Panna Miskolczi

Comparison of Risk Calculation Based on Historical Simulation and the Copula Function

SUMMARY: The fundamental aim of this paper is to compare risk calculation based on historical simulation with risk calculation based on the copula function. In the case of historical simulation it is assumed that future data can be estimated on the basis of historical data. The copula function is a multi-dimensional distribution function with which we can explore the correlations between probability variables (in this case, equities within the portfolio) and simulate or forecast their future development. A method is called "better" if it enables a more accurate estimation of risk, i.e. where actual and estimated values are closer to each other. In this paper, the Value at Risk and Expected Shortfall risk measures are used to determine risk, while the accuracy of the two simulations is tested with the backtesting method. Based on the results of the empirical study of the daily price data of seven equities we may conclude that risk calculation based on the copula function may contribute to a more precise modelling of risk.

KEYWORDS: copula, historical simulation, backtesting, Value at Risk, Expected Shortfall
JEL CODES: C51, C53, G17

Risk is present everywhere, directly or indirectly influencing our decisions and hence our lives. In our modern, globalised world the knowledge, measurement and forecastability of risks is of key importance. Despite the fact that increased emphasis is being placed on the management of financial risk, it is still not entirely clear which mathematical concept or measure could be used to describe or calculate risk precisely. The first chapter presents the major "requirements" connected to risk measures, the definition of risk measure, and the two risk measures used most often these days, Value at Risk and Expected Shortfall.

One of the most important features that risk measures are expected to meet is that diversification should not increase the risk of a portfolio. For example, if the equities included in a portfolio show the same extreme behaviour upon a specific extreme event, this may obliterate the protective effect of diversification. This is why it is necessary to map the relationship between the equities included in the portfolio as precisely as possible. The most widely known and used indicators – such as the Pearson, Kendall or Spearman correlation coefficients – measure correlations "only" in pairs. Copula functions are designed to measure correlations of higher dimensions; for the definition of copula functions, the most important theorem of this topic, and examples, see the chapter "Copula functions".

The aim of this paper is to compare these two methods – historical risk calculation vs. risk calculation based on the copula function. For the comparison, the backtesting method

E-mail address: miskolczipanna@gmail.com

is used. Both methods as well as backtesting are described in the chapter "Backtesting".

The next chapter includes the empirical study, the results, and the conclusions that may be drawn from these. The daily logarithmic returns of seven equities are used for the calculations, for which price data covering ten years have been downloaded from the website of the Budapest Stock Exchange. These are used to examine whether risk calculation based on historical simulation or calculation based on the copula function gives a more accurate view of risks.

RISK MEASURES

According to one possible definition, *"Risk is the potential of losing something of value, weighed against the potential to gain something of value"* (Kungwani, 2014). Of course, this rather general definition of risk can be interpreted in many different ways. It is also clear that the meaning and definition of risk may vary across the fields of science.

A pioneering work of economics from 1921, *Knight's* "Risk, Uncertainty, and Profit" discussed in detail the difference between risk and uncertainty and in doing so, it established the foundations of probability-based economics (Bélyácz, 2010). It is Harry Markowitz who is credited with laying the foundations of financial risk and simultaneously, the modern portfolio theory, defining risk in his article "Portfolio Selection" (Markowitz, 1952) by means of the standard deviation of the portfolio.

The management of financial risk, i.e. the identification, measurement and control of risk factors, remains an indispensable tool for the sustenance of any business activity to this day (Jorion, 1999). Measurement, however, is insufficient in itself, comparability is also essential for decision-making.

Therefore, mathematically the goal is to be able to express risk with the simplest possible indicators, using a single measure. The indicators attached to financial instruments and portfolios and typical of risk are called risk measures (Gáll – Pap, 2010). More precisely, risk measure is any χ functional assuming real values that is interpreted on a set of $\rho{:}\chi\rightarrow\mathbb{R}$, where χ is a set of probability variables (profit of portfolios or financial instruments) (Gáll – Pap, 2010). This means that the measurement of risk in fact means the establishment of a correlation between random variables and real numbers (Szegö, 2002). This is a very broad concept; therefore, in order to get a sensible definition of risk, certain restrictions and properties must be formulated in respect of risk measures.

Naturally several such expectations arise. In the literature, see for example *Artzner et al.* (1999). The features that occur most frequently and are used most often are as follows:

❶ *Monotonicity:* if the return on an investment is never lower than that of another, it is expected that this investment should not be riskier than the other; in other words, if $X \leq Y$ then $\rho(X) \geq \rho(Y)$, $\forall X, Y \in \chi$.

❷ *Subadditivity:* subadditivity is also known as "diversification effect". This means that the resultant risk of two investments must not be larger than the sum of their individual risks, i.e. $\rho(X+Y) \leq \rho(X)+\rho(Y)$, $\forall X, Y, X+Y \in \chi$. Naturally, this feature is valid in the case of several variables as well.

❸ *Positive homogeneity:* if a portfolio is multiplied, with its content remaining the same, it is expected that risk should increase proportionally to size, i.e. $\rho(\lambda X)=\lambda\rho(X)$, $\forall X, \lambda X \in \chi$, $\forall \lambda \geq 0$.

❹ *Translation invariance:* if a risk-free instrument of a given return is added to the portfolio, it is expected that risk should decrease with the size of this cash flow, i.e. $\rho(X+a) = \rho(X) - a$, $\forall X, X+a \in \chi$, $\forall a \in \mathbb{R}$.

If a risk measure fulfils all four of the above-mentioned features, the risk measure is considered to be coherent. It occurs frequently that, for example, the goal is to achieve a certain return and minimise risk at the same time. Thus the existence of convexity is important for optimisation. It can be proven that convexity follows from positive homogeneity and subadditivity. Therefore, in order to measure risk as accurately as possible, there is a need for a risk measure that represents the features of monotonicity, subadditivity, positive homogeneity and translation invariance, and can be applied relatively simply in practice as well.

For example, as regards the standard deviation introduced by Markowitz, it can be seen that although it is subadditive and homogeneous, it is not monotonic or shift invariant. Besides, it fails to make a distinction between gains and losses; therefore, it is unsuitable for measuring risk. Obviously, there are several risk measures, but the two risk measures most frequently used in practice are Value at Risk *(VaR)* and Expected Shortfall *(ES)*. As these two will be used in this paper as well, I will describe them in more detail below.

Value at Risk (VaR)

Value at Risk or *VaR* has been the most frequently used risk measure in recent years. Its use significantly increased after the financial crisis of 1987 (Jorion, 1999).

Value at Risk is designed to answer the naturally arising question of how much one can minimally lose (gain) at the given degree of probability over a specific time period. In other words, Value at Risk is a number showing a potential loss amount relative to which a larger loss may occur with a probability of α or less. More precisely, the Value at Risk of random variable X at a probability degree of α may be defined as follows:

$$VaR_\alpha(X) = -inf\{x \in \mathbb{R} | F_X(x) \geq \alpha\},$$

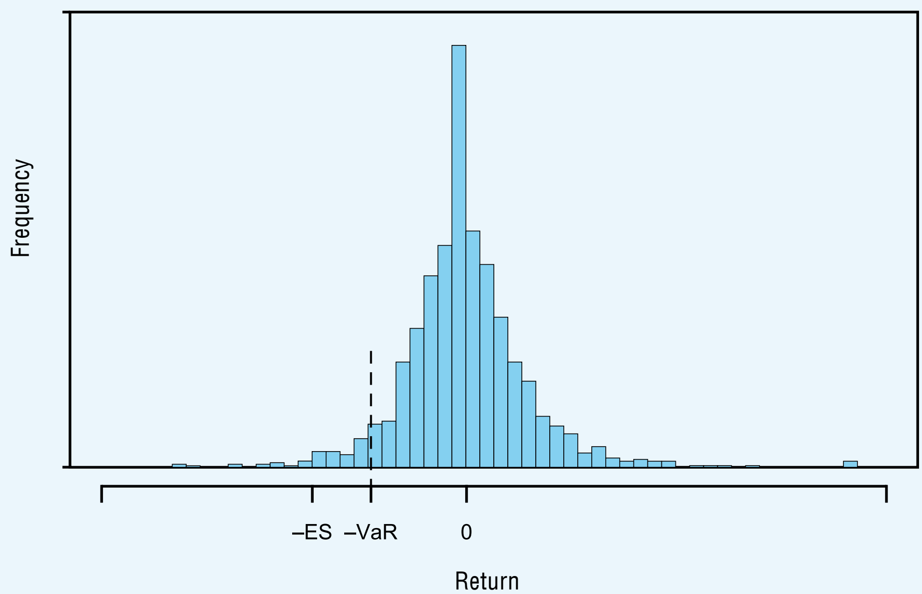
Where $F_X(x) := P(X \leq x)$ signifies the distribution function of random variable X.

Figure 1 shows the histogram drawn on the basis of the returns of the FHB equity, to be presented in detail later. The probability that a value is located to the left of the red line drawn in the histogram equals the confidence level, that is α, while the probability that a value is located to the right of this line equals $1-\alpha$. In the concrete example $VaR = -0.034$, at a level of $\alpha = 5\%$, i.e. Value at Risk is 0.034. This means that the probability that the loss will be 0.034 or larger is 5 per cent or, in other words, there is a 95 per cent $(1-\alpha)$ probability that the loss will be no greater than 0.034.

It can be seen that *VaR* is a monotonic, positive, homogeneous and translation invariant risk measure (Gáll – Pap, 2010). It is a positive feature of *VaR* that it measures risk with a single number and currency; therefore, it can be easily interpreted and the different values are easy to compare (Acerbi at al., 2001). *VaR* is intended to measure downside risk; i.e. it considers values lower than the expected value (expected return) only. On the other hand, it is an enormous problem with *VaR* that it does not support diversification, i.e. it is not subadditive. Furthermore, although *VaR* shows the value that loss will not exceed at a certain probability, it does not deal with the losses that are beyond this value, whereas the knowledge of serious and extreme events would be important. In the case of the example presented in Figure 1 this means that although we know that there is a 95 per cent probability that the loss will not be greater than 0.034, we do not have any information on how large the loss can be in the remaining 5 per cent (where the loss is larger). Therefore, *VaR* is suitable for measuring loss only in the case of elliptical distributions, and in the case of non-elliptical distributions – which is not infrequent in practice – it may result in a misperception of risks (Embrechts et al., 2002). In addition, as

Figure 1

VAR AND ES SHOWN IN A HISTOGRAM



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VaR is not subadditive, it is not convex either, which makes its use difficult if not downright impossible in the case of optimisation problems (Szegö, 2002).

Expected Shortfall (ES)

Thus it can be seen that *VaR* fails to represent the fundamental features of coherent risk measures. Accordingly, the question arises whether there exists any coherent risk measure at all. The answer is yes, for example Expected Shortfall *(ES)*. (Although this term has a Hungarian equivalent, as the Hungarian literature apparently prefers to use the English term, the Hungarian version of this paper also uses “Expected Shortfall”).

VaR seeks to determine the amount of the minimum loss that may occur in the worst $\alpha \times 100$ percentage of cases. In the case of *ES*, the question is modified as follows: how much is the expected shortfall that may occur in the worst $\alpha \times 100$ percentage of cases (*Acerbi,* 2001)? Mathematically it may be formulated as follows: Let X be a random variable and $\alpha \in (0.1)$, and let us assume that $E[(X)^{-}] < \infty$, where $(X)^{-}$ is the negative part of X. Then the expected shortfall of X at level α will be:

$$ES_{\alpha}(X) = \frac{1}{\alpha} \int_{0}^{\alpha} VaR_{u}(X)du.$$

In the case of the returns histogram shown in Figure 1 this actually means that *ES* provides the expected values of the values to the left of the dashed red line. Based on the concrete example: $-ES = 0.055$, at a level of $\alpha = 5\%$, i.e. $ES = 0.055$. This means that at a 5 per cent probability the expected shortfall will be 0.055.

It can be said that since *ES* represents all four properties of coherent risk measures, it qualifies as a coherent risk measure. Despite the fact that *ES* has plenty of positive features, it is still short of perfect as, for example, *VaR* always exists, whereas as a precondition for the existence of *ES* it must be assumed that $E[(X)^{-}] < \infty$.

Later, in the part entitled "Empirical study", the method of calculation of *VaR* and *ES* will be shown using concrete equity data.

COPULA FUNCTIONS

As it has been mentioned in the discussion of the properties of risk measures, diversification has an important role in risk mitigation. Despite this, history has presented us with multiple financial crises to face. One of the reasons for these financial crises could be that, as a result of an event, not only one factor in the portfolio shows extreme behaviour, but several factors jointly, extinguishing the "protective" effect of diversification (Benedek et al., 2002, Aas, 2004).

Therefore, estimating and modelling multi-dimensional distributions can be a problem in the financial and economic field. Mathematical theories and models presume a normal, but at least elliptical distribution in most cases, although it can be seen from empirical data that this condition is in many cases unsatisfied (Härdle et al., 2008). Additionally, dependence structures can also exist where correlation coefficients are identical, but the structures themselves are completely different. Furthermore, the frequently used dependence measures, such as Kendall's tau, Spearman's rho, or the Pearson correlation coefficient "only" measure dependencies in pairs. Copula functions enable us to describe more general dependencies of higher dimensions. To write this chapter, I used, among others, the works of *Durante – Sempi* (2010), Embrechts et al. (2001 and 2002), *Nelsen* (2013), and *Cherubini et al.* (2004).

Copula functions enable the modelling of the dependence structures existing between two or more probability variables. The term itself derives from the Latin word *copulare*, meaning to bind or tie together, unite, join, etc., and its introduction is credited to Abe Sklar (Sklar, 1959). The use of copula functions in financial practice was introduced by the work of Embrechts et al. (1999), which means that we are talking about a rather new and rapidly developing field.

Sklar's theorem established the correlation between copula functions, joint distribution functions and marginal distribution functions. This made it possible to determine new joint distributions and hence, new dependence measures.

Therefore the multi-dimensional copula function $C:[0,1]^{n}\rightarrow[0,1]$ is a multi-dimensional distribution function with uniform marginal distributions.

As it has already been mentioned, within the topic of copula functions one of the most important theorems is Sklar's theorem, which in essence asserts that a joint distribution function can be "broken up" into marginal distributions and the dependence structure between them. This dependence structure is the copula:

Let F be an n-dimensional distribution function, with marginal distributions $F_1, \dots, F_n$. Then a copula C exists, for which:

$$F(x_1, \dots, x_n) = C[F(x_1), \dots, F(x_n)], \quad \forall x_i \in \bar{\mathbb{R}}.$$

The reversal of the theorem is also true, which means that if C is a copula function and $F_1, \dots, F_n$ are single-variable distribution functions, then the $F(x_1, \dots, x_n) = C[F(x_1), \dots, F(x_n)]$ function defined with the formula F is a n-variable distribution function with the marginal distributions $F_1, \dots, F_n$.

The correlation in the theorem is generally the starting point for simulations based on the given copula and marginal distribution functions, whereas the reversal of the theorem provides the theoretical background for how a copula function can be constructed using a multi-dimensional distribution function.

The aim of this paper is to facilitate the modelling of the correlation structure between several equities using certain copula functions and, with the assistance of this structure, to be able to forecast the risk of the portfolio as precisely as possible. For this purpose, four well-known copulas are used, namely the Gaussian, Student, Clayton and Frank copulas, which are described in more detail below.

Implicit copulas

Implicit copulas are derived from some well-known distribution function; consequently, a closed formula by which they can be described does not necessarily exist. If this distribution function is elliptical, we are talking about an elliptical copula. Such are for example the Gaussian copula and Student's *t*-copula, which are frequently used in the financial field. As indicated by their names, the former can be derived from the normal and the latter from Student's t-distribution.

The distribution function of the *Gaussian* copula takes the following form:

$$C_{\Sigma}^{Ga}(u) = \Phi_{\Sigma}[\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)],$$

where Φ signifies the density function of the standard normal distribution, whereas Φ_{Σ} is the density function of an n-dimensional normal distribution with a zero expected value and a linear correlation matrix Σ.

In the case of the Gaussian copula, the probability variables can be considered independent in both tails. This means that no matter how high the selected parameter values are, when one goes far enough along the tails of the distribution curve, extreme events will occur independently of one another. Or putting it another way, if a high value is measured in one of the variables, it is not likely that the other variable will also show a high value (Rakonczai, 2009).

The density function of the 2-variable Gaussian copula and its contours in the case of p–0.3 can be seen in *Figure 2* .

Student's copula – also known as *t-copula* – can be described as follows:

$$C_{v,\Sigma}^{t}(u) = t_{v,\Sigma}[t_v^{-1}(u_1), \dots, t_v^{-1}(u_n)],$$

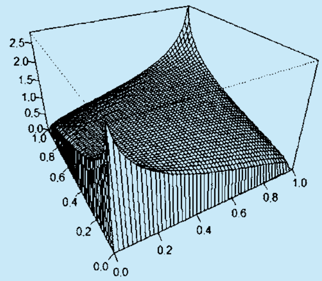
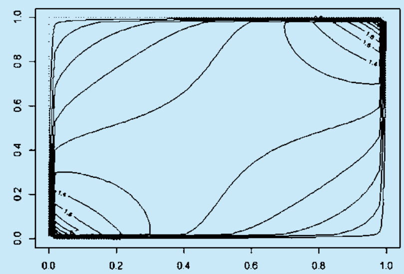
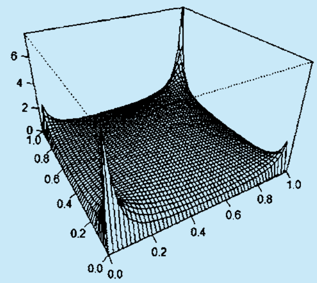
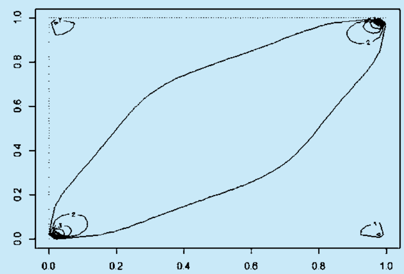
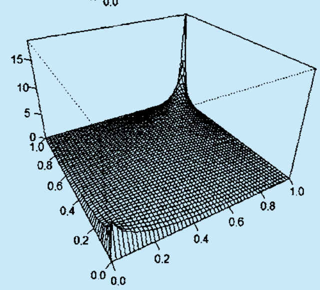
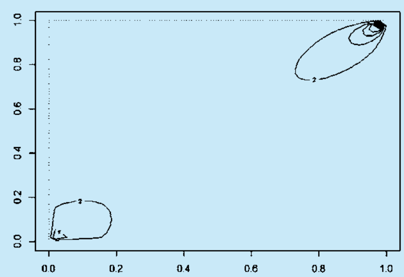
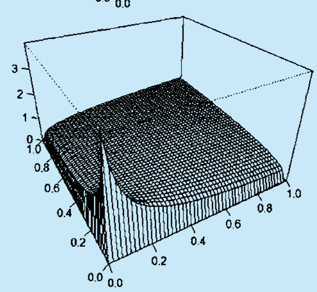
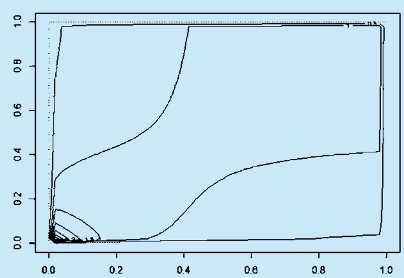
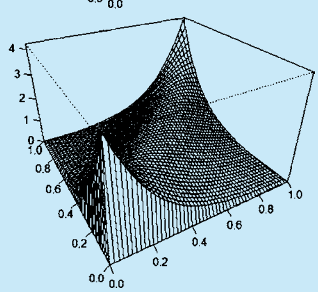
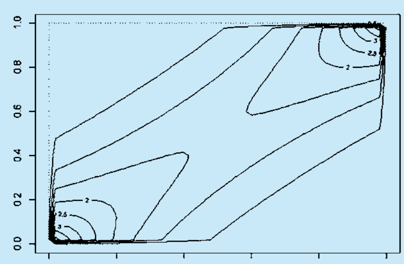
Where t_v is the density function of the 1-dimensional Student's *t*-distribution with a v degree of freedom, and $t_{v,\Sigma}$ is the density function of the n-dimensional Student's *t*-distribution with a correlation matrix of Σ. A *t*-copula has two parameters: v is the number of degrees of freedom, whereas Σ (or in two dimensions p) is the correlation matrix (linear correlation).

In the case of a *t*-copula, the stronger linear correlation and the lower the number of degrees of freedom is, the stronger dependence shall be in the tails. It should be noted that Student's *t*-copula also shows dependence in the tails when the value of p is negative (>-1) or zero. The density function of Student's *t*-copula and its contours can be seen in *Figure 2* in the case of the $\rho = 0.3$ and $v = 2$ parameters.

It is noticeable that Student's *t*-copula and the Gaussian copula behave similarly in the middle of the distribution, but show completely different behaviour in the tails. Despite the fact that in both cases the value of the correlation is identical ($\rho = 0.3$), it can be seen that in the case of Student's *t*-copula some protuberance can be seen in all four "corners", which means that this copula expresses extreme events much better. For example, if the two random variables express losses of a portfolio, then an

Figure 2

DENSITY FUNCTIONS OF THE GAUSSIAN, STUDENT, GUMBEL, CLAYTON AND FRANK COPULAS, AND THEIR CONTOURS

Copula	Density function	Contour
Gauss		
Student		
Gumbel		
Clayton		
Frank		

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outlier in point (0,0) corresponds to significant losses incurred on both equities included in the portfolio (Schmidt, 2007).

Explicit copulas

Contrary to the copulas that can be expressed from their distribution functions, explicit copulas may also be described by means of a relatively simple closed formula. A large part of these copulas belong to the Archimedean copula family.

The Archimedean copula family can be defined as follows:
consider a continuous and strictly decreasing function $\varphi:[0,1]\rightarrow[0,\infty]$ with $\varphi(1)=0$. Then

$$C(u_1,u_2)=\left\{\begin{array}{c}\varphi^{-1}[\varphi(u_1)+\varphi(u_2)],\ if\ \varphi(u_1)+\varphi(u_2)\leq\varphi(0)\\ 0,\ otherwise\end{array}\right\}$$

the function is a copula if and only if φ is convex. A copula provided in this form is called an Archimedean copula, and the function φ is called the generator of the copula.

Using this structure, many different types of copulas can be generated. For example, using the generator $\varphi(u)=(-ln\,u)^{\theta}\,(\theta\geq1)$ we can obtain the *Gumbel* or *Gumbel–Hougaard* copula and using the generator $\varphi(u)=(u^{-\theta}-1)/\theta$ ($\theta\in[-1,\infty]\backslash\{0\}$) the *Clayton* copula, whereas generator $\varphi(u)=-ln\frac{e^{-\theta u}-1}{e^{-\theta}-1}$ leads to the *Frank* copula.

Figure 2 shows the density function of a 2-dimensional Gumbel copula and the pertaining contours in the case of $\theta=2$, the density function of the Clayton copula and the pertaining contours in the case of $\theta=0.3$, and the density function of the Frank copula and the pertaining contours in the case of $\theta=5$.

BACKTESTING

In this chapter I will use the backtesting method to examine whether the historical risk calculation or the copula-based risk calculation proves to be better. A risk calculation method is considered “better” than another if it allows for a more accurate estimation of risk; i.e. where there is a smaller difference between the actual and estimated values.

In order to execute backtesting, time windows must be created. Both in the literature and in practice this value is most frequently one year, that is 250 trading days (T=250). The first window lasts from day 1 to day 250. The second window from day 2 to day 251, and so on. Using the data of the first time window, we can calculate the value of the necessary risk measure for the first 250 days, which at the same time serves as a forecast for day 251. If return data for day 251 are also available, the estimated value calculated on the basis of the first time window can be compared with the actual return value. Then the risk concerning the second time window can be calculated as well, which will serve as an estimated value for the next day, that is day 252. When the return for day 252 is available, the estimated risk can be compared with the actual value. And so forth, going through all the data step by step (Hull, 2006), as also summed up in *Table 1*. It depends on the risk calculation method whether the backtesting is historical or copula-based.

Risk calculation based on historical simulation

In the case of historical simulation it is assumed that future events can be described or estimated by historical events. In such cases backtesting essentially means that using historical data the risk value – in this case the value of *VaR* and *ES* – is calculated, and compared to the next day’s return.

In the case of *VaR* this comparison means that it is examined for each day whether the given return is lower or not than (–1) times

Table 1

TIME WINDOWS

Time window	First day	Last day	$\widehat{VaR}$
1.	1.	250.	251.
2.	2.	251.	252.
3.	3.	252.	253.
⋮	⋮	⋮	⋮
(N–249).	(N–249).	N.	(N+1).

Source: own editing

the *VaR* value estimated for that day, then add up the number of cases where this has been observed, i.e. in how many cases did *VaR* underestimate the value of actual risk. In accordance with the definition of *VaR,* the probability of this should equal the confidence level, that is α (Daníelsson, 2011). This means that in accordance with the definition of *VaR*, alpha can be estimated from the sample realisation with the following correlation:

$$\alpha \approx \frac{1}{M-1}\sum_{m=1}^{M-1} 1_{\{r_{250}^{m+1} < -VaR_{\alpha}^{m}\}},$$

Where VaR_{α}^{m} signifies the *VaR* value belonging to the $m=1, \ldots, M^{th}$ scenario at level α, and r_t^m signifies the value of the logarithmic return belonging to the $t=1, \ldots, T^{th}$ point of time and the m^{th} scenario. The closer the estimated risk is to actual risk, i.e. the more precisely *VaR* is able to estimate risk, the closer this value will be to α.

In 2012 the Basel Committee recommended the introduction of *ES* as a new regulatory risk measure to replace *VaR* (which had been used since 1996). The only problem was that contrary to *VaR ES* was thought to be unsuitable for backtesting due to its elicitability. Recent research has evidently solved this problem as well, as it has been demonstrated that *ES* can also be backtested. Based on the article of *Acerbi and Székely* (Acerbi, 2014), in the case of *ES* alpha can be estimated from the sample realisation using the following correlation:

$$\alpha \approx -\frac{1}{M}\sum_{m=1}^{M}\frac{1}{T}\sum_{t=1}^{T}\frac{r_t^m 1_{\{r_t^m < -VaR_{\alpha}^{m}\}}}{ES_{\alpha}^{m}},$$

Where VaR_{α}^{m}, and ES_{α}^{m} are the *VaR* and *ES* values belonging to the m^{th} scenario at level α, and r_t^m is still the value of the logarithmic return belonging to the t^{th} point of time and the m^{th} scenario.

Risk calculation based on the copula model

In this case – contrary to historical simulation – the *VaR* and *ES* values pertaining to the given date are no longer calculated from historical data, but using a copula-based Monte Carlo simulation.

The steps of risk calculation using the copula-based Monte Carlo simulation are as follows (*Xu,* 2012):

❶ selection of copula model (Gauss, Student, Archimedean, etc.),

❷ selection of marginal distributions and estimation of the parameters of these distributions,

❸ transformation of data into the range of interpretation of the copula function,

❹ adjustment of the copula function to the (transformed) data and estimation of the parameters of this copula function,

❺ generation of random variables having a joint density function corresponding to the estimated copula function,

❻ generation of random variables using the inverse function of the marginal distributions,

❼ calculation of the gain/loss of the portfolio using the simulated data,

❽ repetition of steps 1 to 7 as many times as needed to ensure that the simulated distribution is sufficiently close to the "real" distribution,

❾ calculation of *VaR* and *ES* using the distribution function.

Since backtesting is being conducted at the same time, these nine steps must be carried out for each time window and in the case of each time window the conditions must be true.

EMPIRICAL STUDY

Presentation of the data

The data for the calculations were downloaded from the website of the Budapest Stock Exchange (bet.hu). The data comprise the daily price data of seven equities – namely, FHB, MOL, MTELEKOM, OTP, Pannergy, Raba and Richter – covering a period of 10 years between 01/07/2005 and 29/06/2015. Some missing values were filled in with the price data of the previous day (except for weekends), resulting in a data series consisting of 2,607 values. The R programme was used for the examination and analysis of these data and for the calculations.

For the sake of comparability – as the price data move on different scales – the calculations were based on the logarithmic returns generated from the price data. The logarithmic return of an equity at the t^{th} point of time can be defined as follows:

$$R_t^L := ln\left(\frac{P_t}{P_{t-1}}\right) = ln(1+R_t^S),$$

where P_t signifies the price of the equity and $R_t^S := \frac{p_t - p_{t-1}}{p_{t-1}}$ the simple return on the equity at the t^{th} point of time. Similarly to equities, the logarithmic return on a portfolio consisting of n equities can be calculated as follows:

$$R_t^L := ln\left(\frac{S_t}{S_{t-1}}\right),$$

where $S_t = \sum_{i=1}^n S_{t,i} = \sum_{i=1}^n k_i \, P_{t,i}$ is the amount of money invested in the portfolio at the t^{th} point of time, $S_{t,i}$ is the amount of money invested in the i^{th} equity at the t^{th} point of time, whereas k_i is the quantity of the i^{th} equity in the portfolio, and $P_{t,i}$ is the price of the i^{th} equity at the t^{th} point of time.

The basic statistics calculated from the logarithmic returns expressed in percentages are shown in *Table 2* . It can be seen that the return of the FHB equities has the highest minimum, and the return of the Raba equities the highest maximum. It can also be deduced that from the seven equities only Raba and Richter have a positive average, and that standard deviation is the largest in the case of OTP.

For the calculations, a portfolio including exactly one piece of each equity is created. The logarithmic return on the portfolio so created over a period of ten years is presented in *Table 3*. For transparency reasons, these returns

Table 2

BASIC STATISTICS OF THE LOGARITHMIC RETURNS OF EQUITIES IN PERCENTAGES

	FHB	MOL	MTELEKOM	OTP	Pannergy	Raba	Richter
Minimum	–19.720	–16.220	–12.573	–16.230	–16.140	–16.229	–12.189
Lower quantile	–1.140	–1.184	–0.905	–1.314	–0.786	–0.900	–0.954
Median	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Mean	–0.024	–0.007	–0.031	–0.009	0.000	0.021	0.012
Upper quantile	0.970	1.176	0.860	1.351	0.645	0.852	0.961
Maximum	20.891	14.030	11.680	20.920	13.960	24.701	9.074

Source: own editing

(i.e. the data) are arranged in a table: the first column accommodates the first 250 elements, the second column the second 250 elements, and so on *(Table 3)*. Thus each column is in fact a time window that matches the relevant scenario. Scenarios describe possible future states. Consider r_t^m to signify the value of the return belonging to the m^{th} scenario and the t^{th} point of time, where $m = 1, \ldots, M$ and $t = 1, \ldots, T$. In the case of the data to be analysed, *N*–2606 and *T*–250, therefore *M*–2357. Note that due to the scheme, for example $r_2^1 = r_1^2$ or $r_2^2 = r_1^3$.

Historical backtesting

As it has already been shown, the first step of historical backtesting is to calculate the value of *VaR* and *ES* for each time window. In practice we can calculate with the sample realisation of the returns as continuous probability variables. Consider that r_t^m still signifies the value of the return belonging to the m^{th} scenario and the t^{th} point of time. For the calculation of *VaR* and *ES,* the data must be arranged in ascending order. Let r_t^{m*} signify within this ascending order of r_t^m elements the t^{th} element: $r_1^{m*} \leq r_2^{m*} \leq \ldots \leq r_T^{m*}$. Then the α quantile will be the k^{th} element, where the one-α^{th} part of the elements is lower and the one-$(1–\alpha)^{th}$ part is higher, that is $k = [T\alpha] = max\{m | m < T\alpha,\ m \in \mathbb{N}\}$.
Therefore the worst α percent will be the r_1^{m*}, r_2^{m*},… r_k^{m*} elements. *VaR* can be approximated as the negative of the α quantile:

$$VaR_\alpha^m \approx -r_k^{m*},$$

while *ES* can be approximated as the average of the worst α percentage (Acerbi, 2001):

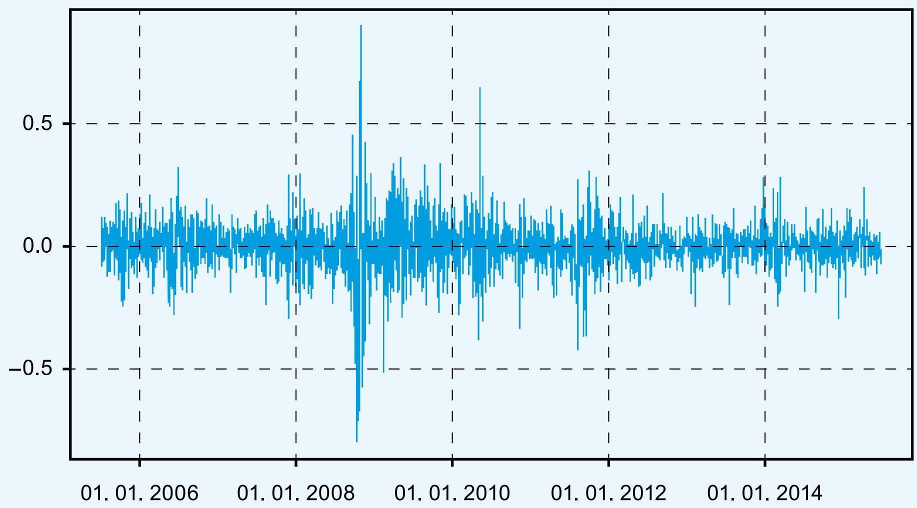
$$ES_\alpha^m \approx -\frac{\sum_{i=1}^{k} r_i^{m*}}{k}.$$

Backtesting based on copula simulation

The analysis has been done as far as implicit copulas are concerned with the Gauss and Student copulas, and as regards explicit copulas with the Frank and Clayton copulas. The Gumbel copula presented in the theoretical section must be excluded from the analysis, as it can only be used in the case of positively correlated data, which should be true for all time windows. Upon the examination of the first time window it shows that the value of Kendall's tau between Raba and Pannergy

Figure 3

DAILY LOGARITHMIC RETURNS ON THE PORTFOLIO BETWEEN 01/07/2005 AND 29/06/2015



Source: own editing

Table 3

ARRANGEMENT OF RETURNS IN TIME WINDOWS

		Scenario *(m)*					
		$m=1$	$m=2$	$m=3$	…	m=2356	m=2357
Time (t)	$t=1$	r_1^1	r_1^2	r_1^3	…	r_1^{2356}	r_1^{2357}
	$t=2$	r_2^1	r_2^2	r_2^3	…	r_2^{2356}	r_2^{2357}
	$t=3$	r_3^1	r_3^2	r_3^3	…	r_3^{2356}	r_3^{2357}
	⋮	⋮	⋮	⋮	…	⋮	⋮
	$t=250$	r_{250}^1	r_{250}^2	r_{250}^3	…	r_1^{2356}	r_1^{2357}
		VaR_α^1	VaR_α^2	VaR_α^3		VaR_α^{2356}	VaR_α^{2357}
		ES_α^1	ES_α^2	ES_α^3		ES_α^{2356}	ES_α^{2357}

Source: own editing

is –0.004, which excludes the possibility of using the Gumbel copula.

In the course of the analysis, the empirical distribution functions are used for the marginal distributions to estimate the distribution of each equity included in the portfolio. Of course, empirical distribution functions should be determined for each time window. In the m^{th} time window, the empirical distribution function of the i^{th} equity is

$$F_i^m(x) := \frac{1}{T+1}\sum_{i=1}^{T} 1_{\{r_{t,i}^m < x\}}(x),$$

where $r_{t,i}^m$ signifies the logarithmic return of the m^{th} equity of the i^{th} time window at the t^{th} point of time. The empirical distribution function can be approximated with the $F_i^m(r_{t,i}) \approx \frac{rank(r_{t,i})}{T+1}$ equation, where $rank(r_{t,i})$ is the rank of the i^{th} element of the t^{th} equity, calculated on the basis of the logarithmic returns of the given equity. The data so transformed are called pseudo-observations, which are already uniformly distributed over the (0.1) interval, i.e. the range of interpretation of the copula function.

For the estimation of the parameters of the (appropriate) copula, the canonical maximum *likelihood* method is used, as this method of estimation is based on the empirical marginal distribution functions. Using this approach, copula parameters can be estimated without the need to specify marginal distributions and their parameters (Yan, 2007).

The generation of random variables having a joint density function corresponding to the estimated copula function is assisted by Sklar's theorem. Based on the theorem, it is known that $C[F_1(x_1),\ldots,F_n(x_n)] = F(x_1, \ldots, x_n)$. Let us assume that marginal distributions are continuous and their inverse exists: F_i^{-1}, $i = 1, \ldots, n$. Additionally, the notation $F_i(x_i) := v_i$ is also introduced, therefore $x_i = F_i^{-1}(v_i)$. Using these notations, Sklar's theorem can be written as follows: $C(v_1, \ldots, v_n) = F[F_i^{-1}(v_1), \ldots, F_i^{-1}(v_n)]$. Therefore in this case the distribution of vector $v = (v_1, \ldots, v_n)$ is appropriate for the C copula function. Using the inverse function of marginal distributions, from vector v we can get simulated logarithmic return $x_1, \ldots, x_n$ for each equity, because: $x_1 = F_I^{-1}(v_1) = F_I^{-1}[F_I(x_1)], \ldots, x_n = F_n^{-1}(v_n) = F_n^{-1}[F_n(x_n)]$. Based on the value of these simulated logarithmic returns and the equity prices, we can calculate the equity prices estimated for the next day. And from the equity prices we can calculate the logarithmic return of a portfolio that includes exactly one piece of each equity.

The steps described above should be repeated as many times as needed to ensure that the simulated distribution is sufficiently close to the "real" distribution. Based on the literature and preliminary own calculations, I believe that repeating the simulation 6,000 times ensures that the difference between the "actual" and the simulated distribution is minimal; therefore, the programme is stable enough. From the distribution so generated the value of *VaR* and *ES* can be calculated.

The nine points presented in the theoretical part must be calculated for each time window, then using the backtesting method discussed in detail in the chapter "Backtesting" (i.e. the comparison of estimated and actual data) the accuracy and efficiency of the simulation can be tested both for *VaR* and *ES*.

Results and conclusions

The simulation was written using the R programme; in particular, the *copula* programme package (Hofert et al., 2014). The results for *VaR* are shown in *Table 4*, and for *ES* in *Table 5*. As the accuracy of the approach is determined by the closeness of the alpha values received from the simulation to the theoretical (original) alpha values, the figures included in the table show these differences

Table 4

RESULTS OF THE BACKTESTING OF COPULA-BASED AND HISTORICAL SIMULATION-BASED RISK CALCULATION FOR VALUE AT RISK (VAR), IN PERCENTAGES (%)

	VaR				
	Gauss	Clayton	Frank	Student	Historical
5.0%	0.7749	0.6900	1.4112	0.9023	0.1761
2.5%	0.8121	0.4299	1.4490	0.8121	0.2153
1.0%	0.5287	0.1890	0.9958	0.4013	0.2728

Source: own editing

Table 5

RESULTS OF THE BACKTESTING OF COPULA-BASED AND HISTORICAL SIMULATION-BASED RISK CALCULATION FOR EXPECTED SHORTFALL (ES), IN PERCENTAGES (%)

	ES				
	Gauss	Clayton	Frank	Student	Historical
5.0%	0.0125	0.0120	0.0128	0.0123	0.0149
2.5%	0.0133	0.0128	0.0135	0.0130	0.0775
1.0%	0.0142	0.0138	0.0144	0.0139	0.1712

Source: own editing0

for the four relevant copula functions and the historical simulation at three risk levels (5%, 2.5% and 1%).

Comparing the results of the backtesting based on historical simulation only, we found that *ES* proved to be a better risk measure than *VaR* (Miskolczi, 2016).

According to the results of the backtesting conducted on the basis of the copula function, of the four copula functions the Clayton copula provided the best approximation at all examined alpha levels both for *VaR* and *ES*. The worst performance was provided by the Frank copula in the case of both risk measures and all examined alpha levels. The *t*-copula proves to be more useful than the normal one in the financial field in general, as *t*-copulas have a thicker tail than normal copulas. In most cases it is not from normal distributions that the data are derived, and the modelling of extreme events is a problem. These tail events occur more frequently in reality than it is assumed by the normal distribution. This is also supported by the values calculated from my own data. It can be seen that the Gaussian copula performed better than the *t*-copula only at the 5 per cent level and only in the case of *VaR*. In the tails and in the case of *ES* the *t*-copula models risk more accurately. If *VaR* is compared to *ES* we find that *ES* is a far more accurate and better risk measure in the case of all copulas and all of the examined alpha levels, which again supports the use of *ES* as opposed to *VaR.*

As we have seen above, certain copulas provide a more accurate, and others a less accurate, modelling of reality. *ES* also proved to be better than *VaR* both in the historical and copula-based simulation. It is another important question, however, whether copula-based simulation is a better approximation of reality than, for example, historical simulation.

As regards *VaR,* we found that in most cases historical simulation provides a more accurate view of risk than the copula-based Monte Carlo simulation. With the data used in this paper, it was only in one case that the copula method proved better: in the case of the Clayton copula, at level $\alpha = 1\%$. Therefore, in this case the Clayton copula – the best of the four examined copulas – provided a better view of tail events (which are typically difficult to measure) and risk than the historical simulation.

As regards *ES,* the results received were far less ambiguous. All four copula models at all three alpha levels estimated risk much more accurately than the risk calculation performed with historical simulation.

Therefore, we found overall, that from the perspective of risk calculation, it is very important to examine the relationship between the instruments included in the portfolio, and their dependence structure. Risks cannot be mitigated by diversification if the instruments included in the portfolio behave, in whole or in part, the same way after a shock, obliterating the effect of the diversification. By using copula functions, however, we can shed light on these correlations, which may improve the accuracy of risk modelling. Simulation by means of copula functions can be tested with the backtesting method. In comparing the results we found that in the case of historical simulation, the copula function may help us formulate a more accurate view of the risk.

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