



Unified Approach of Interior-Point Algorithms for $P_*(\kappa)$ -LCPs Using a New Class of Algebraically Equivalent Transformations

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Abstract

We propose new short-step interior-point algorithms (IPAs) for solving $P_*(\kappa)$ -linear complementarity problems (LCPs). In order to define the search directions, we use the algebraic equivalent transformation (AET) technique of the system describing the central path. A novelty of the paper is that we introduce a whole, new class of AET functions for which a unified complexity analysis of the IPAs is presented. This class of functions differs from the ones used in the literature for determining search directions, like the class of concave functions determined by Haddou, Migot and Omer, self-regular functions, eligible kernel and self-concordant functions. We prove that the IPAs using any member φ of the new class of AET functions have polynomial iteration complexity in the size of the problem, in starting point's duality gap, in the accuracy parameter and in the parameter κ .

Keywords Interior-point algorithm · $P_*(\kappa)$ -Linear complementarity problems · Algebraic equivalent transformation technique · New class of AET functions

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1 Introduction

LCPs have been extensively studied in recent years. Linear programming (LP) and linearly constrained (convex) quadratic programming (QP) problems are special cases of LCPs. Several applications of LCPs arise in different fields, such as engineering, computational mechanics, game theory, economics, see [5, 13]. It was shown that solvability of LCPs related to quitting games ensures the existence of different ε -equilibrium solutions, see [34]. Bimatrix games can be also formulated as LCPs, see [25]. The Arrow-Debreu competitive market equilibrium problem with linear and Leontief utility functions can be transformed to LCP [38]. For detailed study on LCPs, see the books of Cottle, Pang, Stone [5] and Kojima et al. [23]. In the book of Kojima et al. [23], the theory of IPAs for solving LCPs is highlighted.

LCPs belong to the class of NP-complete problems, see [4]. However, the properties of the problem's matrix have influence on the solvability of the LCPs. It is known that if the problem's matrix is skew-symmetric [33, 36, 37] or positive semidefinite [24], IPAs can find approximate solutions of LCPs in polynomial time. Cottle, Pang, and Venkateswaran [6] introduced the class of sufficient matrices. The class of $P_*(\kappa)$ -matrices was proposed by Kojima et al. [23]. If we consider the union of the sets $P_*(\kappa)$ for all nonnegative κ , we obtain the class P_* , see [23]. Väliäho [35] proved that the class of P_* -matrices is equivalent to the class of sufficient matrices. In general, IPAs for solving $P_*(\kappa)$ -LCPs have polynomial iteration complexity in the size of the problem, starting point's duality gap, in the accuracy parameter and in the special parameter $\kappa \geq 0$. However, De Klerk and E.-Nagy [12] showed that the handicap of the problem's matrix could be exponential in the bit length of the data. Furthermore, the complexity analyses of IPAs for $P_*(\kappa)$ -LCPs depend on the special parameter κ . In spite of this fact, there are computational results in the literature for LCPs with matrices having exponential value κ , where the iteration numbers are much better than predicted by the complexity results, see [8–10, 21]. The AET technique for defining search directions in case of IPAs for LP was introduced by Darvay, see [7]. He applied a continuously differentiable, monotone increasing function $\varphi : (\xi^2, \infty) \rightarrow \mathbb{R}$, where $0 \leq \xi < 1$, on the modified nonlinear equation of the system defining the central path. Darvay, Illés and Majoros [8] proposed a new short-step IPA for $P_*(\kappa)$ -LCPs by using the function $\varphi(t) = t - \sqrt{t}$ in the AET technique, and they also provided numerical results. In [9], a PC IPA for $P_*(\kappa)$ -LCPs was introduced where the function $\varphi(t) = t - \sqrt{t}$ was applied in the AET approach. The authors presented numerical results for problems with different sufficient matrices. Furthermore, it turned out that the algorithm is a promising tool to detect matrix copositivity, as well. In [10], a new type of AET technique and numerical results regarding this approach were presented. Kheirfam [21] proposed a new PC IPA for $P_*(\kappa)$ -LCPs, where he used the function $\varphi(t) = \sqrt{t}$ in the AET technique and he also provided numerical results.

An important aspect in the theory of IPAs is how we determine the search directions. Several approaches have been proposed in the literature. For example, there are methods that use barrier functions for defining search directions. Peng, Roos and Terlaky [31] considered self-regular functions, and in this way, they reduced the theoretical complexity of long-step IPAs. Beside these, Bai, Ghami and Roos [3] introduced the class of eligible kernel functions. Lešaja and Roos [26] gave a unified analysis

using eligible kernel functions for IPAs solving $P_*(\kappa)$ -LCPs. They proposed a general scheme to derive iteration bounds for the class of eligible kernel functions, see Theorem 6.3 in [26]. For several known specific eligible kernel functions, they calculated the iteration bounds of long-step and short-step algorithms. As mentioned before, Darvay [7] proposed the AET technique for defining search directions in case of IPAs for LP. In the literature, most of the IPAs do not use any transformation of the central path system; hence, these IPAs refer to the case when $\varphi(t) = t$ in the AET technique, see [16, 20]. Darvay [7] was the first who used the function $\varphi(t) = \sqrt{t}$ in the AET technique. In 2016, Darvay, Papp and Takács [11] considered the case when $\varphi(t) = t - \sqrt{t}$ and they proposed small-update IPA for LP using this search direction. Kheirfam and Haghghi [22] introduced an IPA for $P_*(\kappa)$ -LCPs which applies the function $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ in the AET technique. Later on, Haddou, Migot and Omer [15] proposed a class of concave functions in order to determine search directions in case of IPAs for solving monotone LCPs. It should be mentioned that they used a different type of transformation of the central path system. IPAs using the AET approach for determining search directions have been also extended to LCPs, see [2, 8, 10, 21, 27]. Rigó [32] presented different IPAs using the AET technique for LP, sufficient LCPs and symmetric cone optimization. Furthermore, a comparison between the AET approach and other methods for determining search directions is also provided in [32].

The aim of this paper is to introduce a new class of AET functions and to give a unified analysis of IPAs for $P_*(\kappa)$ -LCPs for the new class of AETs. We compare the new class of functions to the class of concave functions given by Haddou, Migot, Omer [15] and to other AET functions used in this approach. We analyze the relationship of the corresponding kernel functions belonging to this new class of AET functions to the class of eligible kernel functions. We give an example for a function belonging to our new class of AET functions, whose corresponding kernel function is neither eligible, nor self-regular kernel, nor self-concordant function. We also present a kernel function, for which the corresponding AET function is not a member of our class.

The paper is organized in the following way. In Sect. 2, we present several results related to the theory of $P_*(\kappa)$ -LCPs, the classical AET approach. We also introduce a new class of AET functions used in this paper. We compare the proposed class of functions to other techniques for determining search directions. Section 3 is devoted to the introduction of the new class of short-step IPAs and to the complexity analysis of the IPAs that are based on the new class of AET functions. Furthermore, in Sect. 4 some concluding remarks and further research topics are enumerated.

2 A New Class of AET Functions for Interior-Point Algorithms Solving $P_*(\kappa)$ -Linear Complementarity Problems

The aim of the LCPs is to find vectors $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$, that satisfy the following constraints:

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \quad \mathbf{x}, \mathbf{s} \geq \mathbf{0}, \quad \mathbf{x}\mathbf{s} = \mathbf{0}, \quad (LCP)$$

where $M \in \mathbb{R}^{n \times n}$, $\mathbf{q} \in \mathbb{R}^n$ and $\mathbf{x}\mathbf{s}$ is the componentwise product of vectors \mathbf{x} and \mathbf{s} . The feasible region, its interior and the solution set of the LCP are given as follows:

$$\begin{aligned} \mathcal{F} &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}_{\oplus}^n \times \mathbb{R}_{\oplus}^n : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \\ \mathcal{F}^+ &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \\ \mathcal{F}^* &:= \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F} : \mathbf{x}\mathbf{s} = \mathbf{0}\}. \end{aligned}$$

Note that \mathbb{R}_{\oplus}^n denotes the n -dimensional nonnegative orthant and \mathbb{R}_+^n the positive orthant, respectively.

In the first part of this section, we present some basic concepts related to the theory of $P_*(\kappa)$ -LCPs and $P_*(\kappa)$ -matrices.

2.1 $P_*(\kappa)$ -Matrices

Cottle, Pang and Venkateswaran [6] introduced the class of *sufficient matrices*.

Definition 2.1 (Cottle, Pang and Venkateswaran [6]) A matrix $M \in \mathbb{R}^{n \times n}$ is a *column sufficient matrix* if for all $\mathbf{x} \in \mathbb{R}^n$

$$X(M\mathbf{x}) \leq 0 \text{ implies } X(M\mathbf{x}) = 0,$$

where $X = \text{diag}(\mathbf{x})$. Analogously, a matrix M is *row sufficient* if M^T is column sufficient. The matrix M is *sufficient* if it is both row and column sufficient.

Kojima et al. [23] proposed the notion of $P_*(\kappa)$ -matrices.

Definition 2.2 (Kojima et al. [23]) Let $\kappa \geq 0$ be a nonnegative real number. A matrix $M \in \mathbb{R}^{n \times n}$ is a $P_*(\kappa)$ -matrix if

$$(1 + 4\kappa) \sum_{i \in I_+(\mathbf{x})} x_i(M\mathbf{x})_i + \sum_{i \in I_-(\mathbf{x})} x_i(M\mathbf{x})_i \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

where

$$I_+(\mathbf{x}) = \{1 \leq i \leq n : x_i(M\mathbf{x})_i > 0\} \text{ and } I_-(\mathbf{x}) = \{1 \leq i \leq n : x_i(M\mathbf{x})_i < 0\}.$$

A problem is called $P_*(\kappa)$ -LCP if the problem's matrix is a $P_*(\kappa)$ -matrix. Throughout the paper, we assume that $\mathcal{F}^+ \neq \emptyset$ and M is a $P_*(\kappa)$ -matrix. Hence, we are dealing with $P_*(\kappa)$ -LCPs. The handicap of a matrix M [35] is defined by $\hat{\kappa}(M) := \inf\{\kappa \mid M \in P_*(\kappa)\}$.

Definition 2.3 (Kojima et al. [23]) A matrix $M \in \mathbb{R}^{n \times n}$ is a P_* -matrix if it is a $P_*(\kappa)$ -matrix for some $\kappa \geq 0$.

Let $P_*(\kappa)$ denote the set of $P_*(\kappa)$ -matrices. Analogously, we also use P_* to denote the set of all P_* -matrices, i.e.,

$$P_* = \bigcup_{\kappa \geq 0} P_*(\kappa).$$

Kojima et al. [23] proved that a P_* -matrix is column sufficient and Guu and Cottle [14] showed that it is row sufficient, too. This means that each P_* -matrix is sufficient. Väliäho [35] demonstrated the other inclusion, as well, proving that the class of P_* -matrices is the same as the class of sufficient matrices.

2.2 Algebraic Equivalent Transformation Technique

The *central path problem* in this case is

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \quad \mathbf{x}, \mathbf{s} > \mathbf{0}, \quad \mathbf{x}\mathbf{s} = \mu\mathbf{e}, \tag{CPP}$$

where \mathbf{e} denotes the n -dimensional all-one vector and $\mu > 0$. Kojima et al. [23] proved that if M is a $P_*(\kappa)$ -matrix, then (CPP) has unique solution for every $\mu > 0$.

We present the AET technique in case of $P_*(\kappa)$ -LCPs. Let $\varphi : (\bar{\xi}, \infty) \rightarrow \mathbb{R}$, with $0 \leq \bar{\xi} < 1$, be a continuously differentiable and invertible function, such that $\varphi'(t) > 0 \forall t > \bar{\xi}$, see [7]. We use the notation $\varphi(\mathbf{x}) = [\varphi(x_1), \varphi(x_2) \dots, \varphi(x_n)]^T$. System (CPP) can be written as

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \quad \mathbf{x}, \mathbf{s} > \mathbf{0}, \quad \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right) = \varphi(\mathbf{e}). \tag{CPP_\varphi}$$

Applying Newton’s method, we obtain the following system, see [9]:

$$\begin{aligned} -M\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0}, \\ S\Delta\mathbf{x} + X\Delta\mathbf{s} &= \mathbf{a}_\varphi, \end{aligned} \tag{3}$$

where

$$\mathbf{a}_\varphi = \mu \frac{\varphi(\mathbf{e}) - \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)}{\varphi'\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)} \tag{4}$$

and X, S are the diagonal matrices formed by the components of the vectors \mathbf{x} and \mathbf{s} . System (3) has unique solution, see Lemma 4.1 in [23].

Scaling plays important role in the theory of IPAs. Consider the following notations:

$$\mathbf{v} = \sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}, \quad \mathbf{d} = \sqrt{\frac{\mathbf{x}}{\mathbf{s}}}, \quad \mathbf{d}_x = \frac{\mathbf{d}^{-1}\Delta\mathbf{x}}{\sqrt{\mu}} = \frac{\mathbf{v}\Delta\mathbf{x}}{\mathbf{x}}, \quad \mathbf{d}_s = \frac{\mathbf{d}\Delta\mathbf{s}}{\sqrt{\mu}} = \frac{\mathbf{v}\Delta\mathbf{s}}{\mathbf{s}},$$

where all operations are understood componentwise. From these, we obtain

$$\Delta\mathbf{x} = \frac{\mathbf{x}\mathbf{d}_x}{\mathbf{v}} \quad \text{and} \quad \Delta\mathbf{s} = \frac{\mathbf{s}\mathbf{d}_s}{\mathbf{v}}. \tag{5}$$

Table 1 AET functions used in the literature

| φ | \mathbf{a}_φ | \mathbf{p}_v |
|---|---|---|
| $\varphi(t) = t$ | $\mu\mathbf{e} - \mathbf{x}\mathbf{s}$ | $\mathbf{v}^{-1} - \mathbf{v}$ |
| $\varphi(t) = \sqrt{t}$ | $2(\sqrt{\mu\mathbf{x}\mathbf{s}} - \mathbf{x}\mathbf{s})$ | $2(\mathbf{e} - \mathbf{v})$ |
| $\varphi(t) = t - \sqrt{t}$ | $\frac{\sqrt{\mu\mathbf{x}\mathbf{s}}}{2\sqrt{\mathbf{x}\mathbf{s}} - \sqrt{\mu\mathbf{e}}} - \mathbf{x}\mathbf{s}$ | $\frac{2(\mathbf{v} - \mathbf{v}^2)}{2\mathbf{v} - \mathbf{e}}$ |
| $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ | $\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}(\mu\mathbf{e} - \mathbf{x}\mathbf{s})$ | $\mathbf{e} - \mathbf{v}^2$ |

After substituting these into (3), we obtain the scaled system:

$$\begin{aligned}
 -\bar{M}\mathbf{d}_x + \mathbf{d}_s &= \mathbf{0}, \\
 \mathbf{d}_x + \mathbf{d}_s &= \mathbf{p}_v,
 \end{aligned}
 \tag{6}$$

where $\bar{M} = DMD$, $D = \text{diag}(\mathbf{d})$ and

$$\mathbf{p}_v = \frac{\varphi(\mathbf{e}) - \varphi(\mathbf{v}^2)}{\mathbf{v}\varphi'(\mathbf{v}^2)}.
 \tag{7}$$

Table 1 contains the classical AET functions used in the literature and the corresponding vectors \mathbf{a}_φ and \mathbf{p}_v .

Haddou, Migot and Omer [15] proposed a class of smooth concave functions for monotone LCPs. However, it should be mentioned that they used a different type of transformation of the central path system. They used functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that satisfy the following conditions

- H1. $\varphi(0) = 0$;
- H2. $\varphi \in C^3([0, +\infty))$;
- H3. $\varphi'(t) > 0 \forall t \geq 0$;
- H4. $\varphi''(t) \leq 0 \forall t \geq 0$;
- H5. $\varphi'''(t) \geq 0 \forall t \geq 0$.

It should be mentioned that conditions H1-H5 are satisfied only in case of $\varphi(t) = t$ from Table 1. In the following subsection, we introduce the new class of AET functions used in this paper.

2.3 A New Class of AET Functions

We present the new class of AET functions which will be used in order to determine search directions.

Definition 2.4 Let $\varphi : (\xi, \infty) \rightarrow \mathbb{R}$ be a continuously differentiable, invertible function, such that $\varphi'(t) > 0 \forall t > \xi$, where $0 \leq \xi < 1$. All functions φ satisfying the conditions

AET 1. $\exists c_1 \in \mathbb{R}_+$, such that

$$\left| \frac{\varphi(1) - \varphi(t^2)}{2t(1 - t^2)\varphi'(t^2)} \right| \leq c_1,$$

for all $t > \xi$.

AET 2. $\exists c_2 \in \mathbb{R}_+$, such that

$$\left| \frac{4t^2\varphi'(t^2) [(1 - t^2)\varphi'(t^2) - \varphi(1) + \varphi(t^2)]}{(\varphi(1) - \varphi(t^2))^2} \right| \leq c_2,$$

for all $t > \xi$.

AET 3. $\exists c_3 \in \mathbb{R}_+$ such that the inequality

$$\begin{aligned} 4t^2(\varphi(1) - \varphi(t^2))\varphi'(t^2) - c_3 (\varphi(1) - \varphi(t^2))^2 &\leq 4t^2(1 - t^2) (\varphi'(t^2))^2 \\ &\leq 4t^2(\varphi(1) - \varphi(t^2))\varphi'(t^2) + (\varphi(1) - \varphi(t^2))^2 \end{aligned}$$

holds for all $t > \xi$,

belong to the new class of AET functions.

It should be mentioned that the boundedness of the functions appearing in conditions AET1 and AET2 shows that the point $t = 1$ and the point(s) where $\varphi(t^2) = \varphi(1)$ are removable discontinuities of the functions given in AET1 and AET2, respectively.

Remark 2.1 Condition AET1 can be rewritten in the following form:

$$\left| \frac{\frac{\varphi(1) - \varphi(t^2)}{2t\varphi'(t^2)}}{1 - t^2} \right| \leq c_1.$$

Conditions AET2 and AET3 can be rewritten in a similar way, and these forms show that the functions belonging to this class of AET functions have transformations with growth proportional to the identical function. Hence, we may call these functions *AET functions with bounded proportional growth rate*.

Let us introduce the following function: $f : (\xi, \infty) \rightarrow \mathbb{R}$:

$$f(t) := \frac{\varphi(1) - \varphi(t^2)}{t\varphi'(t^2)}. \tag{8}$$

By using the function given in (8), we can give the definition of the new class of AET functions in the following way.

Table 2 Examples for φ belonging to the new class of AET functions

| $\varphi(t)$ | ξ | c_1 | c_2 | c_3 |
|----------------------|-------|-------|-------|-------|
| t | 0.25 | 2 | 6 | 1 |
| \sqrt{t} | 0 | 2 | 6 | 1 |
| $t - \sqrt{t}$ | 0.7 | 2 | 6 | 1 |
| $t^2 - t + \sqrt{t}$ | 0 | 2 | 8 | 8 |
| $t^2 + \sqrt{t}$ | 0 | 2 | 6 | 8 |

Proposition 2.1 Let $\varphi : (\xi, \infty) \rightarrow \mathbb{R}$ be a continuously differentiable, invertible function, such that $\varphi'(t) > 0 \forall t > \xi$, where $0 \leq \xi < 1$. Consider the function f given in (8). The conditions given in Definition 2.4 can be formulated in the following equivalent form:

AET a. $\exists c_1 \in \mathbb{R}_+$, such that $g(t) = \frac{f(t)}{2(1-t^2)}$ and $|g(t)| \leq c_1$, holds for all $t > \xi$;

AET b. $\exists c_2 \in \mathbb{R}_+$, such that $h(t) = \frac{4(1-t^2-tf(t))}{f(t)^2} = \frac{1-2tg(t)}{(1-t^2)g(t)^2}$ and $|h(t)| \leq c_2$, holds for all $t > \xi$;

AET c. $\exists c_3 \in \mathbb{R}_+$ such that the inequality

$$tf(t) - c_3 \frac{f(t)^2}{4} \leq 1 - t^2 \leq tf(t) + \frac{f(t)^2}{4}$$

holds for all $t > \xi$.

Proof By using the function given in (8), after some calculations we obtain that conditions AET1-3 can be formulated as the ones given in AETa-c. □

Remark 2.2 The values of the parameters c_1, c_2 and c_3 will have influence on the well-definedness of the algorithm. For this, we will give a relation between these parameters.

Table 2 contains examples for φ belonging to this new class of functions, the value of ξ and the values c_1, c_2 and c_3 . From Table 1, we can see that almost all AET functions used in the literature belong to our new class of AET, except the function $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$, where condition AET3 is not satisfied. This means that several short-step IPAs proposed in the literature (see for example [2, 8]) are special cases of our unified approach. The values given in Table 2 will be clear in the second part of the paper when we study the well-definedness of the algorithm. It should be mentioned that for the given values from Table 2, the introduced short-step IPAs are well defined and the complexity analyses work. However, there are several other acceptable values for these parameters.

Table 2 shows that most of the functions used in the literature from Table 1 belong to the new class of AET functions. However, it should be mentioned that the intervals on which the functions φ are defined play important role in this approach. For example, $\varphi(t) = t$ is only a member of this new class of AET functions if it is defined on a (ξ, ∞) interval, where ξ is strictly positive. If ξ would be zero, then condition AET1

would not be satisfied for this function. Similar remark can be formulated in case of $\varphi(t) = t - \sqrt{t}$.

We can compare our new class of AET functions to the class of concave functions proposed by Haddou, Migot and Omer [15].

Example 2.1 Consider the function $\varphi(t) = \log(1+t)$, a member of the class of concave functions introduced by Haddou, Migot and Omer [15]. By using Definition 2.4, we can check that for this function condition AET3 is not satisfied.

In the following subsection, we present the class of eligible kernel functions proposed in [3] and the relationship between the kernel function approach and the AET technique.

2.4 Eligible Kernel Functions

The determination of the search directions in case of IPAs can be realized by using kernel functions. It needs to be mentioned that in the literature we can find several properties and definitions of kernel functions. We require that the kernel function ψ has properties similar to those of the logarithmic kernel function. These properties are given in Definition 2.5.

Definition 2.5 A function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_\oplus$ is called *kernel function* if it is twice continuously differentiable and if the following conditions hold:

- K1. $\psi(1) = \psi'(1) = 0$;
- K2. $\psi''(t) > 0$, for all $t > 0$;
- K3. $\lim_{t \downarrow 0} \psi(t) = \lim_{t \rightarrow \infty} \psi(t) = \infty$.

Remark 2.3 Condition K3. is used to define the notion of coercive kernel function, see [3].

We can construct a barrier function $\Psi : \mathbb{R}_+^n \rightarrow \mathbb{R}$ in the following form:

$$\Psi(\mathbf{v}) := \sum_{i=1}^n \psi(v_i),$$

where $\mathbf{v} \in \mathbb{R}_+^n$.

Peng, Roos and Terlaky [31] modified the second equation of the scaled system to

$$\mathbf{d}_x + \mathbf{d}_s = -\nabla \Psi(\mathbf{v}).$$

Using this and the scaled system (6), we have

$$\mathbf{d}_x + \mathbf{d}_s = -\nabla \Psi(\mathbf{v}) = \mathbf{p}_v = \frac{\varphi(\mathbf{e}) - \varphi(\mathbf{v}^2)}{\mathbf{v}\varphi'(\mathbf{v}^2)}. \tag{9}$$

Hence, we can assign a corresponding kernel function to several functions φ appeared in the AET technique in the following way, see [30]:

$$\psi(t) = \int_1^t \frac{\varphi(\bar{\tau}^2) - \varphi(1)}{\bar{\tau}\varphi'(\bar{\tau}^2)} d\bar{\tau}, \quad (10)$$

where the function ψ satisfies the properties *K1.-K3*.

Peng, Roos and Terlaky [31] considered self-regular functions, and in this way, they reduced the theoretical complexity of long-step IPAs. The definition of self-regular functions is given below.

Definition 2.6 (Peng, Roos and Terlaky [31]) A function $\psi : (0, \infty) \rightarrow \mathbb{R}$, $\psi \in \mathcal{C}^2$ is *self-regular* if it satisfies the conditions

SR1. $\psi(t)$ is strictly convex with respect to $t > 0$ and $\psi(t) = 0$ at its global minimal point $t = 1$, i.e., $\psi(1) = \psi'(1) = 0$. Further, there exist positive constants $v_2 \geq v_1 > 0$ and $p \geq 1$, $q \geq 1$ such that

$$v_1 \left(t^{p-1} + t^{-1-q} \right) \leq \psi''(t) \leq v_2 \left(t^{p-1} + t^{-1-q} \right) \quad \forall t \in (0, \infty); \quad (11)$$

SR2. For any $t_1, t_2 > 0$,

$$\psi \left(t_1^r t_2^{1-r} \right) \leq r\psi(t_1) + (1-r)\psi(t_2) \quad \forall r \in [0, 1]. \quad (12)$$

The prototype self-regular kernel function is given by

$$\Upsilon_{p,q}(t) = \frac{t^{p+1} - 1}{p(p+1)} + \frac{t^{1-q} - 1}{q(q-1)} + \frac{p-q}{pq}(t-1), \quad (13)$$

where $p \geq 1$ and $q > 1$.

Bai, Ghami and Roos [3] defined the class of *eligible kernel functions*.

Definition 2.7 (Bai, Ghami and Roos [3]) We call a kernel function *eligible kernel function* if it satisfies the following conditions:

- EK1. $t\psi''(t) + \psi'(t) > 0 \quad \forall t < 1$;
- EK2. $\psi'''(t) < 0 \quad \forall t > 0$;
- EK3. $2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0 \quad \forall t < 1$;
- EK4. $\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0 \quad \forall t > 1, \beta > 1$.

Note that the class of eligible kernel functions contains some self-regular functions, as well as many non-self-regular functions as special cases, see [26]. However, it should be mentioned that self-regular kernel functions $\Upsilon_{p,q}(t)$ with growth $p \leq 1$ belong to the class of eligible kernel functions.

Remark 2.4 By using (8) and (10), we have

$$\psi'(t) = -f(t). \quad (14)$$

From (14) and Proposition 2.1, we obtain that if we have a kernel function, then using AETa-c we can check whether the corresponding function(s) φ do(es) belong to the new class of AET without calculating the functions φ . Furthermore, if we have a function φ , then by using the conditions given in Proposition 2.1 we can check whether the conditions given in EK1-4 and K1-3 hold for the corresponding kernel function. Hence, (14) gives the opportunity to compare AET functions to corresponding kernel functions.

Example 2.2 The function $\psi(t) = \frac{1}{2} \left(t - \frac{1}{t}\right)^2$ is an eligible kernel function, see [26]. It is also a self-regular kernel function. From (10), we have

$$f(t) = \frac{\varphi(1) - \varphi(t^2)}{t\varphi'(t^2)} = -\psi'(t) = \frac{1}{t^3} - t.$$

This leads to

$$\varphi(1) - \varphi(t^2) = \left(\frac{1}{t^2} - t^2\right) \varphi'(t^2);$$

hence, we get the following differential equation

$$\varphi'(t^2) = \frac{t^2}{1 - t^4} \left(\varphi(1) - \varphi(t^2)\right). \quad (15)$$

Easy computations show that $f(t) = \frac{1}{t^3} - t$ does not satisfy condition AETb from Proposition 2.1. Thus, none of the solutions of (15) belongs to the new class of AET functions. This discussion shows that in many cases we do not need to know φ , because Proposition 2.1 gives the opportunity to check whether a corresponding AET function is member of the new class or not.

2.5 Self-Concordant Functions

The theory of self-concordant functions was developed primarily by Nesterov and Nemirovskii [29]. They realized that among the different properties of the logarithmic barrier for a polytope, only two are responsible for the polynomiality of the path-following methods associated with this polytope. One is the self-concordance of the barrier function, and the other is the finiteness of the barrier parameter. The notion of self-concordant function can be found in [29].

Definition 2.8 Let Q be a nonempty open convex set in \mathbb{R}^n and F be a C^3 smooth convex function defined on Q . F is called self-concordant on Q , if it possesses the following two properties:

- (SC1) $F(x_i) \rightarrow \infty$ along every sequence $\{x_i \in Q\}$ converging, as $i \rightarrow \infty$, to a boundary point of Q ;
- (SC2) F satisfies the inequality $|D^3 F(x)[h, h, h]| \leq 2 (D^2 F(x)[h, h])^{\frac{3}{2}} \forall x \in Q$ and $h \in \mathbb{R}^n$.

Thus, a one-dimensional convex barrier function Ψ is called self-concordant if

$$|\psi'''(t)| \leq 2\psi''(t)^{\frac{3}{2}}, \tag{16}$$

for every t in the interior of the function's domain.

By taking into consideration (14), we can write this condition using the AET approach:

$$|f''(t)| \leq 2(-f'(t))^{\frac{3}{2}}. \tag{17}$$

Example 2.3 The kernel functions corresponding to the functions $\varphi(t) = t, \varphi(t) = \sqrt{t}$ are self-concordant functions. However, we have seen that the new class of AET functions contains functions having inflection points, as well. In this case, the corresponding kernel function will not be convex. This means, that in case of $\varphi(t) = t^2 - t + \sqrt{t}$ the corresponding kernel function is not self-concordant.

We showed example for functions that belong to other classes of functions, but do not belong to the new class of AET functions. In the following proposition, we summarize some important properties of two AET functions belonging to the new class that do not belong to other existing classes of functions.

Proposition 2.2 Consider the functions $\varphi_1, \varphi_2 : (0, \infty) \rightarrow \mathbb{R}, \varphi_1(t) = t^2 - t + \sqrt{t}$ and $\varphi_2(t) = t^2 + \sqrt{t}$. Then,

1. Both φ_1 and φ_2 have an inflection point in $t = \frac{1}{4}$.
2. The functions φ_1, φ_2 belong to the new class of AET functions given in Definition 2.4.
3. The functions φ_1 and φ_2 do not belong to the class of concave functions proposed by Haddou, Migot and Omer [15].
4. The kernel functions ψ_1 and ψ_2 corresponding to these AET functions are neither eligible, nor self-regular kernel, nor self-concordant functions.

Proof 1. Elementary analyses give the result.

2. In case of $\varphi_1(t) = t^2 - t + \sqrt{t}$, we have

$$f_1(t) = \frac{2(1 - t^4 + t^2 - t)}{4t^3 - 2t + 1}, \tag{18}$$

and in case of $\varphi_2(t) = t^2 + \sqrt{t}$ we get

$$f_2(t) = \frac{4 - 2t^4 - 2t}{4t^3 + 1}. \tag{19}$$

After some calculations, it can be proven that conditions AET1-3. of Definitions 2.4 are satisfied for these AET functions.

3. It follows from 1.
4. Functions ψ_1 and ψ_2 are not convex on \mathbb{R}_+ .

□

2.6 Recent Trends in Unified Approaches for AET Technique

M. E.-Nagy and A. Varga [28] presented on a conference a family of Ai-Zhang type IPAs for sufficient LCPs. Their class of functions is influenced by the wide neighborhood, Ai-Zhang type search directions. Our new class of AET functions works for small-update IPAs that have a different type of complexity analysis. Hence, up to our best knowledge, these two classes of functions differ and the symmetric difference of these two classes is nonempty.

In the following subsection, we present short-step IPAs for solving $P_*(\kappa)$ -LCPs, that use the new class of functions in the AET technique to determine search directions.

3 New Short-Step Interior-Point Algorithms for Solving $P_*(\kappa)$ -Linear Complementarity Problems

Firstly, we deal with the determination of the search directions. For this, we consider system (3), where \mathbf{a}_φ depends on the function φ used, which is a member of the class of new AET functions.

We define the centrality measure $\delta : \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{\infty\}$ as

$$\delta(\mathbf{x}, \mathbf{s}; \mu) := \delta(\mathbf{v}) = \frac{\|\mathbf{p}_v\|}{2}, \tag{20}$$

where $\|\cdot\|$ denotes the standard Euclidean norm, and \mathbf{p}_v is given in (7). Consider the τ -neighborhood of a fixed point of the central path as

$$\mathcal{N}_2(\tau, \mu) := \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \delta(\mathbf{x}, \mathbf{s}; \mu) \leq \tau\}, \tag{21}$$

where $\delta(\mathbf{x}, \mathbf{s}; \mu)$ is given in (20), $\mu > 0$ is fixed and τ is a threshold parameter.

In Algorithm 3.1, we define a whole class of IPAs for solving $P_*(\kappa)$ -LCPs, which is based on our new class of AET functions.

Algorithm 3.1 : Short-step IPAs for $P_*(\kappa)$ -LCPs based on a new class of AET functions

Let $\epsilon > 0$ be the accuracy parameter, $0 < \theta < 1$ be the update parameter and τ be the proximity parameter. Furthermore, a known upper bound κ of the handicap $\hat{\kappa}(M)$ is given. Assume that for $(\mathbf{x}^0, \mathbf{s}^0)$ we have $(\mathbf{x}^0)^T \mathbf{s}^0 = n\mu^0$ with $\mu^0 > 0$, such that $\delta(\mathbf{x}^0, \mathbf{s}^0; \mu^0) \leq \tau$. Suppose that $\mathbf{v}^0 > \xi \mathbf{e}$.

begin

$k := 0;$

while $(\mathbf{x}^k)^T \mathbf{s}^k > \epsilon$ **do**

begin

(determination of the search directions)

compute $(\Delta x^k, \Delta s^k)$ from (3) with φ from the new class of AETs;

let $\mathbf{x}^k := \mathbf{x}^k + \Delta x^k$ and $\mathbf{s}^k := \mathbf{s}^k + \Delta s^k;$

(update of the parameter μ)

$\mu^{k+1} := (1 - \theta) \mu^k;$

$k := k + 1;$

end

end

Remark 3.1 The default values of the parameters τ and θ will be given later in the complexity analysis of the algorithms.

In the following section, we present the complexity analysis of the IPAs using the new class of AET functions defined by conditions AET1-3 of Definition 2.4.

3.1 Complexity Analysis of the Short-Step IPAs

The first lemma shows the strict feasibility of the full-Newton step. We denote by $\mathbf{x}^+ = \mathbf{x} + \Delta \mathbf{x}$ and $\mathbf{s}^+ = \mathbf{s} + \Delta \mathbf{s}$ the vectors obtained after a full-Newton step.

Lemma 3.1 Let $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+$ be given, such that $\delta(\mathbf{x}, \mathbf{s}; \mu) < \frac{1}{\sqrt{1+4\kappa}}$ and $\mathbf{v} > \xi \mathbf{e}$. Let φ be a function satisfying AET3 of Definition 2.4. Then, we have $(\mathbf{x}^+, \mathbf{s}^+) \in \mathcal{F}^+$.

Proof It should be mentioned that only the second inequality of AET3 should be satisfied in this lemma, namely:

$$4t^2(1 - t^2) \left(\varphi'(t^2) \right)^2 \leq 4t^2(\varphi(1) - \varphi(t^2))\varphi'(t^2) + \left(\varphi(1) - \varphi(t^2) \right)^2, \quad (22)$$

where $t > \xi$. For each $0 \leq \alpha \leq 1$, let us denote $\mathbf{x}(\alpha) = \mathbf{x} + \alpha \Delta \mathbf{x}$ and $\mathbf{s}(\alpha) = \mathbf{s} + \alpha \Delta \mathbf{s}$. We have

$$\begin{aligned} \mathbf{x}(\alpha)\mathbf{s}(\alpha) &= (\mathbf{x} + \alpha \Delta \mathbf{x})(\mathbf{s} + \alpha \Delta \mathbf{s}) = \mathbf{x}\mathbf{s} + \alpha(\mathbf{s}\Delta \mathbf{x} + \mathbf{x}\Delta \mathbf{s}) + \alpha^2 \Delta \mathbf{x}\Delta \mathbf{s} \\ &= \mu \mathbf{v}^2 + \mu \mathbf{v}\alpha(\mathbf{d}_x + \mathbf{d}_s) + \alpha^2 \mu \mathbf{d}_x \mathbf{d}_s \\ &= \mu((1 - \alpha)\mathbf{v}^2 + \alpha(\mathbf{v}^2 + \mathbf{v}\mathbf{p}_v + \alpha \mathbf{d}_x \mathbf{d}_s)). \end{aligned} \quad (23)$$

Our aim is to show that $\alpha(\mathbf{v}^2 + \mathbf{v}\mathbf{p}_v + \alpha \mathbf{d}_x \mathbf{d}_s) > 0$. We have $\mathbf{p}_v = \mathbf{d}_x + \mathbf{d}_s$ and using the notations $\mathbf{q}_v = \mathbf{d}_x - \mathbf{d}_s$ it can be seen that

$$\mathbf{d}_x \mathbf{d}_s = \frac{\mathbf{p}_v^2 - \mathbf{q}_v^2}{4}. \quad (24)$$

By using the definition of \mathbf{p}_v given in (7), we obtain that (22) is equivalent to

$$\mathbf{v}\mathbf{p}_v + \frac{\mathbf{p}_v^2}{4} \geq \mathbf{e} - \mathbf{v}^2, \tag{25}$$

which leads to

$$\mathbf{v}^2 + \mathbf{v}\mathbf{p}_v \geq \mathbf{e} - \frac{\mathbf{p}_v^2}{4}. \tag{26}$$

From (23), (24) and (26), we have

$$\begin{aligned} \frac{\mathbf{x}(\alpha)\mathbf{s}(\alpha)}{\mu} &= (1 - \alpha)\mathbf{v}^2 + \alpha \left(\mathbf{v}^2 + \mathbf{v}\mathbf{p}_v + \alpha \frac{\mathbf{p}_v^2}{4} - \alpha \frac{\mathbf{q}_v^2}{4} \right) \\ &\geq (1 - \alpha)\mathbf{v}^2 + \alpha \left(\mathbf{e} - (1 - \alpha) \frac{\mathbf{p}_v^2}{4} - \alpha \frac{\mathbf{q}_v^2}{4} \right). \end{aligned}$$

We have $\frac{\mathbf{x}(\alpha)\mathbf{s}(\alpha)}{\mu} > \mathbf{0}$, if $\left\| (1 - \alpha) \frac{\mathbf{p}_v^2}{4} - \alpha \frac{\mathbf{q}_v^2}{4} \right\|_\infty < 1$ holds.

By using the results given in [9], we obtain

$$\|\mathbf{q}_v\| \leq \sqrt{1 + 4\kappa} \|\mathbf{p}_v\| = 2\sqrt{1 + 4\kappa} \delta. \tag{27}$$

Then, we have

$$\begin{aligned} \left\| (1 - \alpha) \frac{\mathbf{p}_v^2}{4} - \alpha \frac{\mathbf{q}_v^2}{4} \right\|_\infty &\leq (1 - \alpha) \frac{\|\mathbf{p}_v\|^2}{4} + \alpha \frac{\|\mathbf{q}_v\|^2}{4} \\ &\leq (1 - \alpha) \frac{\|\mathbf{p}_v\|^2}{4} + \alpha(1 + 4\kappa) \frac{\|\mathbf{p}_v\|^2}{4} \\ &= (1 + 4\alpha\kappa)\delta^2 \leq (1 + 4\kappa)\delta^2. \end{aligned}$$

Thus, $\left\| (1 - \alpha) \frac{\mathbf{p}_v^2}{4} - \alpha \frac{\mathbf{q}_v^2}{4} \right\|_\infty < 1$ holds if we have $\delta(\mathbf{x}, \mathbf{s}; \mu) < \frac{1}{\sqrt{1 + 4\kappa}}$. Hence, we have that $\mathbf{x}(\alpha)\mathbf{s}(\alpha) > \mathbf{0}$ for each $0 \leq \alpha \leq 1$. Thus, $\mathbf{x}(\alpha)$ and $\mathbf{s}(\alpha)$ do not change sign on $0 \leq \alpha \leq 1$. From $\mathbf{x}(0) = \mathbf{x} > \mathbf{0}$ and $\mathbf{s}(0) = \mathbf{s} > \mathbf{0}$, we obtain that $\mathbf{x}(1) = \mathbf{x}^+ > \mathbf{0}$ and $\mathbf{s}(1) = \mathbf{s}^+ > \mathbf{0}$. \square

In the following lemma, we analyze the conditions under which the Newton-type iteration is quadratically convergent.

Lemma 3.2 *Let $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+$, $\mathbf{v} > \xi\mathbf{e}$ and $\bar{\mathbf{v}} = \sqrt{\frac{\mathbf{x}^+\mathbf{s}^+}{\mu}}$ be given and suppose that*

$\delta(\mathbf{x}, \mathbf{s}; \mu) < \sqrt{\frac{1 - \xi^2}{1 + 4\kappa}}$. Let φ satisfy conditions AET1-3 of Definition 2.4 with constants

c_1, c_2 and $c_3 \in \mathbb{R}_+$. Then, after a primal-dual Newton barrier step we have $\bar{\mathbf{v}} > \xi \mathbf{e}$ and

$$\delta(\mathbf{x}^+, \mathbf{s}^+; \mu) \leq c_1(c_2 + 2 + 4\kappa)\delta(\mathbf{x}, \mathbf{s}; \mu)^2.$$

Proof By using the proof of Lemma 5.6 in [1] and condition AETc of Proposition 2.1, we get

$$\min(\bar{\mathbf{v}}) \geq \sqrt{1 - (1 + 4\kappa)\delta^2}.$$

From this and $\delta < \sqrt{\frac{1-\xi^2}{1+4\kappa}}$, we obtain $\bar{\mathbf{v}} > \xi \mathbf{e}$. It should be noted that in the next part of the proof we will use the form AETa and AETb of Proposition 2.1. This and (20) yields

$$\delta(\mathbf{x}^+, \mathbf{s}^+; \mu) := \frac{\|\mathbf{p}\bar{\mathbf{v}}\|}{2} = \left\| (\mathbf{e} - \bar{\mathbf{v}}^2)g(\bar{\mathbf{v}}) \right\|, \tag{28}$$

where the function g is given in Proposition 2.1. By using the assumption that $\exists c_1 \in \mathbb{R}_+$, for which $|g(t)| \leq c_1$ for all $t > \xi$, we obtain

$$\delta(\mathbf{x}^+, \mathbf{s}^+; \mu) \leq c_1 \left\| \mathbf{e} - \bar{\mathbf{v}}^2 \right\|. \tag{29}$$

We know from (23) that

$$\left\| \mathbf{e} - \bar{\mathbf{v}}^2 \right\| = \left\| \mathbf{e} - \frac{\mathbf{x}^+\mathbf{s}^+}{\mu} \right\| = \left\| \mathbf{e} - \mathbf{v}^2 - \mathbf{v}\mathbf{p}_v - \frac{\mathbf{p}_v^2}{4} + \frac{\mathbf{q}_v^2}{4} \right\|. \tag{30}$$

From condition AETb of Proposition 2.1, we have

$$\mathbf{e} - \mathbf{v}^2 - \mathbf{v}\mathbf{p}_v = h(\mathbf{v})\frac{\mathbf{p}_v^2}{4}. \tag{31}$$

By using (27), (30) and (31), we obtain

$$\left\| \mathbf{e} - \bar{\mathbf{v}}^2 \right\| \leq \left\| \mathbf{e} - \mathbf{v}^2 - \mathbf{v}\mathbf{p}_v \right\| + \left\| \frac{\mathbf{p}_v^2}{4} \right\| + \left\| \frac{\mathbf{q}_v^2}{4} \right\| \leq (2 + c_2 + 4\kappa)\delta^2. \tag{32}$$

From (29) and (32), we have

$$\delta(\mathbf{x}^+, \mathbf{s}^+; \mu) \leq c_1(c_2 + 2 + 4\kappa)\delta(\mathbf{x}, \mathbf{s}; \mu)^2,$$

which proves the lemma. □

In the next lemma, we investigate the effect of the full-Newton step on the duality gap.

Lemma 3.3 Let $\delta := \delta(\mathbf{x}, \mathbf{s}; \mu)$ and suppose that the vectors \mathbf{x}^+ and \mathbf{s}^+ are obtained using a full-Newton step, thus $\mathbf{x}^+ = \mathbf{x} + \Delta\mathbf{x}$ and $\mathbf{s}^+ = \mathbf{s} + \Delta\mathbf{s}$. Let φ be a function satisfying AET3 of Definition 2.4 with $c_3 \in \mathbb{R}_+$. Then, we have

$$(\mathbf{x}^+)^T \mathbf{s}^+ \leq \mu \left(n + (c_3 + 1)\delta^2 \right).$$

Proof Note that only the first inequality of AET3 will be used in this proof, namely

$$\begin{aligned} 4t^2(\varphi(1) - \varphi(t^2))\varphi'(t^2) - c_3 \left(\varphi(1) - \varphi(t^2) \right)^2 \\ \leq 4t^2(1 - t^2) \left(\varphi'(t^2) \right)^2, \end{aligned} \tag{33}$$

where $t > \xi$. By using the definition of \mathbf{p}_v in (7), we get that (33) is equivalent to

$$\mathbf{v}\mathbf{p}_v - c_3 \frac{\mathbf{p}_v^2}{4} \leq \mathbf{e} - \mathbf{v}^2. \tag{34}$$

From (23) and (34), we obtain

$$\frac{1}{\mu} \mathbf{x}^+ \mathbf{s}^+ = \mathbf{v}^2 + \mathbf{v}\mathbf{p}_v + \mathbf{d}_x \mathbf{d}_s \leq \mathbf{e} + \frac{c_3}{4} \mathbf{p}_v^2 + \mathbf{d}_x \mathbf{d}_s.$$

After some calculations, we have

$$\begin{aligned} (\mathbf{x}^+)^T \mathbf{s}^+ &= \mathbf{e}^T (\mathbf{x}^+ \mathbf{s}^+) \leq \mu \left(\mathbf{e}^T \mathbf{e} + \frac{c_3}{4} \mathbf{e}^T \mathbf{p}_v^2 + \mathbf{e}^T \mathbf{d}_x \mathbf{d}_s \right) \\ &= \mu \left(n + \frac{c_3}{4} \|\mathbf{p}_v\|^2 + \mathbf{d}_x^T \mathbf{d}_s \right) \\ &\leq \mu \left(n + c_3 \delta^2 + \delta^2 \right) = \mu \left(n + (c_3 + 1)\delta^2 \right). \end{aligned}$$

The last inequality holds, since

$$\mathbf{d}_x^T \mathbf{d}_s = \frac{\|\mathbf{d}_x + \mathbf{d}_s\|^2 - \|\mathbf{d}_x - \mathbf{d}_s\|^2}{4} \leq \frac{\|\mathbf{d}_x + \mathbf{d}_s\|^2}{4} = \frac{\|\mathbf{p}_v\|^2}{4},$$

which proves the result. □

The next lemma examines the effect which a Newton step followed by an update of the parameter μ has on the proximity measure.

Lemma 3.4 Let $\mathbf{v} > \xi \mathbf{e}$ and $\mathbf{v}^+ = \sqrt{\frac{\mathbf{x}^+ \mathbf{s}^+}{\mu^+}}$, where $\mu^+ = (1 - \theta)\mu$ and let $\eta = \sqrt{1 - \theta}$. Assume that $\delta(\mathbf{x}, \mathbf{s}; \mu) < \sqrt{\frac{1 - \xi^2}{1 + 4\kappa}}$. Let φ be a function satisfying conditions AET1-3

of Definition 2.4 with c_1, c_2 and $c_3 \in \mathbb{R}_+$. Then, after a primal-dual Newton step we have $\mathbf{v}^+ > \xi \mathbf{e}$ and

$$\delta(\mathbf{x}^+, \mathbf{s}^+; \mu^+) \leq \frac{c_1}{\eta^2} \left(\theta \sqrt{n} + ((c_2 + 2) + 4\kappa) \delta^2 \right).$$

Proof From Lemma 3.2, we have $\bar{\mathbf{v}} > \xi \mathbf{e}$. Using this and $0 < \theta < 1$, we obtain that $\mathbf{v}^+ > \xi \mathbf{e}$. Now, we will use the form AETa and AETb of Proposition 2.1. By using the definition of the proximity measure given in (20), we have

$$\delta(\mathbf{x}^+, \mathbf{s}^+; \mu^+) = \left\| (\mathbf{e} - (\mathbf{v}^+)^2) g(\mathbf{v}^+) \right\|.$$

From condition AETa of Proposition 2.1, we get

$$\delta(\mathbf{x}^+, \mathbf{s}^+; \mu^+) \leq c_1 \left\| \mathbf{e} - (\mathbf{v}^+)^2 \right\|. \tag{35}$$

Furthermore,

$$\begin{aligned} \left\| \mathbf{e} - (\mathbf{v}^+)^2 \right\| &= \left\| \mathbf{e} - \frac{1}{\eta^2} \frac{\mathbf{x}^+ \mathbf{s}^+}{\mu} \right\| = \left\| \mathbf{e} - \frac{1}{\eta^2} \left(\mathbf{v}^2 + \mathbf{v} \mathbf{p}_v + \frac{\mathbf{p}_v^2}{4} - \frac{\mathbf{q}_v^2}{4} \right) \right\| \\ &= \frac{1}{\eta^2} \left\| \eta^2 \mathbf{e} - \left(\mathbf{v}^2 + \mathbf{v} \mathbf{p}_v + \frac{\mathbf{p}_v^2}{4} - \frac{\mathbf{q}_v^2}{4} \right) \right\| \\ &= \frac{1}{\eta^2} \left\| -\theta \mathbf{e} + \mathbf{e} - \mathbf{v}^2 - \mathbf{v} \mathbf{p}_v - \frac{\mathbf{p}_v^2}{4} + \frac{\mathbf{q}_v^2}{4} \right\|. \end{aligned} \tag{36}$$

From (31), (35), (36) and condition AETb of Proposition 2.1, we obtain

$$\begin{aligned} \delta(\mathbf{x}^+, \mathbf{s}^+; \mu^+) &\leq \frac{c_1}{\eta^2} \left(\|\theta \mathbf{e}\| + \left\| \mathbf{e} - \mathbf{v}^2 - \mathbf{v} \mathbf{p}_v \right\| + \left\| \frac{\mathbf{p}_v^2}{4} \right\| + \left\| \frac{\mathbf{q}_v^2}{4} \right\| \right) \\ &\leq \frac{c_1}{\eta^2} \left(\theta \sqrt{n} + ((c_2 + 2) + 4\kappa) \delta^2 \right), \end{aligned} \tag{37}$$

and the lemma is proven. □

In the following lemma, we set the values of the parameters θ and τ and we prove that for these values IPAs using the new class of AET functions are well defined.

Lemma 3.5 *Let $\varphi : (\xi, \infty) \rightarrow \mathbb{R}$ be a function satisfying AET1-3 of Definition 2.4. Assume that $n \geq 1$, $\theta = \frac{2\sqrt{1-\xi^2}}{25c_2(2+\kappa)\sqrt{n}}$ and $\tau = \frac{\sqrt{1-\xi^2}}{2c_2(2+\kappa)}$. Suppose that $\delta(\mathbf{x}, \mathbf{s}; \mu) \leq \tau$. If $c_1 \leq \frac{100c_2-4}{41c_2+50}$ and $c_2 \geq 6$, then we have*

$$\delta(\mathbf{x}^+, \mathbf{s}^+; \mu^+) \leq \tau,$$

hence the IPAs defined in Algorithm 3.1 are well defined.

Proof We have $\tau = \frac{\sqrt{1-\xi^2}}{2c_2(2+\kappa)}$ and $c_2 \geq 6 > \frac{1}{2}$. From here, we have $\frac{\sqrt{1-\xi^2}}{2c_2(2+\kappa)} < \frac{\sqrt{1-\xi^2}}{2+\kappa} < \sqrt{\frac{1-\xi^2}{1+4\kappa}} < \frac{1}{\sqrt{1+4\kappa}}$. By using this and the assumption that conditions AET1-3 of Definition 2.4 are satisfied, from Lemma 3.1, we get that $(\mathbf{x}^+, \mathbf{s}^+) \in \mathcal{F}^+$. By using Lemma 3.4, we have

$$\delta(\mathbf{x}^+, \mathbf{s}^+; \mu^+) \leq \frac{c_1}{\eta^2} \left(\theta\sqrt{n} + ((2 + c_2) + 4\kappa)\delta^2 \right). \tag{38}$$

Considering $\theta = \frac{2\sqrt{1-\xi^2}}{25c_2(2+\kappa)\sqrt{n}}$ and $\tau = \frac{\sqrt{1-\xi^2}}{2c_2(2+\kappa)}$, we get

$$\theta\sqrt{n} = \frac{2\sqrt{1-\xi^2}}{25c_2(2+\kappa)} = \frac{4}{25}\tau. \tag{39}$$

Moreover, by using $n \geq 1$ and $\kappa \geq 0$, we have

$$\frac{1}{1-\theta} \leq \frac{1}{1-\frac{2}{50c_2}} = \frac{25c_2}{25c_2-1}. \tag{40}$$

By using that $\delta \leq \tau$ and from $\eta = \sqrt{1-\theta}$, (38), (39) and (40), we get

$$\begin{aligned} \delta(\mathbf{x}^+, \mathbf{s}^+; \mu^+) &\leq c_1 \frac{25c_2}{25c_2-1} \left(\frac{4}{25}\tau + ((2 + c_2) + 4\kappa)\tau^2 \right) \\ &= c_1 \frac{25c_2}{25c_2-1} \left(\frac{4}{25} + ((2 + c_2) + 4\kappa)\tau \right) \tau \end{aligned} \tag{41}$$

and we want to show that $\delta(\mathbf{x}^+, \mathbf{s}^+; \mu^+) \leq \tau$. For this, we need to show that

$$c_1 \frac{25c_2}{25c_2-1} \left(\frac{4}{25} + ((2 + c_2) + 4\kappa)\tau \right) \leq 1. \tag{42}$$

From $\kappa \geq 0$ and $\delta \leq \tau \leq \frac{1}{2c_2(2+\kappa)}$, we get

$$\begin{aligned} c_1 \frac{25c_2}{25c_2-1} \left(\frac{4}{25} + ((2 + c_2) + 4\kappa)\tau \right) &\leq c_1 \frac{25}{25c_2-1} \left(\frac{4}{25} + \frac{2+c_2+4\kappa}{2c_2(2+\kappa)} \right) \\ &= c_1 \frac{25c_2}{25c_2-1} \left(\frac{4}{25} + \frac{c_2-6}{2c_2(2+\kappa)} + \frac{4(2+\kappa)}{2c_2(2+\kappa)} \right) \\ &\leq c_1 \frac{25c_2}{25c_2-1} \left(\frac{4}{25} + \frac{c_2-6}{4c_2} + \frac{2}{c_2} \right) = \frac{c_1(41c_2+50)}{100c_2-4}. \end{aligned} \tag{43}$$

By using $c_2 \geq 6 > \frac{1}{25}$ and $c_1 \leq \frac{100c_2-4}{41c_2+50}$, we obtain that $\delta(\mathbf{x}^+, \mathbf{s}^+; \mu^+) \leq \tau$, which gives the result. □

Remark 3.2 It should be mentioned that the parameters c_1 and c_2 of the functions given in Table 2 satisfy the conditions $c_2 \geq 6$ and $c_1 \leq \frac{100c_2-4}{41c_2+50}$.

The following lemma gives an upper bound on the number of iterations.

Lemma 3.6 Consider $\varphi : (\xi, \infty) \rightarrow \mathbb{R}$ satisfying AET1-3 of Definition 2.4. Let $n \geq 1$, $\theta = \frac{2\sqrt{1-\xi^2}}{25c_2(2+\kappa)\sqrt{n}}$, $\tau = \frac{\sqrt{1-\xi^2}}{2c_2(2+\kappa)}$, $c_1 \leq \frac{100c_2-4}{41c_2+50}$, and $c_2 \geq 6$ and $c_3 \leq 16c_2^2 - 1$. Assume that the pair $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^+$, $\mu^0 = \frac{(\mathbf{x}^0)^T \mathbf{s}^0}{n}$ and $\delta(\mathbf{x}^0, \mathbf{s}^0; \mu^0) \leq \tau$. Let \mathbf{x}^k and \mathbf{s}^k be the two vectors obtained by the IPAs given in Algorithm 3.1 after k iterations. Then, for

$$k \geq \left\lceil \frac{1}{\theta} \log \frac{\mu^0(n+1)}{\varepsilon} \right\rceil$$

we have $(\mathbf{x}^k)^T \mathbf{s}^k \leq \varepsilon$.

Proof From Lemma 3.3 and $c_3 \leq 16c_2^2 - 1$, we have

$$\begin{aligned} (\mathbf{x}^k)^T \mathbf{s}^k &\leq \mu^k \left(n + (c_3 + 1) \cdot \frac{1}{(2c_2(2 + \kappa))^2} \right) \\ &= (1 - \theta)^k \mu^0 \left(n + (c_3 + 1) \cdot \frac{1}{(2c_2(2 + \kappa))^2} \right) \\ &\leq (1 - \theta)^k \mu^0 (n + 1). \end{aligned} \tag{44}$$

The condition $(\mathbf{x}^k)^T \mathbf{s}^k \leq \varepsilon$ holds if

$$(1 - \theta)^k \mu^0 (n + 1) \leq \varepsilon. \tag{45}$$

By taking the logarithm of both sides of (45), we have

$$k \log(1 - \theta) + \log(\mu^0(n + 1)) \leq \log \varepsilon.$$

By using that $-\log(1 - \theta) \geq \theta$, finally we get

$$k\theta \geq \log(\mu^0(n + 1)) - \log \varepsilon = \log \frac{\mu^0(n + 1)}{\varepsilon},$$

which proves the lemma. □

Remark 3.3 Condition $c_3 \leq 16c_2^2 - 1$ is satisfied for all functions given in Table 2.

Theorem 3.1 Let $\varphi : (\xi, \infty) \rightarrow \mathbb{R}$ satisfying AET1-3 of Definition 2.4. Consider $n \geq 1$, $\theta = \frac{2\sqrt{1-\xi^2}}{25c_2(2+\kappa)\sqrt{n}}$ and $\tau = \frac{\sqrt{1-\xi^2}}{2c_2(2+\kappa)}$. If $c_1 \leq \frac{100c_2-4}{41c_2+50}$, $c_2 \geq 6$ and $c_3 \leq 16c_2^2 - 1$, then the IPAs given in Algorithm 3.1 require no more than

$$\mathcal{O} \left((2 + \kappa)\sqrt{n} \log \frac{(n + 1)\mu^0}{\varepsilon} \right)$$

interior-point iterations.

Corollary 3.1 *Let $\varphi : (\xi, \infty) \rightarrow \mathbb{R}$ satisfying AET1-3 of Definition 2.4. Consider $n \geq 1$, $\theta = \frac{2\sqrt{1-\xi^2}}{25c_2(2+\kappa)\sqrt{n}}$ and $\tau = \frac{\sqrt{1-\xi^2}}{2c_2(2+\kappa)}$. If $c_1 \leq \frac{100c_2-4}{41c_2+50}$, $c_2 \geq 6$ and $c_3 \leq 16c_2^2 - 1$, then the IPAs given in Algorithm 3.1 have polynomial iteration complexity in the size of the problem, in the starting point's duality gap μ^0 , the accuracy parameter and in the parameter κ .*

In the following section, some concluding remarks are presented.

4 Conclusions and Further Research

In this paper, we proposed new short-step IPAs for solving $P_*(\kappa)$ -LCPs. The novelty of the paper is that we introduced a whole new class of AET functions (AET functions with bounded proportional growth rate) in order to determine search directions in case of PD IPAs. Many AET functions used in the literature belong to this new class. This means that several short-step IPAs, see for example [2, 8], can be considered as special cases of the whole class of IPAs proposed in this paper. We proved that IPAs using any member φ of this new class of AET functions with bounded proportional growth rate have polynomial iteration complexity in the size of the problem, in the starting point's duality gap μ^0 , the accuracy parameter and in the parameter κ . As further research, it would be worth analyzing the complexity of IPAs based on other AET function that do not belong to our class.

The new class of AET functions with bounded proportional growth rate defined by conditions AET1-3 of Definition 2.4 differs from the existing classes. We showed examples for functions belonging to our new class that are not members of the existing classes of AET functions. The kernel functions corresponding to these AET functions φ are neither self-regular, nor eligible kernel, nor self-concordant functions. We also provided examples for eligible and self-regular kernel functions for which the corresponding AET functions are not members of our new class AET functions with bounded proportional growth rate.

As further research it would be worth analyzing the system of differential inequalities given in AET1-3. Furthermore, the introduction of new classes of AET functions for other types of IPAs, such as predictor-corrector or affine-scaling IPAs, would be an interesting future research. Numerical results based on members of the new class of AET functions could be also given. However, there are promising results using the functions $\varphi(t) = \sqrt{t}$ and $\varphi(t) = t - \sqrt{t}$ belonging to this new class. These computational results show that the proposed IPAs based on the new class of AET functions with bounded proportional growth rate work efficiently in practice.

Illés et al. [19] proposed a strongly polynomial rounding procedure in case of $P_*(\kappa)$ -linear complementarity problems, which creates exact solution from ϵ -optimal solution, if the value of ϵ is small enough. As future research plan, it would be worth trying to implement the rounding procedure in this case, as well. Another interesting research topic would be to extend the presented IPAs in a similar way that Illés, Nagy and Terlaky did in [17, 18], for general LCPs. It would be good to collect general LCP test problems in order to make the computational performance of the algorithms testable on an interesting set of real-life problems.

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Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

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