Central European Journal of Operations Research
https://doi.org/10.1007/s10100-023-00850-8

How to tighten the control set?

Peter Tallos[1]

Accepted: 4 March 2023

Abstract
We examine the problem of reducing the control set in a dynamical system so that the solution set and the attainable sets remain essentially unchanged. We cite some classical results to exhibit the problem and promote a set-valued approach. A necessary condition is formulated by using a concept of set-valued derivative, which can be regarded as an extension of the classical Relaxation theorem.

Keywords Control systems · Extremal controls · Set-valued maps · Differential inclusions · Relaxation · Set-valued derivative

Mathematics Subject Classification 34A60 · 49J21 · 49K21

1 Control systems

Let $[0, T]$ be the time interval and let X and Y be real Euclidean spaces (with dimensions n and m respectively), where X is the state space, while Y is the control space of the system. Let $x_0 \in X$ be given, the initial state of the system at the instant $t = 0$.

Consider a fixed nonempty convex, compact subset $U \subset Y$, the control set of the system. Then the set of admissible controls is given by

$$\mathcal{U} = \left\{ u \in L^2[0, T] : u(t) \in U \text{ a. e. } t \in [0, T] \right\}$$

Consider a continuously differentiable function $f : X \times Y \to X$. By a control system we mean the system of differential equations

$$x'(t) = f(x(t), u(t)) \quad x(0) = x_0 \quad \text{and} \quad u \in \mathcal{U} \tag{1}$$

for a. e. $t \in [0, T]$ and starting from the initial state x_0.

✉ Peter Tallos
tallosp@gmail.com

[1] Department of Mathematics, Corvinus University of Budapest, Budapest, Hungary

Published online: 24 March 2023

Consider the control system (1) and introduce the following notations. Let $AC[0, T]$ denote the space of all absolutely continuous functions on $[0, T]$, moreover

$$S_U(x_0) = \{x \in AC[0, T] : \exists u \in \mathcal{U},\ x \text{ is a solution to the system}(1)\}$$

which is the collection of all trajectories of the system, and

$$A_U(t, x_0) = \{x(t) : x \in S_U(x_0)\}$$

which is the set of all states attainable from the initial state x_0 at time $t \in [0, T]$.

In this paper we address the following question. How can we find a tighter control set $V \subset U$ with the following properties:

- The attainable set remains unchanged, i.e. $A_V(t, x_0) = A_U(t, x_0)$ for every $t \in [0, T]$, moreover.
- The set of trajectories remains essentially unchanged, i.e. $S_V(x_0)$ is dense in the set $S_U(x_0)$ (with respect to the C-norm).

2 Linear systems

The above question admits a known solution in the case of linear systems.

In particular, let A and B $n \times n$ and $n \times m$ matrices respectively, and consider the linear system

$$x'(t) = Ax(t) + Bu(t) \qquad x(0) = x_0 \quad \text{and} \quad u \in \mathcal{U} \tag{2}$$

for a. e. t in $[0, T]$. Denote by ex U the set of extremal points of U, then we have

$$\text{ex}\,\mathcal{U} = \left\{u \in L^2[0, T] : u(t) \in \text{ex}\,U \text{ a. e. } t \in [0, T]\right\} \tag{3}$$

where on the left-hand side extremal points are meant in the vector space $L^2[0, T]$. The elements of this set are called *extremal controls* of the system.

One direction of equality (3) is simple, the other direction is far from being trivial. In fact, it is the consequence of the so-called *measurable selection theorem* due to Kuratowski and Ryll-Nardzewski (1965), see Theorem 8.1.3 in Aubin and Frankowska (1992), or Theorem 4.7 in Kánnai et al. (2014). This theorem states that a measurable set-valued map with nonempty closed values possesses a measurable selection.

Equality (3) allows us to formulate the following theorem, called the "Bang-bang principle".

Theorem 1 *Introduce the notation $V = \text{ex}\,U$. Then for the linear system (2) we have*

$$A_V(t, x_0) = A_U(t, x_0) \quad \forall t \in [0, T]$$

and $S_V(x_0)$ is a dense subset of $S_U(x_0)$ with respect to the C-norm.

The Bang-bang principle is due to LaSalle and Olech in the early sixties of the last century, for a comprehensive treatment we refer to (Kánnai et al. 2014). Basically, the proof relies on the following facts:

- Solutions to linear systems can be expressed explicitely by the Cauchy-formula,
- Lyapunov's theorem on the convexity of the range of vector measures.

3 Nonlinear systems

For nonlinear systems there is no such result like the Bang-bang principle. For example, if $U = [-1, 1]$ and $x_0 = 0$, then the following simple one dimensional system

$$x'(t) = u(t)^2$$

is kept in the origin by the control $u = 0$. However, this obviously cannot be done by extremal controls.

On the other hand, now consider the linear system

$$x'(t) = u(t)$$

with the same control set and initial state, and put $V = \operatorname{ex} U$. It is easy to see that $S_V(x_0) = S_U(x_0)$ is not true, since for the solution of the original system $x = 0$ we clearly have $0 \notin S_V(x_0)$. However, the constant zero function can be arbitrarily approximated by piecewise straight lines with alternating slopes of -1 and $+1$, and hence $S_V(x_0)$ is a dense subset of $S_U(x_0)$ with respect to the C-norm. This example tells us that equality of the solution sets cannot be expected even in the case of linear systems.

4 Set-valued approach

A possible approach is the following. We do not consider the control functions explicitly, only their collection is important from the problem's point of view. Therefore, we introduce the set-valued mapping F this way:

$$F(x) = \{f(x, u) : \; u \in U\}$$

from X into the subsets of X and consider the relation

$$x'(t) \in F(x(t)) \quad x(0) = x_0 \tag{4}$$

which is called a *differential inclusion*. An absolutely continuous function $x \in AC[0, T]$ is said to be a solution, if this relation is fulfilled almost everywhere. The notations $S_F(x_0)$ and $A_F(t, x_0)$ are interpreted analogously.

Theorem 2 *Under the conditions above the solution sets of the differential inclusion* (4) *and the control system* (1) *coincide.*

One direction is trivial, the other is hard, but ultimately, it is again the consequence of the measurable selection theorem. This theorem is known as “Filippov’s implicit function lemma”, we refer to (Aubin and Frankowska 1992) for the proof and more details.

Consider a set-valued map F defined on X with nonempty compact values in X. We say that F is *locally Lipschitz-continuous*, if at every point $z \in X$ there exists an $\varepsilon > 0$ and a $\lambda > 0$ such that

$$D(F(x), F(y)) \leq \lambda \|x - y\|$$

for every $x, y \in z + \varepsilon B$. Here B is the unit ball in X, and D denotes the Hausdorff-distance.

The theorem below is called the Filippov-Wazewski relaxation theorem (see (Aubin and Frankowska 1992)). The notation $\mathrm{co}\, K$ stands for the convex hull of the set K.

Theorem 3 *Let us denote by $S_F(x_0)$ and $S_{\mathrm{co}\, F}(x_0)$ the solution sets for the set-valued maps F and* $\mathrm{co}\, F$ *respectively in the differential inclusion* (4). *Assume that F is locally Lipschitz-continuous with nonempty compact values in X. Then $S_{\mathrm{co}\, F}(x_0)$ is closed with respect to the C-norm, and*

$$\mathrm{cl}\, S_F(x_0) = S_{\mathrm{co}\, F}(x_0)$$

Moreover $A_F(t, x_0) = A_{\mathrm{co}\, F}(t, x_0)$ for every $t \in [0, T]$.

We may be tempted to think that this theorem answers our problem: it might be enough to focus on the extremal points of the set $f(x, U)$. However, extremal points have bad continuity properties: the extremal points of a Lipschitz-continuous map may fail even to be continuous.

5 A necessary condition

Now we examine the opposite question: if the attainable sets coincide, what can we say about the mappings on right-hand side of the differential inclusion?

The basic idea comes from the classical theory of ordinary differential equations. If F is a locally Lipschitz-continuous single valued map, then $d/dt\, A_F(0, x) = F(x)$. If we want to adapt this observation to set-valued systems, we should introduce the derivative of the set-valued map A_F. In classical analysis the derivative at a given point is the linear map, whose graph (a linear subspace) is tangent to the graph of the function.

For set-valued maps such a tangent space does not necessarily exist. However, the subspace can be replaced by the tangent cone. The tangent cone to a set $K \subset X$ at point $x \in K$ is defined by

$$T_K(x) = \left\{ v \in X : \liminf_{h \to 0+} \frac{1}{h} d_K(x + hv) = 0 \right\}$$

where $d_K(y)$ is the distance of the point y from the set K.

Definition 1 The *derivative* of the set-valued map A_F at the point $(t, x) \in \operatorname{graph} A_F$ is the set-valued map DA_F whose graph is the tangent cone to the set graph A_F at the point (t, x). In other words

$$\operatorname{graph} DA_F(t, x) = T_{\operatorname{graph} A_F}(t, x)$$

The proof of the following theorem can be found in Joó and Tallos (1999).

Theorem 4 *If F is locally Lipschitz-continuous with nonempty compact values, then*

$$F(x) \subset DA_F(0, x) \subset \operatorname{co} F(x)$$

for every $x \in X$.

The following proposition can be verified by a straightforward calculation.

Lemma 1 *Let F and G be compact valued maps, and assume that $A_G(t)$ is a dense subset of $A_F(t)$ for every $t \in [0, T]$. Then*

$$DA_G(0, x) = DA_F(0, x)$$

for each $x \in X$.

As a consequence of our results we formulate an extension of the Relaxation theorem.

Theorem 5 *Let F and G be locally Lipschitz-continuous set-valued maps, and assume that F has nonempty convex, compact values, while G has compact values with $G(x) \subset F(x)$ for every $x \in X$. Then $S_G(x)$ is a dense subset of $S_F(x)$ (with respect to the C-norm) if and only if*

$$\operatorname{co} G(x) = F(x)$$

at each point $x \in X$.

Proof In view of Theorem 4 and Lemma 1 we obtain

$$F(x) \subset DA_F(0, x) = DA_G(0, x) \subset \operatorname{co} G(x)$$

for every $x \in X$. This completes the proof. □

Basically, our theorem tells us that any tighter control set must contain all extremal points of the original control set. However, this condition is by far not sufficient. The problem of finding the appropriate conditions for the sufficiency is still open.

Acknowledgements The author is indebted to the anonymous referees for pointing out some flaws in the earlier version of the paper and making the text more accurate.

Funding Open access funding provided by Corvinus University of Budapest.

Declarations

Conflict of interest There is no conflict of interest regarding this manuscript.

References

Aubin J-P, Frankowska H (1992) Set valued analysis. Birkhäuser Verlag, Boston, Basel

Hu S, Papageorgiou N (1997) Handbook of multivalued analysis. Kluwer, Dordrecht, Boston, London

Joó I, Tallos P (1999) The Filippov–Wazewski relaxation theorem revisited. Acta Math Hung 83:277–283

Kánnai Z, Tallos P (1999) Differential inclusions, selections and economic modeling. Szigma 29:213–220

Kánnai Z, Szabó I, Tallos P (2014) Calculus of variations and optimal control (in Hungarian), tankonyvtar.hu, Typotex

Kánnai Z, Szabó I, Tallos P (2020) Set-valued techniques in dynamic economic models. In: Bischi GI, Szidarovszky F (eds) Games and dynamics in economics. Springer Nature, London, pp 195–203

Kuratowski K, Ryll-Nardzewski C (1965) A general theorem on selectors. Bull Acad Polon Sci 13:397–403

Tallos P (1991) Viability problems for nonautonomous differential inclusions. SIAM J Control Optim 29:253–263