.884259	0.863426	0.841821	0.828597	0.816915
Reverse Copeland	0.870370	0.856481	0.828704	0.821202	0.805413
Reverse Kemény	0.962963	0.887731	0.881173	0.846815	0.842528
Constant	0.666667	0.666667	0.666667	0.666667	0.666667

Table 4: Exact values of BI_1 for $m = 3$

Many profiles fail this criteria and there can be even other missing ranks. Hence, the necessity of resolving ties can destroy the optimality of the simple $MedRank$ algorithm over the set of all profiles and completely different types of rules having ties with a significantly lower probability can perform better as we can observe in Figure 6. For more details we refer to Subsection 5.3.

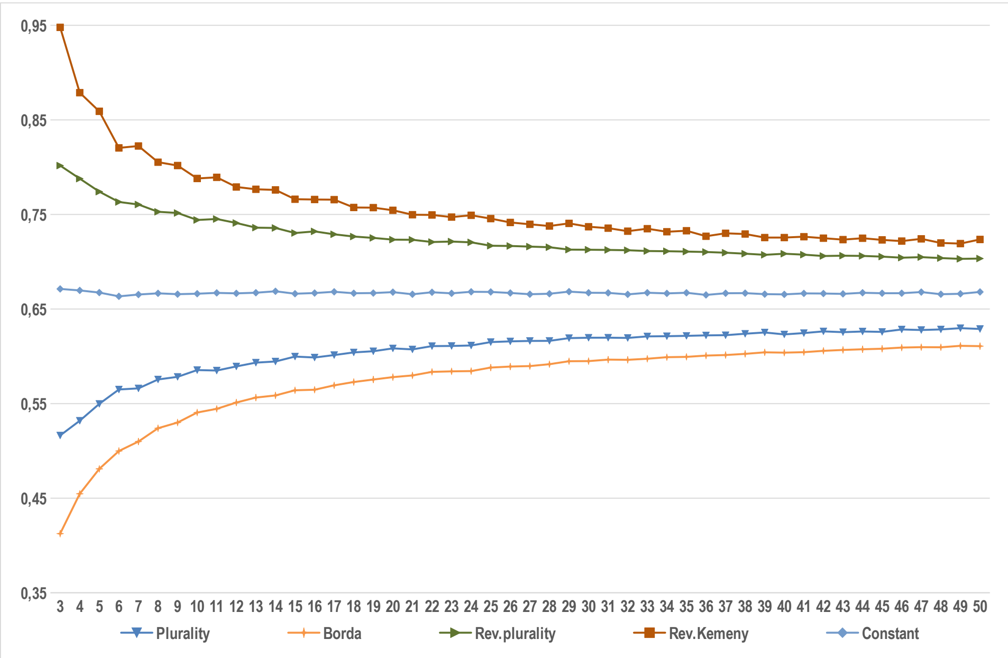

Figure 5: Balancedness in case of ρ_1 and $m = 5$

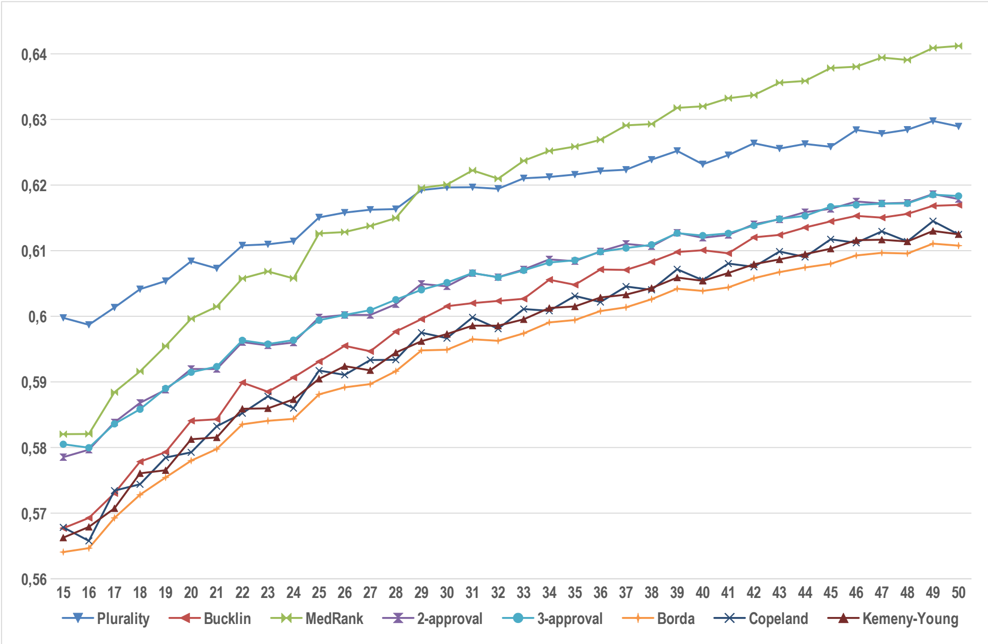

Figure 6: Balancedness in case of ρ_1 and $m = 5$

5 Limits

As we have seen is Section 4 the graphs of the BI_I indices of the investigated SCRs got closer to the BI_I indices of the constant rule. However, the distance for $n = 50$ in the figures and also for $n = 100$ in our simulations (not presented in the paper) remained quite large. In this section we show that the conjectured convergence results hold true. The simplest case is that of the BI_K index. For the BI_2 index we basically need to show that the probability of ties in the sum of ranks determining the social preference of the Borda count tends to zero as the number of voters tends to infinity. The result on the BI_1 index is less clear cut since in this case the probability of ties tends to one for the MedRank rule.

5.1 Kendall τ – Kemény-Young

First, we show that for any given number of alternatives the BI_K index (the same holds for the NDI_K index) of any anonymous SCR F tends to 1/2 as the number of voters tends to infinity.

It follows from the definition of the Kemény-Young rule (KY)[6] that for any anonymous SCR F we must have

$$BI_K(KY) \leq BI_K(F) \leq BI_K(RKY),$$

[6]There can be more than one KY rules, however in this case we can pick anyone of them.

where RKY stands for the reverse Keményi-Young rule. Let $T = \{(a_i, a_j) \in A^2 \mid i < j\}$. Then

$$
\begin{aligned}
BI_K(KY) &= \frac{1}{n}\sum_{i\in N}\frac{1}{(m!)^n C_K}\sum_{\succ\in\mathcal{P}^n}\sum_{(a,b)\in T}\mathbf{1}_{aKY(\succ)b,b\succ_i a} \\
&= \sum_{(a,b)\in T}\frac{1}{n}\sum_{i\in N}\frac{1}{(m!)^n C_K}\sum_{\succ\in\mathcal{P}^n}\mathbf{1}_{aKY(\succ)b,b\succ_i a}, \qquad (5.5)
\end{aligned}
$$

where $\mathbf{1}_{aKY(\succ)b,b\succ_i a}$ equals 1 if $aKY(\succ)b$ and $b \succ_i a$ are true, and zero otherwise. To determine the inner sum of the last sum in (5.5) we approximate it by the following sum

$$\sum_{\succ\in\mathcal{P}^n}\mathbf{1}_{aKY(\succ_{-i})b,b\succ_i a} = \frac{(m!)^n}{2}, \qquad (5.6)$$

where the equality follows from the observation that in half of the preference relations of all profiles $\succ_{-i}\in\mathcal{P}^{n-1}$ the rankings of a and b determined by KY agrees with their rankings in $\succ_i\in\mathcal{P}$ because the former subprofiles and the later preferences are independent. Note that since, as it can be verified, $BI_K(KY) + BI_K(RKY) = 1$ we must have $BI_K(KY) \leq 1/2$.

For notational convenience we set $i = n$, which can be done without loss of generality by the anonymity of KY. Observe that, by taking the LHS of (5.6) as the point of departure, in order to bound the inner sum of (5.5) from below it is sufficient to give an upper bound on the number of cases in which merging $\succ_n$ with $\succ_{-n}$ changes $aKY(\succ_{-n})b$ to $bKY(\succ)a$. The latter change in the ordering of a and b can only occur if the number of cumulative signed inversions[7] between them in $\succ_{-n}$ just differs by at most 1. For a given profile $\succ$ we shall assign 1 to those preferences in which $a \succ_i b$ and -1 to the opposite case. Since the preference relations for each voter can be selected independently in $\mathcal{P}^n$ we can describe the sequence of cumulative signed inversions between a and b for the sequence of subprofiles $(\succ_1, \dots, \succ_j)$ by the Bernoulli random walk. In particular, a sequence of cumulative signed inversions in a subprofile $\succ_{-n}$ can be identified with the first $n-1$ steps of a realization of the Bernoulli random walk. Therefore, the probability that the absolute value of the number of cumulative signed inversions between a and b in $\succ_{-n}$ is at most 1 tends to 0 as n tends to infinity. This implies that the proportion of profiles for which we have

$$\mathbf{1}_{aKY(\succ)b,b\succ_n a} < \mathbf{1}_{aKY(\succ_{-n})b,b\succ_n a} \qquad (5.7)$$

tends to 0. For a given n we shall denote by $\mathcal{N}_n \subset \mathcal{P}^n$ the set of those profiles for which (5.7)

[7]By cumulative signed inversions in a profile $\succ$ we mean the difference between the number of preferences in which we have $a \succ_i b$ and the number of preferences in which we have $b \succ_i a$.

holds true. Then

$$\sum_{\succ\in\mathcal{P}^n} \mathbf{1}_{aKY(\succ_{-n})b, b\succ_n a} - \mid \mathcal{N}_n \mid < \sum_{\succ\in\mathcal{P}^n} \mathbf{1}_{aKY(\succ)b, b\succ_n a},$$

and therefore it follows that

$$BI_K(KY) \to \sum_{(a,b)\in T} \frac{1}{n} \sum_{i\in N} \frac{1}{(m!)^n C_K} \frac{(m!)^n}{2} = \frac{1}{2}\frac{1}{C_K}\frac{1}{2}(m-1)m = \frac{1}{2}$$

as n tends to infinity.

It can be easily verified that $BI_K(CR) = 1/2$. Therefore, we have established the following proposition, which is in line with Figure 1.

Proposition 1. *For any anonymous SCR F we have $BI_K(F) \to BI_K(CR) = 1/2$ as n tends to infinity.*

5.2 Spearman's Rank Correlation – Borda Count

Second, we show that for any given number of alternatives the BI_2 index (the same holds for the NDI_2 index) of any anonymous SCR F tends to $1/2$ as the number of voters tends to infinity.

The following lemma governs our asymptotic results.

Lemma 5.1. *The probability that an additional voter changes the ranking obtained by the Borda count tends to zero as n tends to infinity.*

Proof. For notational convenience we let $i = n$ be the additional voter, which can be assumed without loss of generality by the anonymity of BC. We will show that the ratio of profiles on which $BC(\succ_{-n})$ does not equal $BC(\succ)$ tends to zero as n tends to infinity. In order to show that we start with picking two arbitrary distinct alternatives a_j and a_k. Let us consider the random walk defined by the difference in the sum of ranks of alternatives a_j and a_k. In particular, we consider the random variable $X_n = \sum_{i=1}^{n-1}(rk[a_j, \succ_i] - rk[a_k, \succ_i])$, where $n = 0, 1, \ldots$. Note that $-m < X_n < m$ is a necessary condition for $a_j BC(\succ_{-n}) a_k$ and $a_k BC(\succ) a_j$. The state space of the random walk, which is a Markov process, is $S = \mathbb{Z}$, the initial state is $X_0 = 0$, the transition probabilities are $p_{s,s+d} = P(X_{i+1} = s + d \mid X_i = s) = (m - d)/(m(m - 1))$ if $d = 1, \ldots, m - 1$ and $p_{s,s-d} = P(X_{i+1} = s - d \mid X_i = s) = (m - d)/(m(m - 1))$ if $d = 1, \ldots, m - 1$ for all $i = 0, 1, \ldots, n - 1$. Let us denote by $p_{i,j}^{(t)}$ the transition probability from state i to state j in t steps. The defined random walk is

1. persistent, meaning that it will return to the starting state sometime with probability one, which can be verified by Billingsley (1995) Theorem 8.2.(ii) since $\sum_{t=1}^{\infty} p_{i,j}^{(t)} > \frac{1}{m}\sum_{t=1}^{\infty} \widetilde{p}_{i,j}^{(t)} = \infty$, where $\widetilde{p}_{i,j}^{(t)}$ stands for the respective transition probabilities of the symmetric Bernoulli random walk,

2. irreducible, that is all states can be reached from every other state in a finite number of transitions and
3. aperiodic since with the exception of X_1 we have $P(X_i = 0) > 0$ for all $i \geq 2$.

By Billingsley (1995) Theorem 8.8. we know that $\lim_{n\to\infty} p_{ij} = 0$ for all i and j. Therefore, for any arbitrarily small $\varepsilon > 0$ and for sufficiently large n we have $\varepsilon/(m(m-1)) > P(-m < X_n < m)$. Now even considering all distinct pairs of alternatives the probability that adding an nth voter changes the result of BC is less than ε and tends to zero as n tends to infinity. □

We formulate a simple corollary of Lemma 5.1.

Corollary 1. *The probability that BC has tied alternatives or put it otherwise alternatives with identical rank sums tend to zero as n tends to infinity.*

Dwork et al. (2002) have shown that for any $\succ \in \mathcal{P}^n$ for which the Borda count determines a linear ordering BC minimizes Spearman's rank correlation. Taking their result, Corollary 1 and symmetry into consideration, we know that for any anonymous SCR F we must have

$$\lim_{n\to\infty} BI_2(BC_\tau) \leq \lim_{n\to\infty} BI_2(F) \leq \lim_{n\to\infty} BI_2(RBC_{\tau^{-1}}).$$

We have

$$\begin{aligned} BI_2(BC_\tau) &= \frac{1}{n}\sum_{i\in N}\frac{1}{(m!)^n C_2}\sum_{\succ\in\mathcal{P}^n}\sum_{j=1}^{m}(rk[a_j, BC_\tau(\succ)] - rk[a_j, \succ_i])^2 \\ &= \sum_{j=1}^{m}\frac{1}{n}\sum_{i\in N}\frac{1}{(m!)^n C_2}\sum_{\succ\in\mathcal{P}^n}(rk[a_j, BC_\tau(\succ)] - rk[a_j, \succ_i])^2. \end{aligned} \tag{5.8}$$

To determine the inner sum of the last sum in (5.8) we approximate it by the following sum

$$\overline{BI}_2(BC_\tau) \to \sum_{\succ_{-i}\in\mathcal{P}^{n-1}}\sum_{\succ_i\in\mathcal{P}}(rk[a_j, BC_\tau(\succ_{-i})] - rk[a_j, \succ_i])^2. \tag{5.9}$$

Note that each alternative a_j is ranked l^{th} for any $l = 1, \ldots, m$ with probability $1/m$ by both a random $BC_\tau(\succ_{-i})$ and $\succ_i$, where in the former case only its limit equals $1/m$. Concerning a fixed alternative a, there are $2(m-1), 2(m-2), \ldots, 2(m-l), \ldots, 2\cdot 2, 2\cdot 1$ ways such that the rank distances of a in $BC_\tau(\succ_{-i})$ and $\succ_i$ equal $1, 2, \ldots, l, \ldots, m-2, m-1$, respectively.

Moreover, $\succ_{-i}$ and $\succ_i$ are drawn independently.

$$\begin{aligned}
\overline{BI}_2(BC_\tau) &\to \frac{(m!)^n}{m^2} 2\left((m-1)+(m-2)2^2+\cdots+(m-l)l^2+\cdots+1(m-1)^2\right)\\
&= \frac{(m!)^n}{m^2} 2\left(m-1+m2^2-2^3+\cdots+ml^2-l^3+\cdots+m(m-1)^2-(m-1)^3\right)\\
&= \frac{(m!)^n}{m^2} 2\left(m(1^2+2^2+\cdots+(m-1)^2)-(1^3+2^3+\cdots+(m-1)^3\right)\\
&= \frac{(m!)^n}{m^2} 2\left(m\frac{(m-1)m(2m-1)}{6}-\frac{(m-1)^2m^2}{4}\right)\\
&= (m!)^n(m-1)\left(\frac{2m-1}{3}-\frac{m-1}{2}\right) = (m!)^n\frac{1}{6}(m^2-1) \qquad (5.10)
\end{aligned}$$

Note that since, as it can be verified, $BI_2(BC_\tau)+BI_2(RBC_{\tau^{-1}})=1$ we must have $BI_2(BC_\tau)\leq 1/2$. Therefore, calculating with the approximation (5.10) of the inner sum (5.9), it follows that $BI_2(BC_\tau)$ given by (5.8) tends to

$$\frac{1}{C_2}\frac{1}{6}m(m^2-1) \qquad (5.11)$$

Hence, we have obtained the following proposition.

Proposition 2. *For any anonymous SCR F we have $BI_2(F)\to 1/2$ as n tends to infinity.*

Next we determine the BI_2 index of the constant rule. Note that any alternative a_j is ranked lth with probability $1/m$ by any voter i. Without loss of generality we can assume that the constant rule is given by $a_1 \succ^* a_2 \succ^* \cdots \succ^* a_m$. The possible rank distances from alternative a_l are $l-1, l-2, \ldots, 1, 0, 1, \ldots, m-l$. If we take all alternatives into consideration, then there are $2(m-1), 2(m-2), \ldots, 2(m-l), \ldots, 2\cdot 2, 2\cdot 1$ ways such that the rank distances of an alternative in $\succ^*$ and $\succ_i$ equal $1, 2, \ldots, l, \ldots, m-2, m-1$, respectively. Therefore,

$$\begin{aligned}
BI_2(CR) &= \frac{1}{n}\sum_{i\in N}\frac{1}{(m!)^n C_2}\sum_{j=1}^{m}\sum_{\succ\in\mathcal{P}^n}(rk[a_j,\succ^*]-rk[a_j,\succ_i])^2\\
&= \frac{1}{n}\sum_{i\in N}\frac{1}{(m!) C_2}\sum_{j=1}^{m}\sum_{\succ_i\in\mathcal{P}}(j-rk[a_j,\succ_i])^2\\
&= \frac{1}{mC_2}2\left((m-1)1^2+\cdots+(m-l)l^2+\cdots+1(m-1)^2\right)=\frac{1}{2}, \qquad (5.12)
\end{aligned}$$

where we have abbreviated our calculations since they are similar to those ones in (5.10).

Proposition 3. *$BI_2(CR)=1/2$ for any n.*

Propositions 2 and 3 are in line with Figure 3.

5.3 Spearman's Footrule – MedRank

Third, we turn to the limit of the BI_1 index (the same holds for the NDI_1 index) and show that its limit for any anonymous SCR F tends to 2/3 for odd m and to values close to 2/3 for even m as the number of voters tends to infinity. We cannot directly proceed in the same way as in Subsections 5.1 and 5.2 because it was apparent from Figure 5 and its discussion that the MedRank algorithm does not bound the other anonymous SCRs in the same way as the Kemény-Young method in Subsection 5.1 and the Borda count in Subsection 5.2. The MedRank algorithm only bounds the Spearman footrule distance of other anonymous SCRs on the set of profiles for which all median ranks are different. However, a MedRank* algorithm not necessarily resulting in a linear ordering can be used to bound the other anonymous SCRs.

The MedRank*, henceforth briefly MR^*, algorithm that assigns to each alternative its median rank, that is MR^* maps from $\mathcal{P}^n$ to $\{1, \ldots, m\}$ and is not an SCR, can result in weak preferences. In addition, it assigns ranks to the alternatives and not just a weak ordering of them.[8] Nevertheless, $BI_1(MR^*)$ can be still defined in the same was as for SCRs. Furthermore, since for any $j = 1, \ldots, m$ and any $\succ \in \mathcal{P}^n$ the solution of

$$\min_{r=1,\ldots,m} \sum_{i \in N} |r - rk[a_j, \succ_i]|$$

is the median of the ranks $rk[a_j, \succ_1], \ldots, rk[a_j, \succ_n]$ by the standard property of the minimum of the sum of absolute differences from a set of given numbers. Note that for any j and any $\succ$ we have independent minimization problems, and therefore we get

$$BI_1(MR^*) \leq BI_1(F)$$

for any anonymous SCR F.

We have

$$\begin{aligned} BI_1(MR^*) &= \frac{1}{n} \sum_{i \in N} \frac{1}{(m!)^n C_1} \sum_{\succ \in \mathcal{P}^n} \sum_{j=1}^{m} |rk[a_j, MR^*(\succ)] - rk[a_j, \succ_i]| \\ &= \sum_{j=1}^{m} \frac{1}{n} \sum_{i \in N} \frac{1}{(m!)^n C_1} \sum_{\succ \in \mathcal{P}^n} |rk[a_j, MR^*(\succ)] - rk[a_j, \succ_i]|. \end{aligned} \tag{5.13}$$

To determine the inner sum of the last sum in (5.13) we approximate it by the following sum

$$\overline{BI}_1(MR^*) = \sum_{\succ_{-i} \in \mathcal{P}^{n-1}} \sum_{\succ_i \in \mathcal{P}} |rk[a_j, MR^*(\succ_{-i})] - rk[a_j, \succ_i]|. \tag{5.14}$$

[8]This is more admissive than the usual rank assignments since, for instance, there can be two gold medalists in the olympic games, but there cannot be two silver medalists without a gold medalist.

Note that each alternative $a_j \in A$ is ranked rth for any $r = 1, \ldots, m$ with probability $1/m$ by a random $\succ_i$. However, at first sight may be surprisingly this does not hold true for a random $MR^*(\succ_{-i})$.

Our next step is to determine for notational convenience the asymptotic distribution of $MR^*(\succ)$ as n tends to infinity, which is the same as the asymptotic distribution of $MR^*(\succ_{-i})$. Pick an arbitrary alternative a and note that for each $\succ_l$, where $l = 1, \ldots, n$, alternative a is ranked in the top r positions with probability r/m. If for a given n we denote by $X_n^{(r)}$ the number of voters ranking a in the top r positions, then $X_n^{(r)}$ is binomially distributed with parameters n and r/m. Hence, the probability that a is ranked in the top r positions by MR^* equals

$$P(X_n^{(r)} \geq \lfloor n/2 \rfloor + 1) = \sum_{i=\lfloor n/2 \rfloor+1}^{n} \binom{n}{i} \left(\frac{r}{m}\right)^i \left(1 - \frac{r}{m}\right)^{n-i} \tag{5.15}$$

and the probability that a is ranked rth by MR^* equals

$$P(MR^* = r) = P(X_n^{(r)} \geq \lfloor n/2 \rfloor + 1) - P(X_n^{(r-1)} \geq \lfloor n/2 \rfloor + 1).$$

We recall the result on the binomial distribution in Arratia and Gordon (1989, Theorem 1), which we will employ to give an upper bound of the probability in (5.15). Let Y_n be binomially distributed with parameters n and p. Then for any $n = 1, 2, \ldots$ and any $p < a < 1$ we have

$$P(Y_n \geq an) \leq e^{-n\left(a \log \frac{a}{p} + (1-a) \log \frac{1-a}{1-p}\right)}. \tag{5.16}$$

In our case p takes values $1/m$, $2/m$,...,$(m-1)/m$ and because of symmetry we only need to give an upper bound of the probabilities in (5.15) for values $1/m$, $2/m$,...,$\lfloor m/2 \rfloor$ since we can give related required lower bounds on the probabilities in (5.15) for values $\lfloor m/2 \rfloor + 1$,...,$(m-1)/m$ by considering the complementary event. For the missing case $p = m/2$ if m is even, we will determine (5.15) directly. Therefore, we start with the case of $p < a = 1/2$. Substituting these values into (5.16), we get

$$P(Y_n \geq \frac{1}{2}n) \leq e^{-\frac{1}{2}n\left(\log \frac{1}{2p} + \log \frac{1}{2(1-p)}\right)} = e^{-\frac{1}{2}n \log \frac{1}{4p(1-p)}},$$

which in turn implies by $p(1-p) < 1/4$ that $P(Y_n \geq \frac{1}{2}n)$ tends to zero as n tends to infinity. Therefore, we also have that for $r = 1, 2, \ldots, \lfloor (m-1)/2 \rfloor$ the probabilities in (5.15) tend to zero as n tends to infinity. Now if m is even, that is $m = 2k$, then

$$\sum_{i=\lfloor n/2 \rfloor+1}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^i \left(1 - \frac{1}{2}\right)^{n-i} = \frac{1}{2^{n+1}} 2^n = \frac{1}{2}.$$

From our limiting results and symmetry (by considering the complementary events) we conclude that when n tends to infinity that the probabilities in (5.15) tend to 1/2 and 1 for $r = m/2$ and $r = m/2+1$,[9] respectively, while for r smaller than the median(s) and for r larger than the median(s) they tend to zero and one as n tends to infinity, respectively.

Now we are ready to determine the limit of $\overline{BI}_1(MR^*)$ as n tends to infinity. From the results on MR^* summarized in the previous paragraph we know that for odd $m = 2k+1$ each alternative is ranked $k+1$th with probability one in the limit and for even $m = 2k$ each alternative is ranked kth and $k+1$th with probability one half each in the limit. Therefore, for odd $m = 2k+1$ the possible rank distances for any alternative in $MR^*(\succ_{-i})$ and $\succ_i$ are $k, k-1, \ldots, 1, 0, 1, \ldots, k-1, k$. Hence, for odd $m = 2k+1$ we get

$$\begin{aligned} \overline{BI}_1(MR^*) &\to \frac{(m!)^n}{m}\left(k + (k-1) + \cdots + 1 + 0 + 1 + \cdots + (k-1) + k\right) \\ &= \frac{(m!)^n}{m} 2\frac{1}{2}k(k+1) = \frac{(m!)^n}{4}\frac{m^2-1}{m}. \end{aligned} \tag{5.17}$$

which in turn implies that $BI_1(MR^*)$ given by (5.13) tends to

$$BI_1(MR^*) \to \frac{1}{C_1} m \frac{1}{4}\frac{m^2-1}{m} = \frac{1}{4}\frac{m^2-1}{m-1+(m-1)^2/2} = \frac{1}{2}\frac{m+1}{m+3} \tag{5.18}$$

For even $m = 2k$ the possible rank distances are still $k, k-1, \ldots, 1, 0, 1, \ldots, k-1, k$ if m is even, however the rank distance k has half the probability of the other rank distances. Hence, we get

$$\begin{aligned} \overline{BI}_1(MR^*) &\to \frac{(m!)^n}{m}\left[\frac{1}{2}\left((k-1) + (k-2) + \cdots + 1 + 0 + 1 + \cdots + (k-1) + k\right)\right. \\ &\quad \left. + \frac{1}{2}\left(k + (k-1) + \cdots + 1 + 0 + 1 + \cdots + (k-2) + (k-1)\right)\right] \\ &= \frac{(m!)^n}{m}\left[2\frac{1}{2}(k-1)k + k\right] = (m!)^n \frac{m}{4}. \end{aligned} \tag{5.19}$$

which in turn implies that $BI_1(MR^*)$ given by (5.13) tends to

$$BI_1(MR^*) \to \frac{1}{C_2} m \frac{m}{4} = \frac{1}{2}. \tag{5.20}$$

From Figure 5 we see that 1/2 can only be a nontight lower bound. However, as we will see later after investigating the constant rule, it conveys a substantial message.

We determine the BI_1 value of the constant rule. Without loss of generality we can assume that $a_1 \succ^* a_2 \succ^* \cdots \succ^* a_m$. The possible rank distances from alternative a_l are $l-1, l-$

[9] We have these cases only if m is even.

$2, \ldots, 1, 0, 1, \ldots, m-l$. If we take all alternatives into consideration, then there are $2(m-1), 2(m-2), \ldots, 2(m-l), \ldots, 2\cdot 2, 2\cdot 1$ ways such that the rank distances of an alternative in $\succ^*$ and $\succ_i$ equal $1, 2, \ldots, l, \ldots, m-2, m-1$, respectively. Moreover, $\succ^*$ is fixed, while $\succ_i$ are drawn independently. Furthermore, each alternative is ranked lth with probability $1/m$ in $\succ_i$.

$$\begin{aligned}
BI_1(CR) &= \frac{1}{C_1}\frac{1}{m}2\left((m-1)+(m-2)2+\cdots+(m-l)l+\cdots+1(m-1)\right)\\
&= \frac{1}{C_1 m}2\left(m-1+m2-2^2+\cdots+ml-l^2+\cdots+m(m-1)-(m-1)^2\right)\\
&= \frac{1}{C_1 m}2\left(m(1+2+\cdots+(m-1))-(1^2+2^2+\cdots+(m-1)^2\right)\\
&= \frac{1}{C_1 m}2\left(m\frac{1}{2}m(m-1)-\frac{(m-1)m(2m-1)}{6}\right)\\
&= \frac{m-1}{C_1}\left(m-\frac{2m-1}{3}\right)=\frac{m^2-1}{3\lfloor m^2/2\rfloor},
\end{aligned} \tag{5.21}$$

which equals $2/3$ if m is odd and $2/3(1-1/m^2)$ if m is even. We formulate our result as a proposition.

Proposition 4. *For any n the balancedness index $BI_1(CR)=2/3$ if m is odd and $BI_1(CR)=2/3(1-1/m^2)$ if m is even.*

From Figure 5 (and also from the ones in the Appendix for 3 and 4 alternatives) we see that for the given tie-breaking rule the Borda count has a lower BI_1 index than the other investigated SCRs including both the MedRank algorithm and the Bucklin rule, where the latter one is a refined version of the former one. Therefore, we determine the limit of the $BI_1(BC_\tau)$. We have

$$\begin{aligned}
BI_1(BC_\tau) &= \frac{1}{n}\sum_{i\in N}\frac{1}{(m!)^n C_1}\sum_{\succ\in\mathcal{P}^n}\sum_{j=1}^{m}|rk[a_j, BC_\tau(\succ)]-rk[a_j,\succ_i]|\\
&= \sum_{j=1}^{m}\frac{1}{n}\sum_{i\in N}\frac{1}{(m!)^n C_1}\sum_{\succ\in\mathcal{P}^n}|rk[a_j, BC_\tau(\succ)]-rk[a_j,\succ_i]|.
\end{aligned} \tag{5.22}$$

To determine the inner sum of the last sum in (5.22) we approximate it by the following sum

$$\overline{BI}_1(BC_\tau)\to\sum_{\succ_{-i}\in\mathcal{P}^{n-1}}\sum_{\succ_i\in\mathcal{P}}|rk[a_j, BC_\tau(\succ_{-i})]-rk[a_j,\succ_i]|. \tag{5.23}$$

Note that each alternative a_j is ranked l^{th} for any $l=1,\ldots,m$ with probability $1/m$ by both a random $BC_\tau(\succ_{-i})$ and $\succ_i$, where in the former case only its limit equals $1/m$. Concerning a fixed alternative a, there are $2(m-1), 2(m-2), \ldots, 2(m-l), \ldots, 2\cdot 2, 2\cdot 1$ ways such that the rank distances of a in $BC_\tau(\succ_{-i})$ and $\succ_i$ equal $1, 2, \ldots, l, \ldots, m-2, m-1$, respectively.

Moreover, $\succ_{-i}$ and $\succ_i$ are drawn independently.

$$
\begin{aligned}
\overline{BI}_1(BC_\tau) &\to \frac{(m!)^n}{m^2} 2\left((m-1)+(m-2)2+\cdots+(m-l)l+\cdots+1(m-1)\right) \\
&= \frac{(m!)^n}{m^2} 2\left(m\frac{m(m-1)}{2} - \frac{(m-1)m(2m-1)}{6}\right) \\
&= \frac{(m!)^n}{m}(m-1)\left(m - \frac{2m-1}{3}\right) = (m!)^n \frac{m^2-1}{3m}, \qquad (5.24)
\end{aligned}
$$

where we have skipped a few steps since they are similar to those ones in (5.21). Therefore, calculating with the approximation of the inner sum (5.24), it follows that $BI_1(BC_\tau)$ given by (5.22) tends to

$$\frac{1}{C_1}\frac{1}{3}(m^2-1). \qquad (5.25)$$

From (5.25) for odd m we get $BI_1(BC_\tau) = 2/3$ and for even m we get $BI_1(BC_\tau) = 2/3(1 - 1/m^2)$. These values are the same as for the constant rule. Furthermore, since we have to do exactly the same calculations for determining $BI_1(RBC_{\tau^{-1}})$ we obtain the same values, and thus we have obtained the following proposition.

Proposition 5. *For any anonymous SCR F for which*

$$BI_1(BC_\tau) \leq BI_1(F) \leq BI_1(RBC_{\tau^{-1}})$$

holds for any n we have $BI_1(BC_\tau) \to BI_1(F) \to BI_1(CR)$ as n tends to infinity.

The *main message* from this subsection is that while in Subsections 5.1 and 5.2 the chosen anonymous tie-breaking had no effect on the orderings obtained by an anonymous SCR and also on their limits, for the Spearman's footrule it has a substantial effect and changes its limit extremely. When considering BI_1, for large n basically the tie-breaking rule enforces the chosen linear ordering and the rules themselves only play a negligible role.

6 Conclusion

In this paper we have ranked common SCRs based on the Kendall τ, the Spearman rank correlation and the Spearman footrule distances. In line with these distances we have defined respective balancedness and non-dictatorship indices by considering the distance of an SCR from the dictatorial rules. In this way we have also given a new interpretation of the traditional literature which derives SCRs as a solution of distance minimization problems.

For three alternatives and at most seven voters we have determined the indices by employing the brute force algorithm, while for a larger number of voters and 3 to 5 alternatives we took a random sample of 2500 profiles. Our main findings were that the Borda count, the Copeland method and the Kemény-Young method were the most balanced rules, or put it otherwise for the three distances we have considered these rules have the smallest average sum of normalized distances. We have carried out the same calculations and estimations for the distance function

$$\rho_\infty(F,G) = \sum_{\succ \in \mathcal{P}^n} \max_{i=1,\ldots,m} |rk[a_i, F(\succ)] - rk[a_i, G(\succ)]|$$

and found the same results. We have not included these results into this paper since ρ_∞ is not related to a known SCR and the paper contains already a large number of figures.

We found that minimization with respect to the Spearman footrule is sensitive on how ties are broken. The usually associated MedRank algorithm with this distance function, as its optimal solution, performs poorly if ties have to be broken. One of its refined version, the Bucklin rule behaves significantly better. Another escape route is to allow for indifferences in the socially chosen preference relation. However, even in this case the way how we assign ranks to tied alternatives matters.

Finally, we outline possible further research directions. In our analysis we presented our results for a given number of alternatives m and let the number of voters n vary. Clearly, this is the more interesting case in the social choice context. However, in computer science applications the other case in which we fix n and let m vary can be equally interesting. Since m alternatives imply $m!$ rankings, this analysis is far less tractable. Already determining the Kemény-Young ranking is NP-hard (Bartoldi et al., 1989). Therefore, we expect less results from this research direction. Furthermore, additional limits, in which we tend only with m to infinity or with both m and n, may be determined. Another strand of research would be not to simply take the uniform distribution above the set of profiles, but consider other manageable alternative distributions.

Acknowledgements

We are grateful Ágnes Cseh and Andreas Orland for helpful comments and suggestions.

Appendix

Figures 7-18 contain the same figures for $m = 3$ and $m = 4$ as those ones in Section 4 for $m = 5$. There is one interesting difference worthwhile to mention: for four alternatives and the Spearman footrule distance the Bucklin rule is the second best after the Borda count and the MedRank algorithm itself does perform somewhat better than in the other two Figures for 3 and 5 alternatives. This might indicate a difference in the performance of the Bucklin method between an odd and even number of alternatives. However, it also might be the case that this only holds for small even number of alternatives. To get more insight we have also included for the Spearman footrule distance the case of six alternatives in Figures 19-20. We can observe that the Borda rule is clearly the best, but now we cannot really distinguish between the Kemény-Young method, the Copeland method and the Bucklin rule. We conjecture that for eight alternatives the Bucklin rule will be outperformed by the Kemény-Young and Copeland methods. In several zoomed in versions of our figures we made only parts of the BI_I graph of the MedRank algorithm visible to improve the clarity of the figures.

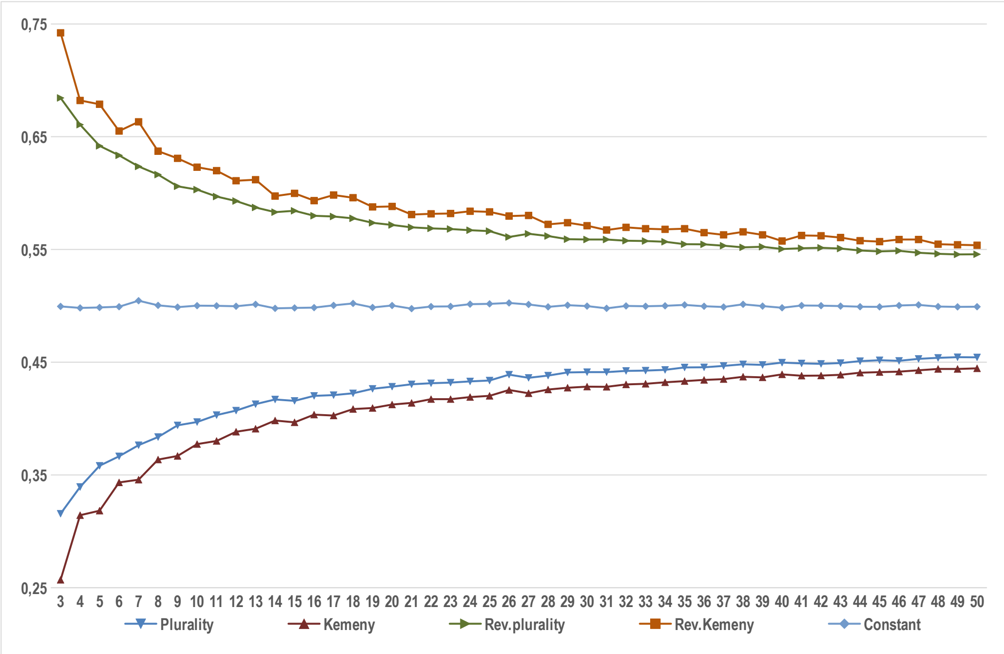

Figure 7: Balancedness in case of ρ_K and $m = 3$

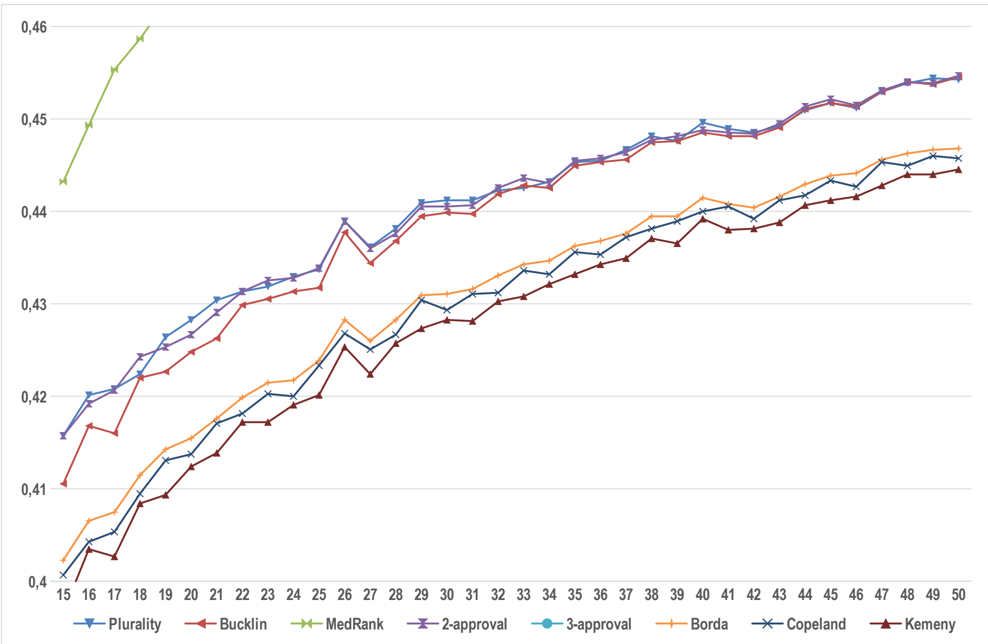

Figure 8: Balancedness in case of ρ_K and $m = 3$

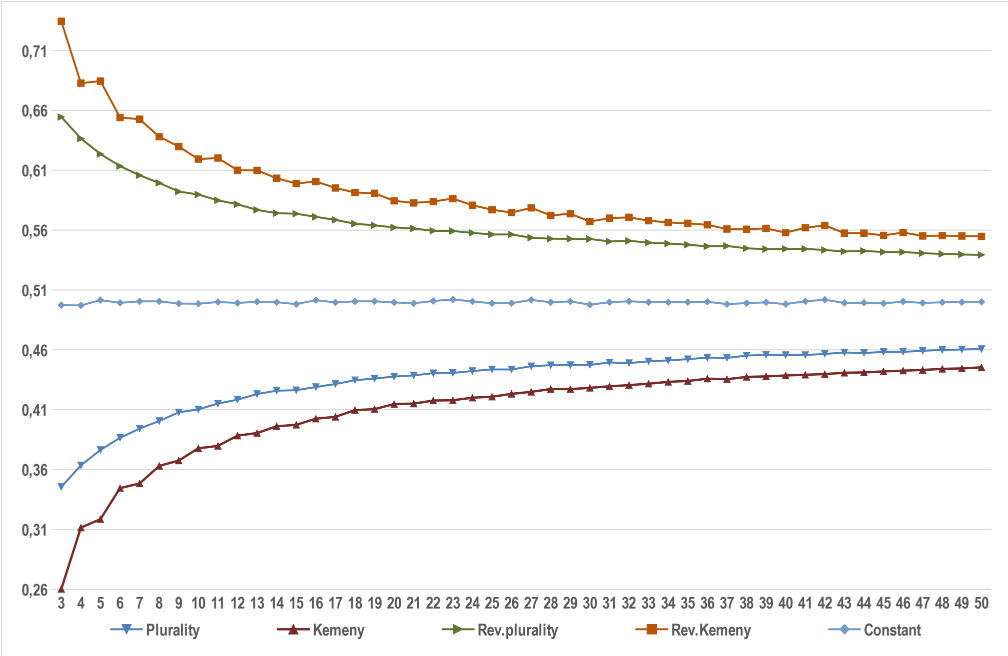

Figure 9: Balancedness in case of ρ_K and $m = 4$

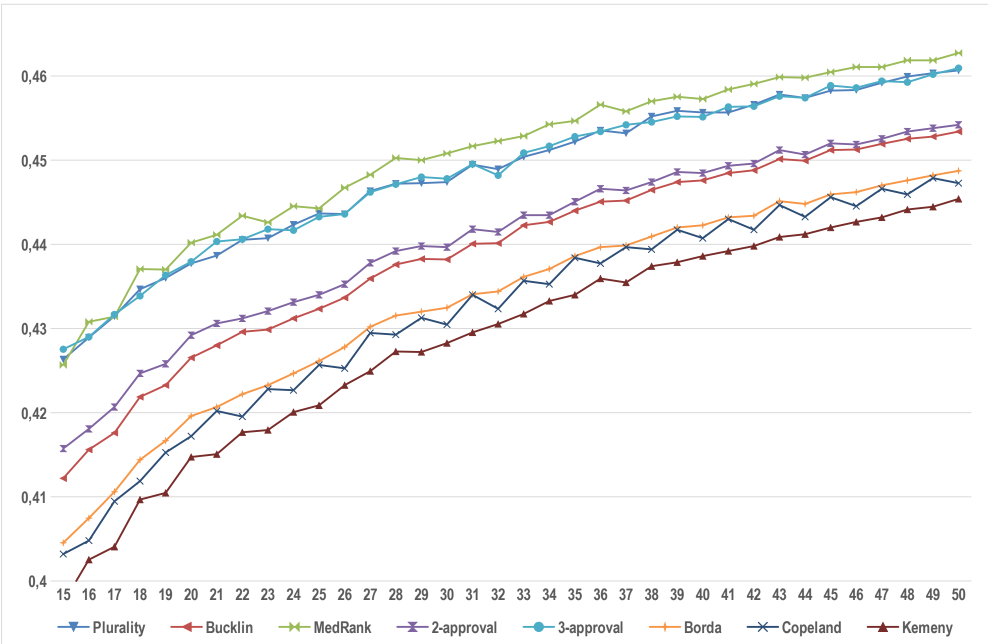

Figure 10: Balancedness in case of ρ_K and $m = 4$

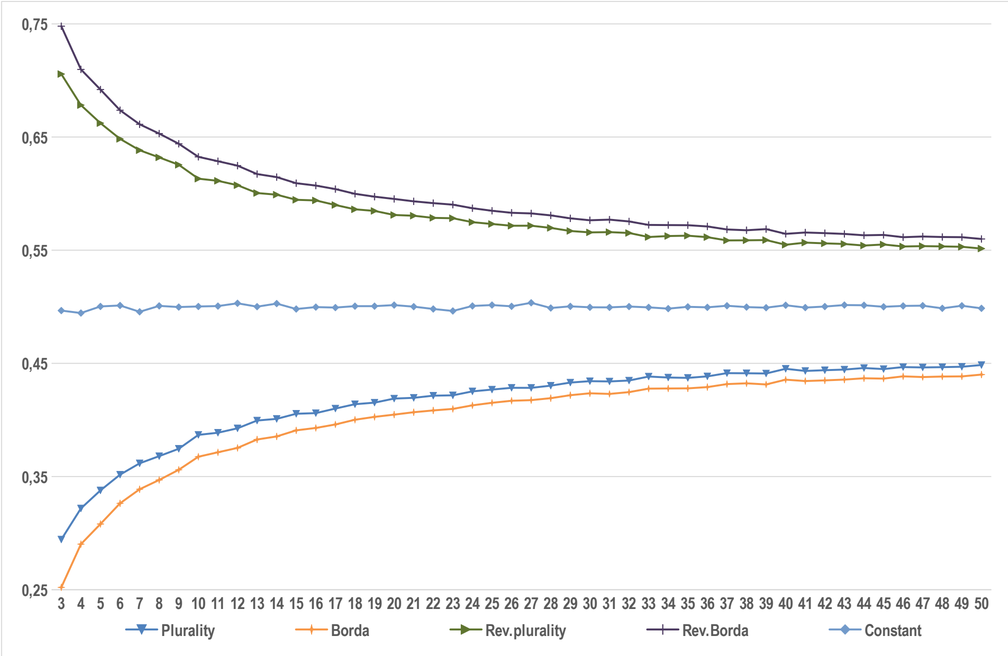

Figure 11: Balancedness in case of ρ_2 and $m = 3$

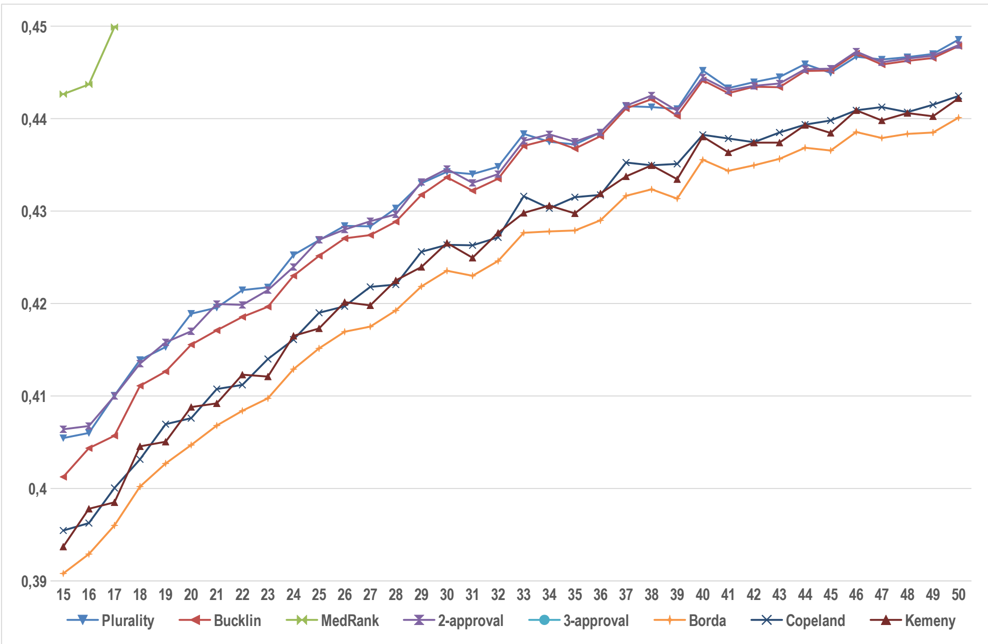

Figure 12: Balancedness in case of ρ_2 and $m = 3$

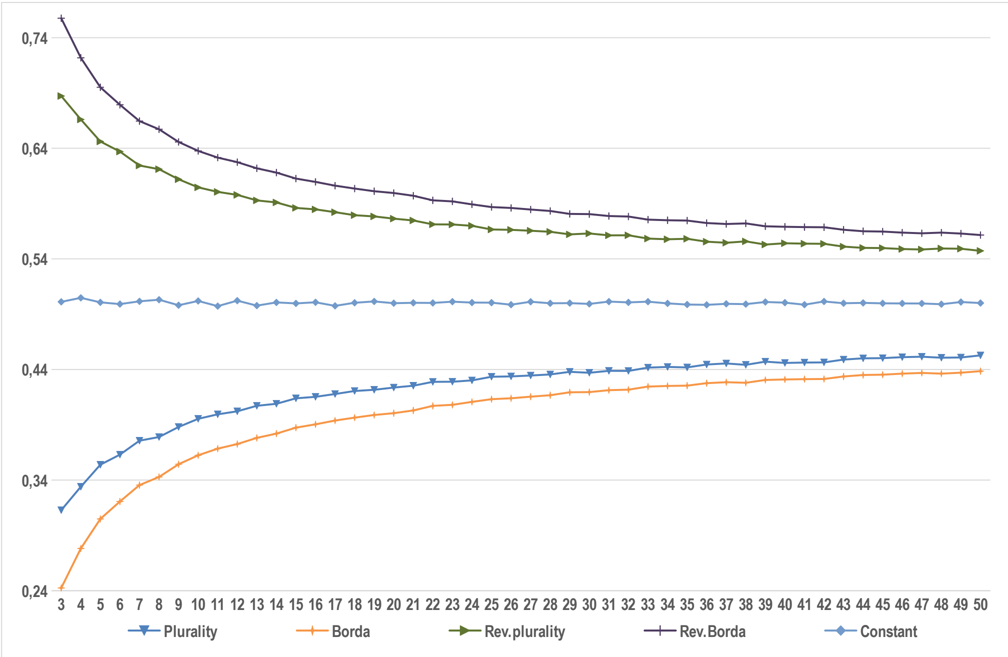

Figure 13: Balancedness in case of ρ_2 and $m = 4$

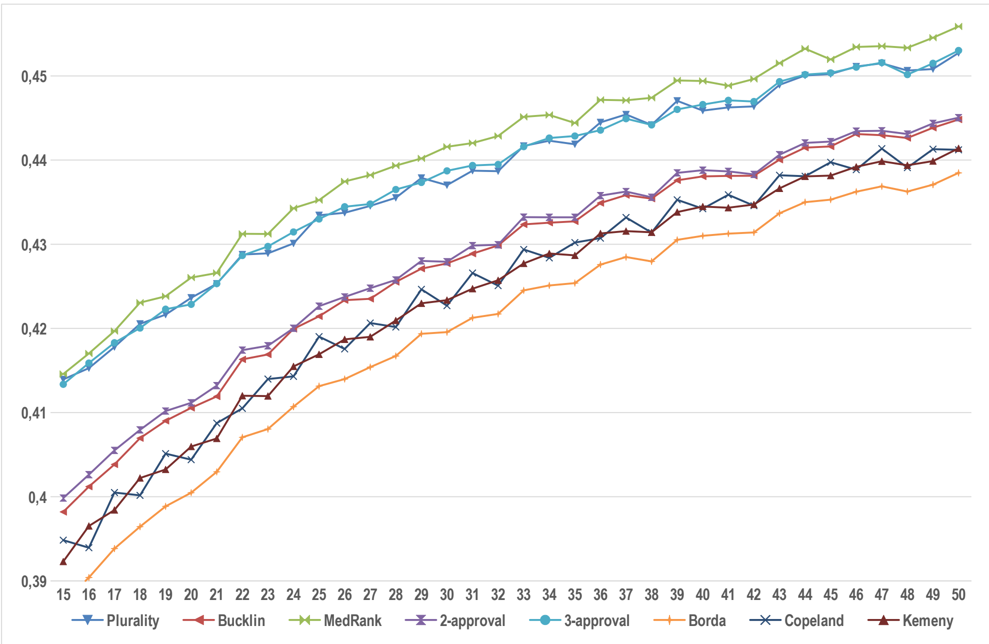

Figure 14: Balancedness in case of ρ_2 and $m = 4$

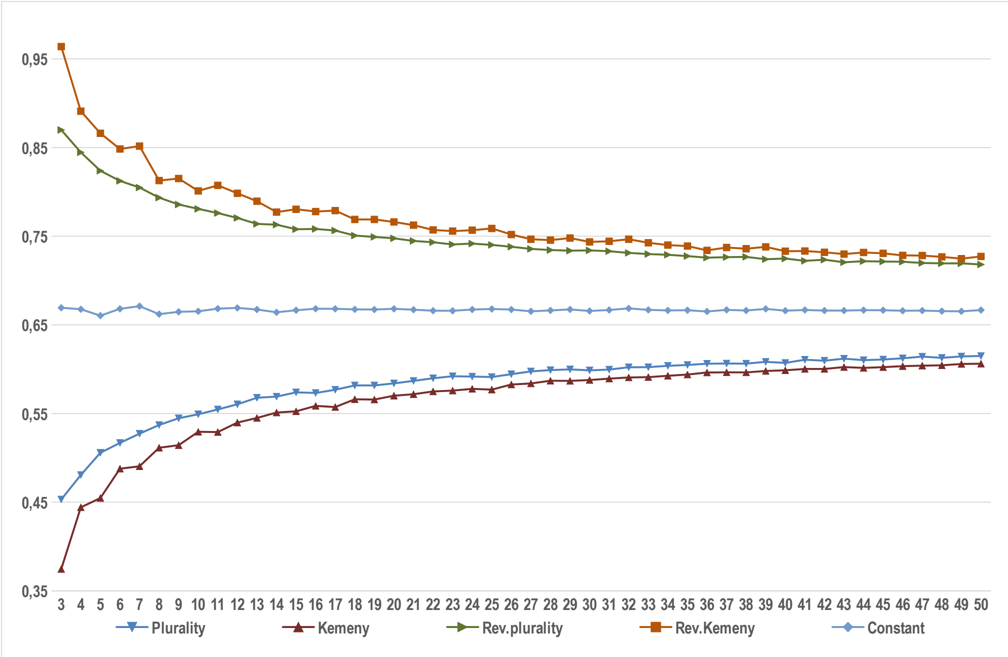

Figure 15: Balancedness in case of ρ_1 and $m = 3$

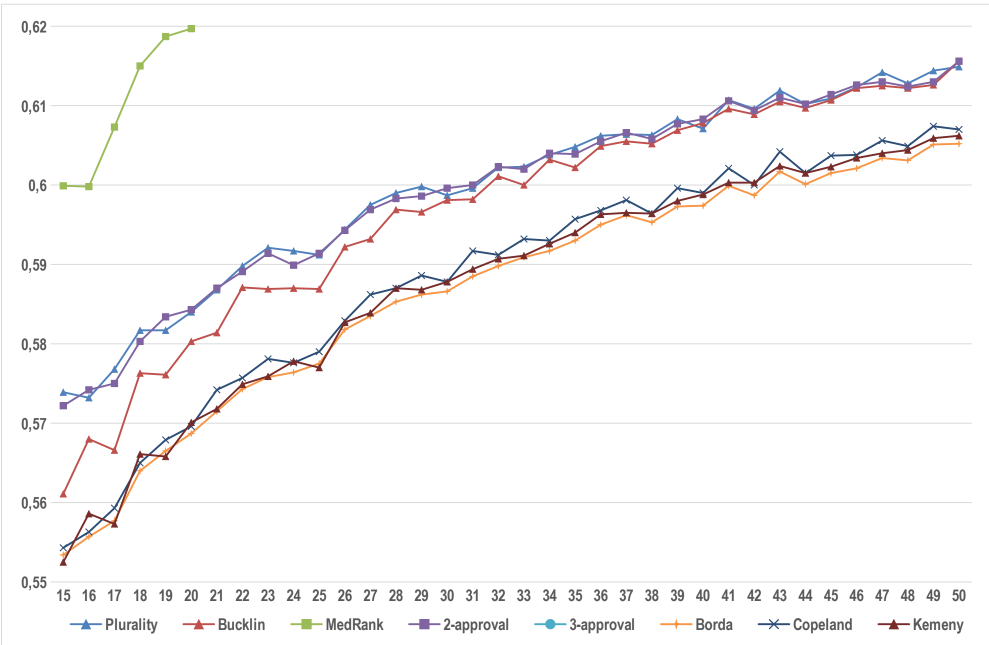

Figure 16: Balancedness in case of ρ_1 and $m = 3$

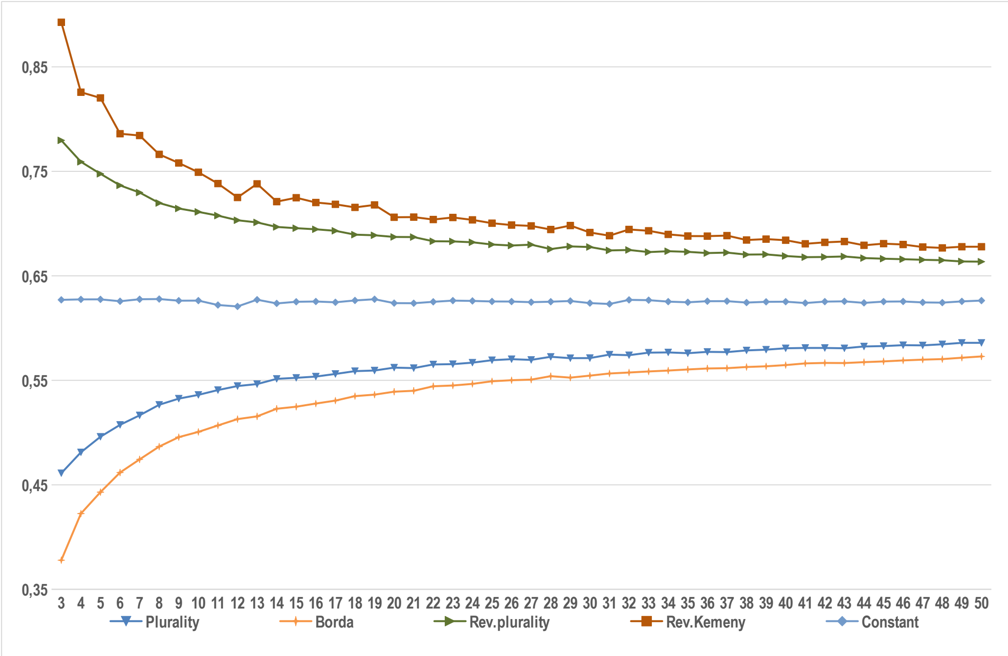

Figure 17: Balancedness in case of ρ_1 and $m = 4$

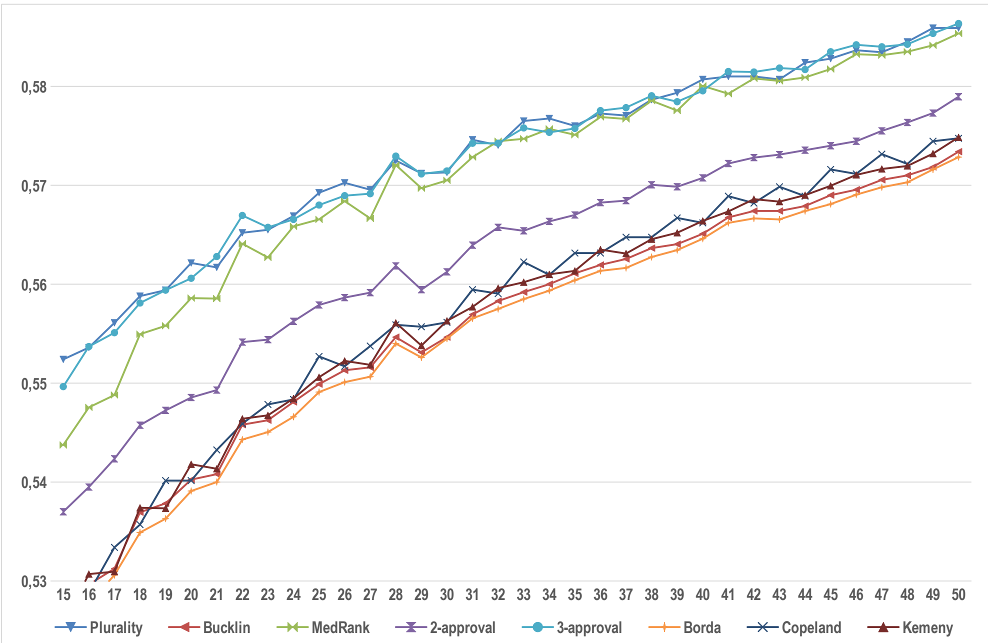

Figure 18: Balancedness in case of ρ_1 and $m = 4$

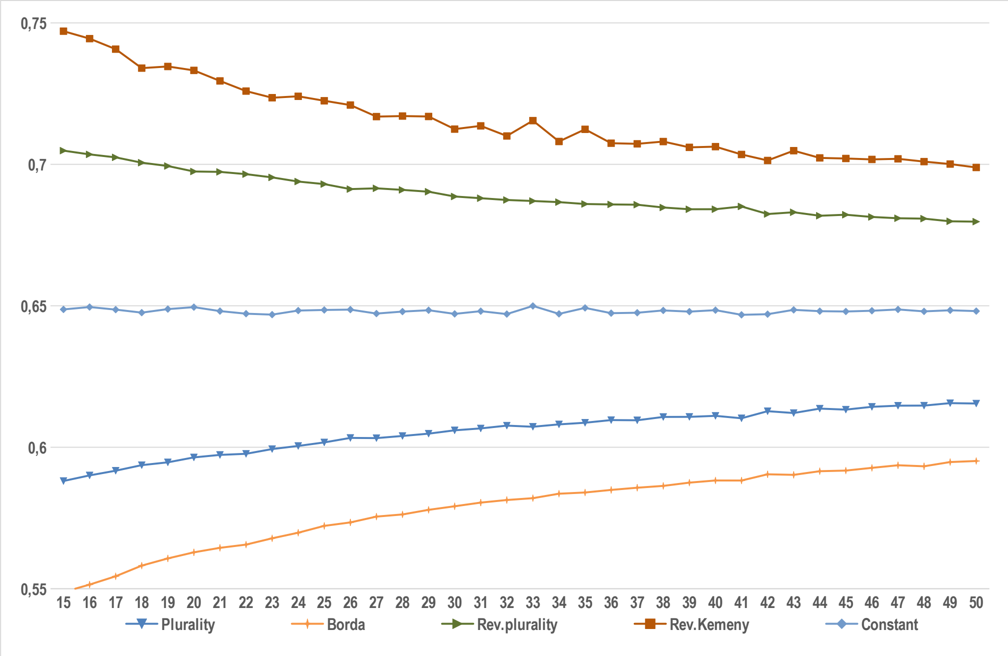

Figure 19: Balancedness in case of ρ_1 and $m = 6$

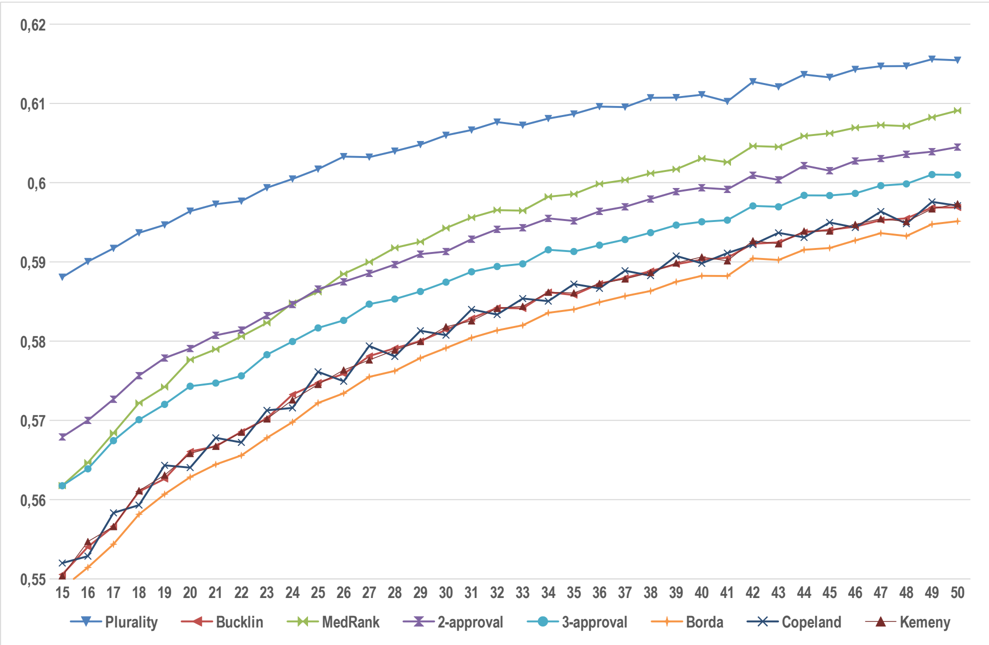

Figure 20: Balancedness in case of ρ_1 and $m = 6$

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