EURO Journal on Computational Optimization 11 (2023) 100072

Contents lists available at ScienceDirect

EURO Journal on Computational Optimization

journal homepage: www.elsevier.com/locate/ejco

Large-step predictor-corrector interior point method for sufficient linear complementarity problems based on the algebraic equivalent transformation

Tibor Illés [a], Petra Renáta Rigó [a,*], Roland Török [b]

[a] *Corvinus University of Budapest, Hungary*
[b] *Doctoral School of Economics, Business and Informatics, Corvinus University of Budapest, Hungary*

A R T I C L E I N F O

Keywords:
Predictor-corrector interior-point algorithm
$P_*(\kappa)$-linear complementarity problems
Wide neighbourhood
Algebraic equivalent transformation technique

A B S T R A C T

We introduce a new predictor-corrector interior-point algorithm for solving $P_*(\kappa)$-linear complementarity problems which works in a wide neighbourhood of the central path. We use the technique of algebraic equivalent transformation of the centering equations of the central path system. In this technique, we apply the function $\varphi(t) = \sqrt{t}$ in order to obtain the new search directions. We define the new wide neighbourhood $\mathcal{D}_\varphi$. In this way, we obtain the first interior-point method, where not only the central path system is transformed, but the definition of the neighbourhood is also modified taking into consideration the algebraic equivalent transformation technique. This gives a new direction in the research of interior-point algorithms. We prove that the interior-point method has $\mathcal{O}\left((1+\kappa)n \log\left(\frac{(\mathbf{x}^0)^T \mathbf{s}^0}{\epsilon}\right)\right)$ iteration complexity. Furthermore, we show the efficiency of the proposed predictor-corrector algorithm by providing numerical results. To our best knowledge, this is the first predictor-corrector interior-point algorithm which works in the $\mathcal{D}_\varphi$ neighbourhood using $\varphi(t) = \sqrt{t}$.

* Corresponding author.
E-mail addresses: tibor.illes@uni-corvinus.hu (T. Illés), petra.rigo@uni-corvinus.hu (P.R. Rigó), roland.torok@stud.uni-corvinus.hu (R. Török).

https://doi.org/10.1016/j.ejco.2023.100072

1. Introduction

Starting from the field of *linear optimization* (LO), *interior-point algorithms* (IPAs) have spread around different fields of mathematical programming, returning to nonlinear (convex) programming, as well. For analysis of IPAs see the monographs of Roos et al. [1], Wright [2], Ye [3], Klerk [4], Kojima et al. [5], and Nesterov and Nemirovskii [6].

IPAs for LO have been extended to more general classes of problems, such as *linear complementarity problems* (LCP) [5,7–12], *semidefinite programming problems* (SDP) [4,13–15], *smooth convex programming problems* (CPP) [6], and *symmetric cone optimization* (SCO) problems [16–21].

LCPs have several applications in different fields, such as optimization theory, engineering, business and economics, game theory etc [8,22,23]. For example, the Arrow-Debreu competitive market equilibrium problem with linear and Leontief utility functions are formulated as LCP [24,25]. Testing copositivity of matrices also has connection with solvability of special LCPs [26]. In 2020, Darvay et al. [27] introduced a *predictor-corrector* (PC) IPA for $P_*(\kappa)$-LCPs and obtained very promising numerical results for testing copositivity of matrices using LCPs. Note that the notion of $P_*(\kappa)$-matrix and $P_*(\kappa)$-LCPs will be defined in Section 2.

The monographs written by Cottle et al. [8] and Kojima et al. [5] summarize the most important results related to the theory and applications of LCPs. The solvability of the LCP is influenced by the properties of the problem's matrix. If the problem's matrix is skew-symmetric, see [1–3], or positive semidefinite, see [28], then LCPs can be solved in polynomial time by using IPAs. However, there is still an open question, whether the LCPs with other types of matrices can be solved in polynomial time [13]. In general, LCPs belong to the class of NP-complete problems, see [29]. The most important class of LCPs from the point of view of the complexity theory is the class of sufficient LCPs. This class was introduced by Cottle, Pang, and Venkateswaran [30]. The name *sufficient* comes from the observation that if the matrix of the LCP is sufficient, then that condition is sufficient to ensure that the solution set of the LCP is a convex, closed, bounded polyhedron [30]. The union of the sets $P_*(\kappa)$ for all nonnegative κ gives the class P_* [5]. Väliaho [31] demonstrated that the class of P_*-matrices is equivalent to the class of sufficient matrices introduced by Cottle et al. [30]. Most of the IPAs for LCPs with sufficient (P_*) matrices have polynomial iteration complexity in the size of the problem n; handicap of the matrix, $\kappa \geq 0$; the starting point's duality gap and in the accuracy parameter. As de Klerk and E.-Nagy [13] pointed out, the handicap of the matrix could be exponential in the bit length of the data, therefore it is still unknown whether for sufficient LCPs there exists a polynomial time algorithm in the size of the problem and the bit length.

Theoretical complexity analysis of IPAs for $P_*(\kappa)$-LCPs depends on parameter κ. However, the preliminary computational results [27,32] with different types of IPAs, even with matrices that have exponential value κ, show much better iterations number than it is predicted by their complexity results.

The PC IPAs have shown to be an efficient tool for solving LO and LCPs, respectively. In a main iteration they perform a predictor step and several corrector steps. One of the first PC IPAs for LO was proposed by Sonnevend et al. [33]. Later on, Mizuno, Todd and Ye [34] introduced such PC IPA for LO in which only a single corrector step is performed in each iteration of the algorithm and whose iteration complexity is the best known in the LO literature. These types of methods are called Mizuno-Todd-Ye (MTY) PC IPAs. It should be mentioned that in order to use only one corrector step in each iteration, the centrality parameter and the update parameter should be properly synchronized. Illés and Nagy [35], Potra and Sheng [7,36] and Gurtuna et al. [37] also introduced MTY-type PC IPAs for $P_*(\kappa)$-LCPs.

We can classify the IPAs based on the length of the steps, as well. In this way, there exist short- and long-step IPAs. The short-step algorithms generate the new iterates in a smaller neighbourhood, while the long-step ones work in a wider neighbourhood of the central path. Potra and Liu [38,39] presented first order and higher order PC IPAs for solving $P_*(\kappa)$-LCPs using the $\mathcal{N}_\infty^-$ wide neighborhood of the central path. It should be mentioned that there was a gap between theoretical and practical behavior of these IPAs in the sense that in theory, short-step algorithms had better theoretical complexity, while the long-step algorithms turned out to be more efficient in practice. Peng et al. [40] were the first who reduced this gap by using self-regular barriers. Similar results have been obtained by using a different class of kernel functions, the so-called eligible kernel functions, see [11,41]. After that, Potra [42] proposed a PC IPA for degenerate LCPs working in a wide neighbourhood of the central path having the same complexity as the best known short-step IPAs. Later on, Ai and Zhang [43] introduced a long-step IPA for monotone LCPs which has the same complexity as the currently best-known short-step IPAs. They decomposed the classical Newton direction as the sum of two other directions, corresponding to the negative and positive parts of the right-hand side. After that, Potra [44] generalized this algorithm to $P_*(\kappa)$-LCPs.

An important aspect in the analysis of the IPAs is the determination of the search directions. Peng et al. [40] used self-regular kernel functions and they reduced the theoretical complexity of large-update IPAs. Darvay [45] presented the technique of *algebraic equivalent transformation* (AET) of the centering equations of the central path system. The idea of this method is to apply a continuously differentiable, invertible, monotone increasing function φ on the nonlinear equation of the central path problem. The first PC IPAs using the AET method for determining search directions were given by Darvay [46,47] for LO and linearly constrained convex optimization. Kheirfam [48] generalized these algorithms to $P_*(\kappa)$-horizontal LCPs. Note that the most widely used function for finding search directions using the AET technique is the identity map. Darvay [45,49] used the square root function in the AET technique. Subsequently, Darvay et al. [50] proposed an IPA for LO based on the direction generated by using the function $\varphi(t) = t - \sqrt{t}$. In 2020, Darvay et al. [27,51] introduced PC IPAs for LO and $P_*(\kappa)$-LCPs, that are based on this search direction. They also provided a new approach for introducing PC IPAs using the AET technique, by decomposing the right hand side of the Newton-system into two terms: one depending and the other not depending on the parameter μ. Kheirfam and Haghighi [52] defined IPA for solving $P_*(\kappa)$-LCPs which uses the function $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ in the AET technique. Rigó [53] presented several IPAs that are based on the search directions obtained by using the function $\varphi(t) = t - \sqrt{t}$ in the AET technique. Haddou et al. [54] proposed a class of concave functions in the AET technique. However, they transformed the central path system in a different way. Illés et al. [55] defined a new class of AET functions in order to define primal-dual IPAs for solving $P_*(\kappa)$-LCPs. Note that $\varphi(t) = t$, $\varphi(t) = \sqrt{t}$ and $\varphi(t) = t - \sqrt{t}$ are members - among many other AET functions - of this class.

The purpose of this paper is to generalize the wide neighbourhoods $\mathcal{D}$ and $\mathcal{N}_\infty^-$ taking into consideration the transformed central path system obtained by using the AET approach. We also analyse the relationship between the new generalized neighbourhoods $\mathcal{D}_\varphi$ and $\mathcal{N}_{\infty,\varphi}^-$. We prove that in case of $\varphi(t) = t$ and $\varphi(t) = \sqrt{t}$ these neighbourhoods are the same. However, in case of $\varphi(t) = t - \sqrt{t}$, only the relation $\mathcal{D}_\varphi \subseteq \mathcal{N}_{\infty,\varphi}^-$ holds. Moreover, using the method given by Potra and Liu in [39] and the approach proposed by Darvay et al. [27], we introduce a first order PC IPA which works in the new wide neighbourhood $\mathcal{D}_\varphi$ using the function $\varphi(t) = \sqrt{t}$. This is the first PC IPA which works in the $\mathcal{D}_\varphi$ neighbourhood of the central path using $\varphi(t) = \sqrt{t}$ in the AET technique. We prove that this algorithm has $\mathcal{O}\left((1+\kappa)n \log\left(\frac{(\mathbf{x}^0)^T \mathbf{s}^0}{\epsilon}\right)\right)$ iteration complexity, similarly to that of Potra and Liu [39]. Following the results of Potra and Liu [39], our algorithm keeps the property that the predictor and corrector steplengths can be computed as a solution of some optimization problems.

Furthermore, by providing numerical results we also show the efficiency of the proposed PC IPA. We implemented a version of the proposed PC IPA and compared our PC IPA to the PC IPA using the function $\varphi(t) = \sqrt{t}$ in the AET technique and the neighbourhood $\mathcal{N}_{\infty,\varphi}^-(1-\beta)$ with the PC IPA of Potra and Liu proposed in [39], which corresponds to the $\varphi(t) = t$ case in our generalization of the wide neighbourhood.

The paper is organized in the following way. In Section 2 the $P_*(\kappa)$-LCPs and the central path problem is presented. Section 3 contains the AET technique and the new generalized wide neighbourhoods used in this paper. In Section 4 we present the new PC IPA for solving $P_*(\kappa)$-LCPs. Section 5 is devoted to the analysis of the proposed PC IPA. In Section 6 we propose a new version of the PC IPA which does not depend on κ. In Section 7 we provide numerical results that show the efficiency of the introduced IPA. Section 8 contains concluding remarks.

We use the following notations throughout the paper. Let $\mathbf{x}$ and $\mathbf{s}$ be two n-dimensional vectors. Then, $\mathbf{xs}$ denotes the componentwise product of the vectors $\mathbf{x}$ and $\mathbf{s}$. Furthermore, $\frac{\mathbf{x}}{\mathbf{s}} = \left[\frac{x_1}{s_1}, \frac{x_2}{s_2}, ..., \frac{x_n}{s_n}\right]^T$, where $s_i \neq 0$ for all $1 \leq i \leq n$. In case of an arbitrary function f and a vector $\mathbf{x}$ we use $f(\mathbf{x}) = [f(x_1), f(x_2), \ldots, f(x_n)]^T$. The vector $\mathbf{e} = [1, 1, \ldots, 1]^T$ denotes the n-dimensional all-one vector. The diagonal matrix obtained by the elements of the vector $\mathbf{x}$ is denoted by $diag(\mathbf{x})$. We denote by $\|\mathbf{x}\|$ the Euclidean norm and by $\|\mathbf{x}\|_\infty$ the infinity norm. Furthermore, $\mathbb{R}^n_\oplus$ denotes the nonnegative orthant, while $\mathbb{R}^n_+$ is the positive orthant.

2. Linear complementarity problems and matrix classes

In this section we present the linear complementarity problem and some well known matrix classes. In the linear complementarity problem (LCP) we would like to find such $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$ that satisfy the following constraints

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \qquad \mathbf{x}, \mathbf{s} \geq \mathbf{0}, \qquad \mathbf{xs} = \mathbf{0}, \quad (LCP)$$

where $M \in \mathbb{R}^{n\times n}$ and $\mathbf{q} \in \mathbb{R}^n$ are given.

In general, LCPs belong to the class of NP-complete problems, see [29]. However, if we suppose that the problem's matrix is a $P_*(\kappa)$-matrix, then IPAs solve the LCPs in polynomial time in the size of the problem n; handicap of the matrix, $\kappa \geq 0$; in the starting point's duality gap and in the accuracy parameter. Kojima et al. [5] defined the notion of $P_*(\kappa)$-matrices.

Definition 2.1. (Kojima et al. [5]) Let $\kappa \geq 0$ be a real number. A matrix $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$-*matrix* if

$$(1 + 4\kappa) \sum_{i\in I_+(\mathbf{x})} x_i(Mx)_i + \sum_{i\in I_-(\mathbf{x})} x_i(Mx)_i \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \tag{2.1}$$

where

$$I_+(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i > 0\} \text{ and } I_-(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i < 0\}.$$

It should be mentioned that $P_*(0)$ is the set of positive semidefinite matrices. The handicap of the matrix M is defined in the following way:

$$\hat{\kappa}(M) := \min\{\kappa : \kappa \geq 0, M \text{ is } \mathcal{P}_*(\kappa)\text{-matrix}\}.$$

Definition 2.2. (Kojima et al. [5]) A matrix $M \in \mathbb{R}^{n\times n}$ is a P_*-*matrix* if it is a $P_*(\kappa)$-matrix for some $\kappa \geq 0$. Let $P_*(\kappa)$ denote the set of $P_*(\kappa)$-matrices. Analogously, we also use P_* to denote the set of all P_*-matrices, i.e.,

$$P_* = \bigcup_{\kappa \geq 0} P_*(\kappa).$$

If M is $P_*(\kappa)$-matrix, then the corresponding LCP is called $P_*(\kappa)$-LCP. If the problem's matrix is not $P_*(\kappa)$-matrix, then we speak about general LCPs. We define the *feasible solution set* of the LCP as follows

$$\mathcal{F} := \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^{2n}_{\oplus} : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\},$$

the *set of interior points* as

$$\mathcal{F}^+ := \mathcal{F} \cap \mathbb{R}^{2n}_+$$

and the *set of feasible, complementarity solutions* in the following way:

$$\mathcal{F}^* := \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F} : \mathbf{x}\mathbf{s} = \mathbf{0}\}.$$

Throughout the paper we will assume that M is a $P_*(\kappa)$-matrix. We also suppose that $\mathcal{F}^+ \neq \emptyset$. The *central path problem* is defined as finding $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$ for all $\mu > 0$, for which

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \qquad \mathbf{x}, \mathbf{s} > \mathbf{0}, \qquad \mathbf{x}\mathbf{s} = \mu\mathbf{e}, \qquad (CPP)$$

where $\mathbf{e}$ denotes the n-dimensional all-one vector. If M is a $P_*(\kappa)$-matrix, then the central path system has unique solution for each $\mu > 0$, see [5]. The unique solution to the central path system for certain μ is called μ-center. The set of μ-centers form a path toward the solution when μ is running through positive real numbers.

Now, we present some other matrix classes. A matrix $M \in \mathbb{R}^{n \times n}$ is a *P-matrix* (*P_0-matrix*), if all of its principal minors are positive (nonnegative) [56,57]. Furthermore, Cottle et al. [30] defined the class of *sufficient matrices* as a subclass of P_0-matrices.

Definition 2.3. (Cottle et al. [30]) A matrix $M \in \mathbb{R}^{n \times n}$ is a *column sufficient matrix* if for all $\mathbf{x} \in \mathbb{R}^n$

$$X(M\mathbf{x}) \leq 0 \text{ implies } X(M\mathbf{x}) = 0,$$

where $X = diag(\mathbf{x})$. Analogously, matrix M is *row sufficient* if M^T is column sufficient. The matrix M is *sufficient* if it is both row and column sufficient.

Kojima et al. [5] showed that a P_*-matrix is column sufficient and Guu and Cottle [58] proved that it is row sufficient, too. Therefore, each P_*-matrix is sufficient. Moreover, Väliaho [31] proved the other inclusion as well, showing that the class of P_*-matrices is the same as the class of sufficient matrices.

3. Generalized wide neighbourhoods

In this section we define some new generalized neighbourhoods. Firstly, we present the AET technique of the centering equations of the central path system [45].

Let $\varphi : (\xi^2, \infty) \to \mathbb{R}$ be a continuously differentiable and invertible function, such that $\varphi'(t) > 0$, for each $t \geq \xi^2$, where $\xi \in [0, 1)$. Then, the transformed central path system is

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \qquad \mathbf{x}, \mathbf{s} > \mathbf{0}, \qquad \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right) = \varphi(\mathbf{e}), \qquad (CPP_\varphi)$$

where $\mu > 0$ is a target central path parameter and we use the notation $\varphi(\mathbf{x}) = [\varphi(x_1), \ldots, \varphi(x_n)]^T$. Let $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}$, then the *average duality gap* is defined as

$$\mu(\mathbf{x}, \mathbf{s}) \coloneqq \frac{\mathbf{x}^T\mathbf{s}}{n}. \tag{3.1}$$

Let us introduce a simplified notation for the argument of the function φ in (CPP_φ) as

$$\mathbf{u} \coloneqq \frac{\mathbf{x}\,\mathbf{s}}{\mu(\mathbf{x}, \mathbf{s})}. \tag{3.2}$$

Consider the following generalized *proximity measure*

$$\delta^-_{\infty,\varphi}(\mathbf{x}, \mathbf{s}) \coloneqq \left\| [\varphi(\mathbf{u}) - \varphi(\mathbf{e})]^- \right\|_\infty .$$

Using the introduced proximity measure and the AET approach, we introduce the generalized *wide neighbourhood* of the (CPP_φ):

$$\mathcal{N}^-_{\infty,\varphi}(\alpha) \coloneqq \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \delta^-_{\infty,\varphi}(\mathbf{x}, \mathbf{s}) \leq \alpha\}. \tag{3.3}$$

It should be mentioned that in case of $\varphi(t) = t$ we get the wide neighbourhood used by Potra and Liu [39]:

$$\mathcal{N}^-_{\infty}(\alpha) \coloneqq \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \delta^-_{\infty}(\mathbf{x}, \mathbf{s}) \leq \alpha\}. \tag{3.4}$$

We also introduce another, generalized *wide neighbourhood* for (CPP_φ):

$$\mathcal{D}_\varphi(\beta) \coloneqq \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \varphi(\mathbf{u}) \geq \beta\varphi(\mathbf{e})\}. \tag{3.5}$$

Note, that in the special case when $\varphi(t) = t$, we get the wide neighbourhood used in [39]:

$$\mathcal{D}(\beta) \coloneqq \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \mathbf{u} \geq \beta\mathbf{e}\}. \tag{3.6}$$

The following lemma represents a novelty of the paper. It plays an important role in this theory, because it shows which function used in the AET technique in the literature can be applied in this approach for introducing PC IPAs working in the generalized wide neighbourhood given in (3.3). However, the complexity analysis of the algorithm could be done in a simplier wide neighnourhood (3.5).

Lemma 3.1. Let $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+$ and $\alpha \in (0,1)$. Then, in case of $\varphi(t) = t$ and $\varphi(t) = \sqrt{t}$ we have $\mathcal{N}^-_{\infty,\varphi}(\alpha) = \mathcal{D}_\varphi(1-\alpha)$. In case of $\varphi(t) = t - \sqrt{t}$ we have $\mathcal{D}_\varphi(1-\alpha) \subseteq \mathcal{N}^-_{\infty,\varphi}(\alpha)$.

Proof. Firstly, we consider the case when $\varphi(t) = t$:

$$\begin{aligned}
(\mathbf{x}, \mathbf{s}) \in \mathcal{D}(1-\alpha) &\iff \mathbf{xs} \geq (1-\alpha)\mu(\mathbf{x}, \mathbf{s})\mathbf{e} = \mu(\mathbf{x}, \mathbf{s})\mathbf{e} - \alpha\mu(\mathbf{x}, \mathbf{s})\mathbf{e} \\
&\iff \mathbf{u} - \mathbf{e} \geq -\alpha \mathbf{e} \iff \left\| [\mathbf{u} - \mathbf{e}]^- \right\|_\infty \leq \alpha \\
&\iff (\mathbf{x}, \mathbf{s}) \in \mathcal{N}^-_\infty(\alpha).
\end{aligned}$$

Next, consider the other cases. Then, we have

$$\begin{aligned}
(\mathbf{x}, \mathbf{s}) \in \mathcal{N}^-_{\infty,\varphi}(\alpha) &\iff \left\| [\varphi(\mathbf{u}) - \varphi(\mathbf{e})]^- \right\|_\infty \leq \alpha \\
&\iff \varphi(\mathbf{u}) - \varphi(\mathbf{e}) \geq -\alpha\, \mathbf{e} \iff \varphi(\mathbf{u}) \geq \varphi(\mathbf{e}) - \alpha \mathbf{e}
\end{aligned}$$

and

$$(\mathbf{x}, \mathbf{s}) \in \mathcal{D}_\varphi(1-\alpha) \iff \varphi(\mathbf{u}) \geq (1-\alpha)\varphi(\mathbf{e}) = \varphi(\mathbf{e}) - \alpha\varphi(\mathbf{e}).$$

It is easy to see, that in case of $\varphi(t) = \sqrt{t}$ the $\varphi(\mathbf{e}) = \mathbf{e}$ holds, so we obtain $\mathcal{N}^-_{\infty,\varphi}(\alpha) = \mathcal{D}_\varphi(1-\alpha)$. In case of $\varphi(t) = t - \sqrt{t}$ only $\mathcal{D}_\varphi(1-\alpha) \subseteq \mathcal{N}^-_{\infty,\varphi}(\alpha)$ holds. □

Note that the typical MTY-type PC IPAs are similar to the one proposed in [35]. In the following section we present the new PC IPA, which works in wide neighbourhood and uses the function $\varphi(t) = \sqrt{t}$ in the AET technique.

4. New predictor-corrector interior-point algorithm

If we apply Newton's method to the system (CPP_φ) we obtain

$$\begin{aligned}
-M\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0}, \\
\mathbf{s}\Delta\mathbf{x} + \mathbf{x}\Delta\mathbf{s} &= \mathbf{a}_\varphi,
\end{aligned} \tag{4.1}$$

where

$$\mathbf{a}_\varphi = \mu \frac{\varphi(\mathbf{e}) - \varphi\left(\frac{\mathbf{xs}}{\mu}\right)}{\varphi'\left(\frac{\mathbf{xs}}{\mu}\right)} \tag{4.2}$$

and $\mu > 0$. In this paper we consider the $\varphi(t) = \sqrt{t}$ case in this generalized wide neighbourhood approach. Substituting $\varphi(t) = \sqrt{t}$ in (4.1) we get

$$\begin{aligned} -M\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0}, \\ \mathbf{s}\Delta\mathbf{x} + \mathbf{x}\Delta\mathbf{s} &= 2\left(\sqrt{\mu\mathbf{xs}} - \mathbf{xs}\right). \end{aligned} \tag{4.3}$$

In the predictor step we use the approach given by Darvay et al. [27], where the right hand side of (4.3) is decomposed into the two terms, one of which depends on μ, while the other does not. Next, we set $\mu = 0$ and obtain

$$\begin{aligned} -M\Delta^p\mathbf{x} + \Delta^p\mathbf{s} &= \mathbf{0}, \\ \mathbf{s}\Delta^p\mathbf{x} + \mathbf{x}\Delta^p\mathbf{s} &= -2\mathbf{xs}, \end{aligned} \tag{4.4}$$

where $(\Delta^p\mathbf{x}, \Delta^p\mathbf{s})$ denote the predictor search directions.

There are important and useful lemmas that give information about the magnitudes of the solutions of the Newton system. Let us recall these lemmas.

Lemma 4.1. (Lemma 3.2 in [39]) Assume that we have a $\mathcal{P}_*(\kappa)$-LCP and let $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the solution of the following linear system:

$$\begin{aligned} -M\,\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0}, \\ \mathbf{s}\,\Delta\mathbf{x} + \mathbf{x}\,\Delta\mathbf{s} &= \mathbf{a}, \end{aligned}$$

where $(\Delta\mathbf{x}, \Delta\mathbf{s}) \in \mathbb{R}^{2n}_+$ and $\mathbf{a} \in \mathbb{R}^n$ are given. Defining

$$\mathcal{K}_+ = \{i : \Delta x_i \Delta s_i > 0\} \qquad \text{and} \qquad \mathcal{K}_- = \{i : \Delta x_i \Delta s_i < 0\}$$

we have

$$\frac{1}{1+4\kappa}\left\|\Delta\mathbf{x}\,\Delta\mathbf{s}\right\|_\infty \le \sum_{i\in\mathcal{K}_+} \Delta x_i \Delta s_i \le \frac{1}{4}\left\|(\mathbf{x}\,\mathbf{s})^{-\frac{1}{2}}\,\mathbf{a}\right\|_2^2. \tag{4.5}$$

Lemma 4.2. (Lemma 3.3 in [39]) Assume that we have a $\mathcal{P}_*(\kappa)$-LCP and let $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the solution of the following linear system:

$$\begin{aligned} -M\,\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0}, \\ \mathbf{s}\,\Delta\mathbf{x} + \mathbf{x}\,\Delta\mathbf{s} &= \mathbf{a}, \end{aligned}$$

where $(\Delta\mathbf{x}, \Delta\mathbf{s}) \in \mathbb{R}^{2n}_+$ and $\mathbf{a} \in \mathbb{R}^n$ are given. Then, the following inequality holds:

$$\Delta\mathbf{x}^T\Delta\mathbf{s} \ge -\kappa\left\|(\mathbf{x}\,\mathbf{s})^{-\frac{1}{2}}\,\mathbf{a}\right\|_2^2. \tag{4.6}$$

In the following subsection we deal with the predictor step.

4.1. Predictor step

Let $(\mathbf{x}, \mathbf{s}) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta)$, where $\beta \in (0,1)$. Then, the predictor search direction $(\Delta^p\mathbf{x}, \Delta^p\mathbf{s})$ can be calculated from system (4.4). Defining the predictor updates as

$$\mathbf{x}^p(\theta) = \mathbf{x} + \theta\Delta^p\mathbf{x} \quad \text{and} \quad \mathbf{s}^p(\theta) = \mathbf{s} + \theta\Delta^p\mathbf{s}, \tag{4.7}$$

and using (3.1) and (4.4), after some calculations we have

$$\mathbf{x}^p(\theta)\mathbf{s}^p(\theta) = (1-2\theta)\mathbf{x}\mathbf{s} + \theta^2\Delta^p\mathbf{x}\Delta^p\mathbf{s}, \qquad \mu_p(\theta) = (1-2\theta)\mu(\mathbf{x},\mathbf{s}) + \frac{\theta^2\Delta^p\mathbf{x}^T\Delta^p\mathbf{s}}{n}, \tag{4.8}$$

where $\mu_p(\theta) = \mu(\mathbf{x}^p(\theta), \mathbf{s}^p(\theta))$. Furthermore, we obtain that $-M\mathbf{x}^p(\theta) + \mathbf{s}^p(\theta) = \mathbf{q}$. We want to determine the step size $\theta > 0$ in such a way, that

$$(\mathbf{x}^p(\theta), \mathbf{s}^p(\theta)) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta+\beta\gamma) = \mathcal{D}_\varphi((1-\gamma)\,\beta)$$

holds, where

$$\gamma = \frac{1-\beta}{5((1+4\kappa)n+1)}. \tag{4.9}$$

It should be mentioned that in [39] the value of this parameter is $\gamma = \frac{1-\beta}{(1+4\kappa)n+1}$. However, the new search direction used in this paper influences the value of γ, hence we chose a γ for which the complexity analysis of the new PC IPA works. It should be mentioned, that several other values of γ can be given for which the complexity analysis of the algorithm could work. This choice of the value γ will become clear from the analysis of the algorithm. We have to calculate the largest θ such that $(\mathbf{x}^p(\theta), \mathbf{s}^p(\theta)) \in \mathcal{F}^+$ and the following inequality holds

$$\sqrt{\frac{\mathbf{x}^p(\theta)\mathbf{s}^p(\theta)}{\mu(\mathbf{x}^p(\theta), \mathbf{s}^p(\theta))}} \geq (1-\gamma)\,\beta\,\mathbf{e}, \tag{4.10}$$

that is exactly the same as

$$\frac{(\mathbf{x}+\theta\Delta^p\mathbf{x})\,(\mathbf{s}+\theta\Delta^p\mathbf{s})}{(\mathbf{x}+\theta\Delta^p\mathbf{x})^T(\mathbf{s}+\theta\Delta^p\mathbf{s})} \geq \frac{(1-\gamma)^2\beta^2}{n}\,\mathbf{e}. \tag{4.11}$$

From the requirements $(\mathbf{x}^p(\theta), \mathbf{s}^p(\theta)) \in \mathcal{F}^+$ follows that $\theta_F = \min\{\theta_x, \theta_s\}$, where

$$\theta_x = \min\left\{-\frac{x_i}{\Delta^p x_i} : \Delta^p x_i < 0\right\} \quad \text{and} \quad \theta_s = \min\left\{-\frac{s_i}{\Delta^p s_i} : \Delta^p s_i < 0\right\}.$$

Thus, $\theta_F > 0$ is the largest step that ensures the feasibility of the new predictor solution. However, from the inequality (4.11) further restrictions on the predictor steplength follow, that after elementary computations can be summarized as

$$a_i\,\theta^2 - b_i\,\theta + c_i \geq 0, \qquad \text{for all } \; i, \tag{4.12}$$

where $a_i = \Delta^p x_i\,\Delta^p s_i - (1-\gamma)^2\beta^2\mu(\Delta^p \mathbf{x}, \Delta^p \mathbf{s})$, $b_i = 2\,(x_i\,s_i - (1-\gamma)^2\beta^2\mu(\mathbf{x},\mathbf{s}))$, and $c_i = \frac{b_i}{2}$ for all indices i. Since $(\mathbf{x},\mathbf{s}) \in \mathcal{D}_\varphi(\beta)$, it follows that $b_i \geq 0$ for all i indices, thus $\theta = 0$ satisfy all inequalities.

After some elementary computations, we obtain that inequality (4.12) is fulfilled if $\theta \in (0, \theta_{p_i}]$, where

$$\theta_{p_i} = \begin{cases} \infty, & \text{if } \Delta_i \leq 0 \\ \frac{1}{2}, & \text{if } a_i = 0 \\ \zeta_i, & \text{if } \Delta_i > 0 \text{ and } a_i \neq 0, \end{cases} \tag{4.13}$$

where

$$\Delta_i = b_i^2 - 4a_i c_i = b_i^2 - 2a_i b_i$$

and

$$\zeta_i = \frac{b_i - \sqrt{\Delta_i}}{2a_i} = \frac{b_i^2 - \Delta_i}{2a_i(b_i + \sqrt{\Delta_i})} = \frac{b_i}{b_i + \sqrt{\Delta_i}} = \frac{1}{1 + \sqrt{1 - 2\,\frac{a_i}{b_i}}}.$$

Taking

$$\theta_N = \min\{\theta_{p_i} : \; 1, \ldots, n\} \tag{4.14}$$

we get an appropriate predictor steplength which ensures that the predictor solution stays in the predictor neighbourhood.

Still we need to ensure that $\mu_p(\theta) > 0$ is satisfied for the computed predictor direction with properly selected $\theta > 0$ steplength. Let $(\mathbf{x},\mathbf{s}) \in \mathcal{D}_\varphi(\beta)$ and $\beta \in (0,1)$, then the vector $\mathbf{u}$ is defined as in (3.2)

$$\mathbf{u} = \frac{\mathbf{x}\,\mathbf{s}}{\mu(\mathbf{x},\mathbf{s})} \quad \text{and} \quad \mathbf{v} = \frac{\Delta^p \mathbf{x}\,\Delta^p \mathbf{s}}{\mu(\mathbf{x},\mathbf{s})}. \tag{4.15}$$

For the vector $\mathbf{v}$, the following bounds can be obtained by using Lemma 4.1 and Lemma 4.2:

$$\|\mathbf{v}\|_\infty \leq (1+4\kappa)n \quad \text{and} \quad -4\kappa n \leq \mathbf{e}^T\mathbf{v} \leq \sum_{i\in\mathcal{I}_+} v_i \leq n. \tag{4.16}$$

Taking into consideration that $\frac{\mu(\Delta^p \mathbf{x}, \Delta^p \mathbf{s})}{\mu(\mathbf{x},\mathbf{s})} = \frac{\mathbf{e}^T\mathbf{v}}{n}$ and using (4.8), the quadratic equation defined by $\mu_p(\theta) = 0$ can be written as

$$\frac{\mathbf{e}^T\mathbf{v}}{n}\,\theta^2 - 2\,\theta + 1 = 0, \tag{4.17}$$

therefore, the discriminant denoted by Δ_μ is

$$\Delta_\mu = 4\left(1 - \frac{\mathbf{e}^T\mathbf{v}}{n}\right)$$

and it is always nonnegative, because $-4\,\kappa \leq \frac{\mathbf{e}^T\mathbf{v}}{n} \leq 1$ follows from (4.16). The smallest positive root will be:

$$\theta_0 = \frac{2 - \sqrt{4 - 4\,\frac{\mathbf{e}^T\mathbf{v}}{n}}}{2\,\frac{\mathbf{e}^T\mathbf{v}}{n}} = \frac{1}{1 + \sqrt{1 - \frac{\mathbf{e}^T\mathbf{v}}{n}}}. \tag{4.18}$$

Therefore,

$$\mu_p(\theta) > \mu_p(\theta_0) = 0, \text{ for all } 0 \leq \theta < \theta_0. \tag{4.19}$$

Now, we are ready to define the predictor steplength θ_p, which satisfies the interior point condition, neighbourhood condition and ensures decrease in the centrality parameter μ, as follows:

$$\theta_p = \min\{\theta_F, \theta_N, \theta_0\}. \tag{4.20}$$

From the definition of θ_F it follows that $\theta_F \geq \theta_N$. Taking into consideration the definition of θ_p, for all $0 \leq \theta < \theta_p$ we have

$$\sqrt{\mathbf{x}^p(\theta)\mathbf{s}^p(\theta)} \geq (1-\gamma)\beta\sqrt{\mu_p(\theta)} > (1-\gamma)\beta\sqrt{\mu_p(\theta_p)} \geq 0. \tag{4.21}$$

Using standard continuity argument it can be shown that $\mathbf{x}^p(\theta) > \mathbf{0}$ and $\mathbf{s}^p(\theta) > \mathbf{0}$, for all $\theta \in (0, \theta_p)$. Therefore $(\mathbf{x}^p, \mathbf{s}^p) \in \mathcal{F}^+$, where $\mathbf{x}^p = \mathbf{x}^p(\theta_p)$ and $\mathbf{s}^p = \mathbf{s}^p(\theta_p)$.

The obtained result can be summarized in the following lemma.

Lemma 4.3. Let $(\mathbf{x}, \mathbf{s}) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta) = \mathcal{D}_\varphi(\beta)$. Then, $\exists\theta_p > 0$ predictor steplength such that

$$(\mathbf{x}^p, \mathbf{s}^p) \in \mathcal{N}^-_{\infty,\varphi}(1 - \beta + \beta\gamma) = \mathcal{D}_\varphi((1-\gamma)\,\beta)$$

holds, where $\beta, \gamma \in (0, 1)$ are given parameters.

It should be mentioned, that we can calculate the exact value of θ_p in each iteration of the algorithm. However, this is very time consuming. Therefore, intuitively, we can define θ_p using an optimization problem in the following way

$$\bar{\theta}_p = \sup\{\hat{\theta} > 0 : (\mathbf{x}^p(\theta), \mathbf{s}^p(\theta)) \in \mathcal{N}^-_{\infty,\varphi}(1 - \beta + \beta\gamma), \forall \theta \in [0, \hat{\theta}]\}, \tag{4.22}$$

as well. Although, someone might think that it is necessary to prove that θ_p and $\bar{\theta}_p$ are equal, we do not deal with this question, because it will be enough for us to know a $\hat{\theta}$ mentioned in (4.22). Our goal is to give a lower bound $\hat{\theta}$ on the value θ_p depending only on κ, n and β. The lower bound $\hat{\theta}$ can be used instead of θ_p to obtain the necessary decrease of the duality gap (see the proof of Theorem 5.1). Before we compute the lower bound $\hat{\theta}$, we need the following technical lemma.

Lemma 4.4. Let $\mathbf{u} = \frac{\mathbf{x}\,\mathbf{s}}{\mu(\mathbf{x}\,\mathbf{s})}$, where $(\mathbf{x}, \mathbf{s}) \in \mathcal{D}_\varphi(\beta)$, $\beta \in (0, 1)$ and $\gamma = \frac{1-\beta}{5((1+4\kappa)n+1)}$. Then, we have

$$u_i - ((1-\gamma)\,\beta)^2 \geq \beta^2\,\gamma.$$

Proof. Before the predictor step $(\mathbf{x}, \mathbf{s}) \in \mathcal{D}_\varphi(\beta)$, hence

$$u_i - ((1-\gamma)\beta)^2 = u_i - \beta^2 + 2\beta^2\gamma - \beta^2\gamma^2 \geq 2\beta^2\gamma - \beta^2\gamma^2.$$

It follows

$$2\beta^2\gamma - \beta^2\gamma^2 \geq \beta^2\gamma, \tag{4.23}$$

hence, we obtain $\gamma \geq \gamma^2$, which holds for all $\gamma < 1$. Using the definition of γ in (4.9) and $0 < \beta < 1$ we obtain the desired inequality. □

Now, we are ready to compute a lower bound $\hat{\theta}$ on θ_p depending only on κ, n and β.

Lemma 4.5. Let $(\mathbf{x}, \mathbf{s}) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta) = \mathcal{D}_\varphi(\beta)$. Then,

$$\theta_p > \hat{\theta},$$

where $\hat{\theta} = \dfrac{\beta\sqrt{1-\beta}}{5((1+4\kappa)n+2)}$.

Proof. Using the results of the Lemma 4.4 it follows

$$\frac{b_i}{\mu(\mathbf{x}, \mathbf{s})} = 2\,(u_i - ((1-\gamma)\,\beta)^2) \geq 2\,\beta^2\,\gamma > 0.$$

Next, we have

$$-\frac{a_i}{\mu(\mathbf{x}, \mathbf{s})} = -v_i + ((1-\gamma)\,\beta)^2\,\frac{\mathbf{e}^T\mathbf{v}}{n} \leq \|\mathbf{v}\|_\infty + ((1-\gamma)\,\beta)^2 \leq \|\mathbf{v}\|_\infty + 1,$$

since $\frac{\mu(\Delta^p\mathbf{x},\Delta^p\mathbf{s})}{\mu(\mathbf{x},\mathbf{s})} = \frac{\mathbf{e}^T\mathbf{v}}{n}$ and we used bounds (4.16) on the norm of $\mathbf{v}$ and on $\mathbf{e}^T\mathbf{v}$. When the discriminant Δ_i of the quadratic equation related to (4.12) is positive and $a_i \neq 0$, then

$$\zeta_i = \frac{1}{1+\sqrt{1+\frac{-v_i+((1-\gamma)\beta)^2\,\frac{\mathbf{e}^T\mathbf{v}}{n}}{u_i-((1-\gamma)\,\beta)^2}}} \geq \frac{1}{1+\sqrt{1+\frac{\|\mathbf{v}\|_\infty+1}{\beta^2\,\gamma}}} \geq \frac{1}{1+\sqrt{1+\frac{(1+4\,\kappa)\,n+1}{\beta^2\,\gamma}}},$$

which follows from the fact that $u_i - ((1-\gamma)\,\beta)^2 \geq \beta^2\,\gamma > 0$ and the consequences of Lemmas 4.1 and 4.2. Using the definition of γ we can rewrite the lower bound as

$$\zeta_i \geq \frac{\beta\,\sqrt{1-\beta}}{\beta\,\sqrt{1-\beta}+\sqrt{\beta^2\,(1-\beta)+5((1+4\,\kappa)\,n+1)^2}}.$$

It can be seen that $\beta\sqrt{1-\beta} \leq \frac{1}{2}$, hence the denominator of the previous fraction can be bounded from above as

$$\begin{aligned}\beta\sqrt{1-\beta}+\sqrt{\beta^2(1-\beta)+5((1+4\kappa)n+1)^2} &\leq \frac{1}{2}+\sqrt{5((1+4\kappa)n+1)^2+\frac{1}{4}}\\ &< 5((1+4\kappa)n+2),\end{aligned}$$

thus we get a bound on ζ_i depending on the parameters β, κ and the problem size n

$$\zeta_i > \hat{\theta} := \frac{\beta\sqrt{1-\beta}}{5((1+4\kappa)n+2)}. \tag{4.24}$$

Using the definition of $\mu_p(\theta)$, the quadratic equation $\mu_p(\theta) = 0$ can be written as in (4.17) and the smallest root θ_0 as in (4.18). Root θ_0 satisfies the required lower bound because (4.16) holds, namely

$$\theta_0 = \frac{1}{1+\sqrt{1-\frac{\mathbf{e}^T\mathbf{v}}{n}}} \geq \frac{1}{1+\sqrt{1+4\kappa}} > \hat{\theta}. \tag{4.25}$$

Therefore, the predictor steplength satisfies $\theta_p > \hat{\theta}$. $\square$

In the following subsection we deal with the corrector step.

4.2. Corrector step

After the predictor step, using (4.7), (4.14) and Lemma 4.3 we have

$$(\mathbf{x}^p, \mathbf{s}^p) = (\mathbf{x}^p(\theta_p), \mathbf{s}^p(\theta_p)) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta+\beta\gamma) = \mathcal{D}_\varphi((1-\gamma)\,\beta). \tag{4.26}$$

The output of the predictor step will be the input of the corrector step. Using system (4.3) we calculate the corrector direction $(\Delta^c\mathbf{x}, \Delta^c\mathbf{s})$ from the following system:

$$
\begin{aligned}
&-M\Delta^c\mathbf{x} + \Delta^c\mathbf{s} = 0, \\
&\mathbf{s}^p\Delta^c\mathbf{x} + \mathbf{x}^p\Delta^c\mathbf{s} = 2\left(\sqrt{\mu_p\,\mathbf{x}^p\,\mathbf{s}^p} - \mathbf{x}^p\,\mathbf{s}^p\right),
\end{aligned} \tag{4.27}
$$

where

$$
\mu_p = \mu(\mathbf{x}^p, \mathbf{s}^p) = \frac{(\mathbf{x}^p)^T\mathbf{s}^p}{n}. \tag{4.28}
$$

The goal of the corrector step is to return the iterate to the narrower, corrector neighbourhood of the central path. Thus, the best steplength of the corrector step is defined in the following way:

$$
\theta_c := \arg\min\{\mu_c(\theta) : (\mathbf{x}^c(\theta), \mathbf{s}^c(\theta)) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta) = \mathcal{D}_\varphi(\beta)\}, \tag{4.29}
$$

where

$$
\mu_c(\theta) = \mu(\mathbf{x}^c(\theta), \mathbf{s}^c(\theta)) = \frac{(\mathbf{x}^c(\theta))^T\mathbf{s}^c(\theta)}{n} \tag{4.30}
$$

and

$$
\mathbf{x}^c(\theta) = \mathbf{x}^p + \theta\Delta^c\mathbf{x}, \quad \mathbf{s}^c(\theta) = \mathbf{s}^p + \theta\Delta^c\mathbf{s}. \tag{4.31}
$$

Instead of solving the optimization problem given in (4.29), for us it is enough to define such corrector steplength $\bar{\theta}$ (see (5.10) in Theorem 5.1) which ensures that in one step we get back into the corrector neighbourhood and gives enough decrease in the duality gap.

Using system (4.27), we determine the corrector search directions $(\Delta^c\mathbf{x}, \Delta^c\mathbf{s})$. We describe the way how to calculate the corrector steplength θ_c. Using (4.27), (4.30) and (4.31) we have

$$
\mathbf{x}^c(\theta)\mathbf{s}^c(\theta) = (1-2\theta)\mathbf{x}^p\mathbf{s}^p + 2\theta\sqrt{\mu_p\mathbf{x}^p\mathbf{s}^p} + \theta^2\Delta^c\mathbf{x}\Delta^c\mathbf{s}, \tag{4.32}
$$

and

$$
\begin{aligned}
\mu_c(\theta) = \frac{\mathbf{x}^c(\theta)^T\mathbf{s}^c(\theta)}{n} &= (1-2\theta)\mu_p + 2\theta\frac{\mathbf{e}^T\sqrt{\mu_p\mathbf{x}^p\mathbf{s}^p}}{n} + \frac{\theta^2\Delta^c\mathbf{x}^T\Delta^c\mathbf{s}}{n} \\
&\le (1-2\theta)\mu_p + 2\theta\frac{\sqrt{\mu_p}}{n}\sqrt{n}\sqrt{\mathbf{e}^T\mathbf{x}^p\mathbf{s}^p} + \frac{\theta^2\Delta^c\mathbf{x}^T\Delta^c\mathbf{s}}{n} \\
&= (1-2\theta)\mu_p + 2\theta\mu_p + \frac{\theta^2\Delta^c\mathbf{x}^T\Delta^c\mathbf{s}}{n} = \mu_p + \frac{\theta^2\Delta^c\mathbf{x}^T\Delta^c\mathbf{s}}{n} =: \bar{\mu}_c(\theta),
\end{aligned} \tag{4.33}
$$

where we used that $\mathbf{e}^T\sqrt{\mathbf{x}^p\mathbf{s}^p} \le \sqrt{n}\sqrt{\mathbf{e}^T\mathbf{x}^p\mathbf{s}^p}$ due to the Cauchy-Schwartz inequality.

Algorithm 1: First-order predictor-corrector algorithm.

Input:
Given $\kappa \geq \hat{\kappa}(M)$, $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta)$, $\beta \in (0,1)$
Calculate $\gamma = \frac{1-\beta}{5((1+4\kappa)n+1)}$
Let $\mu_0 = \mu(\mathbf{x}^0, \mathbf{s}^0)$ and $k = 0$
$\varepsilon > 0$ precision value.
Output: $(\mathbf{x}^k, \mathbf{s}^k) : \mathbf{x}^{kT}\mathbf{s}^k \leq \varepsilon$
begin
  while $n\mu \geq \varepsilon$ **do**
    (Predictor step);
    $\mathbf{x} := \mathbf{x}^k$, $\mathbf{s} := \mathbf{s}^k$;
    Step 1. Calculate affine direction from (4.4);
    Step 2. Calculate the predictor steplength using (4.20);
    Step 3. Calculate $(\mathbf{x}^p, \mathbf{s}^p)$ using (4.26);
    if $\mu(\mathbf{x}^p, s^p) = 0$ **then**
      STOP; Optimal solution found;
    else
      if $(\mathbf{x}^p, \mathbf{s}^p) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta)$ **then**
        $(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}) = (\mathbf{x}^p, \mathbf{s}^p)$, $\mu^{k+1} = \mu(\mathbf{x}^p, \mathbf{s}^p)$, $k = k+1$, RETURN;
      else
        (Corrector step);
        Step 4. Calculate centering direction from (4.27);
        Step 5. Calculate centering steplength using (4.29);
        Step 6. Calculate $(\mathbf{x}^c, \mathbf{s}^c)$ using (4.40);
      end
      $(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}) = (\mathbf{x}^c, \mathbf{s}^c)$, $\mu^{k+1} = \mu(\mathbf{x}^c, \mathbf{s}^c)$, $k = k+1$, RETURN;
    end
  end
end

Moreover, using (4.7) and (4.28) we consider the following notations:

$$\bar{\mathbf{u}} = \frac{\mathbf{x}^p\mathbf{s}^p}{\mu_p}, \qquad \bar{\mathbf{v}} = \frac{\Delta^c\mathbf{x}\Delta^c\mathbf{s}}{\mu_p}. \tag{4.34}$$

We want to reach

$$\sqrt{\frac{\mathbf{x}^c(\theta)\mathbf{s}^c(\theta)}{\mu_c(\theta)}} \geq \beta\,\mathbf{e}, \tag{4.35}$$

that is exactly the same as

$$(\mathbf{x}^p + \theta\Delta^c\mathbf{x})\,(\mathbf{s}^p + \theta\Delta^c\mathbf{s}) \geq \mu_c(\theta)\,\beta^2\,\mathbf{e}. \tag{4.36}$$

Using (4.33) we have

$$\mu_c(\theta) \leq \bar{\mu}_c(\theta). \tag{4.37}$$

Hence, it is enough to reach

$$(\mathbf{x}^p + \theta\Delta^c\mathbf{x})\,(\mathbf{s}^p + \theta\Delta^c\mathbf{s}) \geq \bar{\mu}_c(\theta)\beta^2\mathbf{e} \geq \mu_c(\theta)\beta^2\mathbf{e}. \tag{4.38}$$

From the requirements $(\mathbf{x}^c(\theta), \mathbf{s}^c(\theta)) \in \mathcal{F}^+$ follows that $\theta_F = \min\{\theta_x, \theta_s\}$, where

$$\theta_x = \min\left\{-\frac{x_i^p}{\Delta^c x_i} : \Delta^c x_i < 0\right\} \quad \text{and} \quad \theta_s = \min\left\{-\frac{s_i^p}{\Delta^c s_i} : \Delta^c s_i < 0\right\}.$$

Thus, $\theta_F > 0$ is the largest step that ensures the feasibility of the new corrector solution. Similarly to the computation of the predictor steplength, now in the corrector step, from the inequality (4.38) further restrictions on the corrector steplength follow. Any θ satisfying (4.38) gives a lower bound on θ_F. After elementary computations the constraints (4.38) can be presented in the following form:

$$\bar{a}_i\,\theta^2 + \bar{b}_i\,\theta + \bar{c}_i \geq 0, \qquad \text{for all } \; i, \tag{4.39}$$

where $\bar{a}_i = \bar{v}_i - \beta^2 \frac{\mathbf{e}^T\bar{\mathbf{v}}}{n}$, $\bar{b}_i = 2\left(\sqrt{\bar{u}_i} - \bar{u}_i\right)$ and $\bar{c}_i = \bar{u}_i - \beta^2$. Let us compute the discriminant as $\Delta_i = \bar{b}_i^2 - 4\,\bar{a}_i\,\bar{c}_i$ of the quadratic equation.

In the proof of Theorem 5.1 it will be shown that the inequality (4.35) has solution, thus θ_c satisfying (4.29) exists. Hence the situation $\Delta_i < 0$ and $\bar{a}_i < 0$, $i = 1, \ldots, n$ cannot occur.

When $\Delta_i \geq 0$ and $\bar{a}_i \neq 0$, the smallest and the largest root of the quadratic equation will be denoted as

$$\theta_i^- = \frac{-\bar{b}_i - \operatorname{sgn}(\bar{a}_i)\sqrt{\Delta_i}}{2\,\bar{a}_i}, \qquad \text{and} \qquad \theta_i^+ = \frac{-\bar{b}_i + \operatorname{sgn}(\bar{a}_i)\sqrt{\Delta_i}}{2\,\bar{a}_i}.$$

Now, we are ready to solve inequalities (4.39) in terms of θ. For each $1, \ldots, n$, the solution set is denoted by T_i and it is given below

$$\mathcal{T}_i = \begin{cases} (-\infty, \infty), & \text{if } \Delta_i < 0,\ \bar{a}_i > 0 \\ (-\infty, \theta_i^-] \cup [\theta_i^+, \infty), & \text{if } \Delta_i \geq 0,\ \bar{a}_i > 0 \\ [\theta_i^-, \theta_i^+], & \text{if } \Delta_i \geq 0,\ \bar{a}_i < 0 \\ \left(-\infty, -\frac{\bar{c}_i}{\bar{b}_i}\right], & \text{if } \bar{a}_i = 0,\ \bar{b}_i < 0 \\ \left[-\frac{\bar{c}_i}{\bar{b}_i}, \infty\right), & \text{if } \bar{a}_i = 0,\ \bar{b}_i > 0 \\ (-\infty, \infty), & \text{if } \bar{a}_i = 0,\ \bar{b}_i = 0. \end{cases}$$

It is worth mentioning that the following equality holds

$$-\frac{\bar{c}_i}{\bar{b}_i} = \frac{\beta^2 - \bar{u}_i}{2\left(\sqrt{\bar{u}_i} - \bar{u}_i\right)}.$$

For all $\theta \in \mathcal{T} = \cap_{i=1}^n \mathcal{T}_i \cap \mathbb{R}^n_\oplus$, the inequality given in (4.35) is satisfied, unless $\mathcal{T} = \emptyset$. The set $\mathcal{T}$ contains the feasible solutions of the optimization problem given in (4.29). Thus, it remains to show that $\mathcal{T} \neq \emptyset$, which will be shown in Theorem 5.1.

After the corrector step we get the following:

$$(\mathbf{x}^c, \mathbf{s}^c) = (\mathbf{x}^c(\theta_c), \mathbf{s}^c(\theta_c)) \in \mathcal{N}_{\infty,\varphi}^{-}(1-\beta) = \mathcal{D}_\varphi(\beta), \tag{4.40}$$

where $\mathbf{x}^c(\theta_c) = \mathbf{x}^p + \theta_c \Delta^c \mathbf{x}$ and $\mathbf{s}^c(\theta_c) = \mathbf{s}^p + \theta_c \Delta^c \mathbf{s}$.

Since $(\mathbf{x}^c, \mathbf{s}^c) \in \mathcal{N}_{\infty,\varphi}^{-}(1-\beta) = \mathcal{D}_\varphi(\beta)$, we can set $(\mathbf{x}, \mathbf{s}) := (\mathbf{x}^c, \mathbf{s}^c)$ and start another predictor-corrector iteration.

Instead of computing the exact value of θ_c, we choose a $\bar{\theta} \in \mathcal{T}$ such that $\mu_c(\theta_c) \leq \mu_c(\bar{\theta})$. Theorem 5.1 will show that it is enough to work with $\bar{\theta}$ to ensure a large enough decrease in the duality gap. From the practical point of view this shows that neither the optimization problem given in (4.22), nor the problem given in (4.29) for the predictor and corrector steps, respectively, need to be exactly solved in order to ensure polynomial complexity of the algorithm. On the other hand, if we solve the mentioned optimization problems, then the steplengths will not depend on κ, thus it will only be used in the analysis of the algorithms.

In the following section we analyse Algorithm 1.

5. Analysis of the algorithm

We have already discussed the feasibility of the predictor and corrector step. Furthermore, we obtained a lower bound on the predictor steplength. Now we are ready to state and prove the size of decrease in the central path parameter at each iteration.

Theorem 5.1. Let $n \geq 2$ and $\beta \in (0,1)$. Then, the PC IPA given in Algorithm 1 using the function $\varphi(t) = \sqrt{t}$ in the AET technique is well defined and

$$\mu_{k+1} \leq \left(1 - \frac{(1-\beta)\beta}{20((1+4\kappa)n+2)}\right)\mu_k, \quad k = 0, 1 \ldots$$

Proof. The k^{th} iteration of Algorithm 1 starts with $(\mathbf{x}^k, \mathbf{s}^k) := (\mathbf{x}, \mathbf{s}) \in \mathcal{D}_\varphi(\beta)$ and $\mu_k := \mu(\mathbf{x}, \mathbf{s})$. Solving the predictor Newton-system (4.4) and computing the predictor steplength θ_p, we derive from Lemma 4.3 that $(\mathbf{x}^p, \mathbf{s}^p) \in \mathcal{D}_\varphi((1-\gamma)\,\beta)$. Furthermore, in Lemma 4.5 we obtained a lower bound on θ_p. Hence, we have

$$\sqrt{\mathbf{x}^p(\theta)\mathbf{s}^p(\theta)} \geq (1-\gamma)\beta\sqrt{\mu_p(\theta)} > (1-\gamma)\beta\sqrt{\mu_p(\theta_p)} \geq 0,$$

where $\theta_p > \hat{\theta}$. From (4.16) it follows that $\frac{\mathbf{e}^T\mathbf{v}}{n} \leq 1$ which is equivalent to $(\Delta^p\mathbf{x})^T \Delta^p \mathbf{s} \leq \mu(\mathbf{x}, \mathbf{s})n$. Now, using (4.8) the following inequality holds for $\theta_p > \hat{\theta}$:

$$\mu_p = \mu(\theta_p) < \mu(\hat{\theta}) \leq \left((1 - 2\,\hat{\theta}) + \hat{\theta}^2\right)\mu(\mathbf{x}, \mathbf{s}) = (1 - (2-\hat{\theta})\,\hat{\theta})\,\mu(\mathbf{x}, \mathbf{s}). \tag{5.1}$$

Assuming that $n \geq 2$, $\kappa > 0$ and using (4.24), we obtain

$$2-\hat{\theta}=2-\frac{\beta\sqrt{1-\beta}}{5((1+4\kappa)n+2)}\geq 2-\frac{\beta\sqrt{1-\beta}}{20}\geq 2-\frac{1}{40}=\frac{79}{40},$$

hence, we have

$$\mu_p \leq \left(1-\frac{79\beta\sqrt{1-\beta}}{200((1+4\kappa)n+2)}\right)\mu(\mathbf{x},\mathbf{s}). \tag{5.2}$$

Now, we are analysing a corrector step. In this step $(\mathbf{x}^p,\mathbf{s}^p)\in\mathcal{N}^-_{\infty,\varphi}(1-\beta+\beta\gamma)=\mathcal{D}_\varphi((1-\gamma)\,\beta)$, so

$$\sqrt{\bar{\mathbf{u}}}=\sqrt{\frac{\mathbf{x}^p\mathbf{s}^p}{\mu_p}}\geq(1-\gamma)\,\beta\,\mathbf{e} \qquad\iff\qquad \mathbf{e}^T\sqrt{\bar{\mathbf{u}}}\geq(1-\gamma)\,\beta\,n.$$

Using Lemma 4.1 with $\mathbf{a}=2(\sqrt{\mu_p\mathbf{x}^p\mathbf{s}^p}-\mathbf{x}^p\mathbf{s}^p)$ we get the following inequality

$$\frac{1}{1+4\,\kappa}\,\|\Delta^c\mathbf{x}\Delta^c\mathbf{s}\|_\infty\leq\sum_{i\in\mathcal{I}_+}\Delta^c x_i\,\Delta^c s_i\leq\mu_p\,\left\|\mathbf{e}-\sqrt{\bar{\mathbf{u}}}\right\|_2^2. \tag{5.3}$$

Next, we have

$$\left\|\mathbf{e}-\sqrt{\bar{\mathbf{u}}}\right\|_2^2=(\mathbf{e}-\sqrt{\bar{\mathbf{u}}})^T(\mathbf{e}-\sqrt{\bar{\mathbf{u}}})=(n-2\,\mathbf{e}^T\sqrt{\bar{\mathbf{u}}}+\mathbf{e}^T\bar{\mathbf{u}})\leq 2\,(1-(1-\gamma)\,\beta)\,n:=\xi\,n,$$

where we denote

$$\xi=2\,(1-(1-\gamma)\,\beta). \tag{5.4}$$

The following two upper bounds are derived from (5.3) and (5.4)

$$\|\Delta^c\mathbf{x}\Delta^c\mathbf{s}\|_\infty\leq(1+4\,\kappa)\,\xi\,n\,\mu_p,\qquad \sum_{i\in\mathcal{I}_+}\Delta^c x_i\,\Delta^c s_i\leq\xi\,n\,\mu_p, \tag{5.5}$$

implying $\|\bar{\mathbf{v}}\|_\infty\leq(1+4\,\kappa)\,\xi\,n$. Using (4.32) we obtain

$$\begin{aligned}\frac{\mathbf{x}^c(\theta)\mathbf{s}^c(\theta)}{\mu_p}&=(1-2\,\theta)\,\bar{\mathbf{u}}+2\,\theta\,\sqrt{\bar{\mathbf{u}}}+\theta^2\,\bar{\mathbf{v}}\\ &\geq\left((1-2\theta)\,((1-\gamma)\beta)^2+2\,\theta\,(1-\gamma)\,\beta-\theta^2\,(1+4\,\kappa)\,\xi\,n\right)\,\mathbf{e}\\ &=\left(((1-\gamma)\beta)^2+\theta\,(1-\gamma)\,\beta\,\xi-\theta^2\,(1+4\,\kappa)\,\xi\,n\right)\,\mathbf{e}.\end{aligned} \tag{5.6}$$

Furthermore, from (4.33) and (5.5) we have

$$\mu_c(\theta)\leq\left(1+\theta^2\,\frac{\mathbf{e}^T\bar{\mathbf{v}}}{n}\right)\,\mu_p\leq\left(1+\theta^2\,\xi\right)\,\mu_p. \tag{5.7}$$

Using (5.6) and (5.7) we get

$$\frac{\mathbf{x}^c(\theta)\mathbf{s}^c(\theta) - \beta^2\,\mu_c(\theta)\,\mathbf{e}}{\mu_p} \geq g(\theta)\,\mathbf{e}, \tag{5.8}$$

where

$$g(\theta) := -\gamma\,\beta^2\,(2-\gamma) + \theta\,\xi\,((1-\gamma)\,\beta) - \xi\,\left(\beta^2 + (1+4\,\kappa)\,n\right)\,\theta^2. \tag{5.9}$$

Then, the quadratic function $g(\theta)$ can be written in the form $g(\theta) = a_g\,\theta^2 + b_g\,\theta + c_g$, where

$$a_g := -\xi\,\left(\beta^2 + (1+4\,\kappa)\,n\right) < 0, \qquad b_g := \xi\,\left((1-\gamma)\,\beta\right) > 0,$$

and

$$c_g := -\gamma\,\beta^2\,(2-\gamma) < 0.$$

Using the definition of γ given in (4.9) we get the following value for ξ

$$\xi = 2\,\frac{(1-\beta)\,(\beta + 5((1+4\kappa)n+1))}{5((1+4\kappa)n+1)}.$$

Using the definition of γ and ξ, the parameters a_g, b_g and c_g are computed:

$$c_g = -\frac{\beta^2\,(1-\beta)\,(10\,((1+4\,\kappa)\,n+1) + \beta - 1)}{25((1+4\,\kappa)\,n+1)^2},$$

$$b_g = \frac{2\,\beta\,(1-\beta)\,(\beta + 5((1+4\,\kappa)\,n+1))\,(\beta - 1 + 5((1+4\,\kappa)\,n+1))}{25((1+4\,\kappa)\,n+1)^2}$$

and

$$a_g = -\frac{2\,(1-\beta)\,(\beta + 5((1+4\,\kappa)\,n+1))\,(\beta^2 + (1+4\,\kappa)\,n)}{5((1+4\,\kappa)\,n+1)}.$$

We need to find the values of parameters β, γ for which $g(\theta) \geq 0$ is satisfied. To simplify the search for such pairs we fix

$$\bar{\theta} := \frac{\beta}{2\,((1+4\,\kappa)\,n+1)} \tag{5.10}$$

value and compute the corresponding parameters β. Clearly, $g(\bar{\theta}) \geq 0$ is equivalent to

$$g(\bar{\theta}) := \frac{\beta^2\,(1-\beta)}{((1+4\,\kappa)\,n+1)^3}\,\bar{g}(\bar{\theta}) \geq 0$$

for some $\beta \in (0,1)$. We have

$$\begin{aligned}\bar{g}(\bar{\theta}) = &-\frac{1}{10}\left(\beta + 5((1+4\,\kappa)\,n+1)\right)\left(\beta^2 + (1+4\,\kappa)\,n\right)\\ &+\frac{1}{25}(\beta + 5((1+4\,\kappa)\,n+1))\left(\beta - 1 + 5((1+4\,\kappa)\,n+1)\right)\\ &-\frac{1}{25}\left(\beta - 1 + 10\left((1+4\,\kappa)\,n+1\right)\right)\left((1+4\,\kappa)\,n+1\right).\end{aligned}$$

After some computations we get

$$\begin{aligned}f(\beta) := \bar{g}(\bar{\theta}) = &\left(-\frac{\beta^3}{10} - \frac{23\beta^2}{50} + \frac{8\beta}{25} + \frac{11}{25}\right)\\ &+\left(-\frac{\beta^2}{2} + \frac{13\beta}{50} + \frac{27}{50}\right)(1+4\,\kappa)\,n + \frac{1}{10}\,(1+4\,\kappa)^2\,n^2.\end{aligned}$$

A single variable function f has a domain $(0,1)$. It would be beneficial to find all $\beta \in (0,1)$ that satisfy $f(\beta) \geq 0$. A sufficient condition for $f(\beta) \geq 0$ leads to solving the following system of nonlinear inequalities

$$\begin{aligned}-\frac{\beta^3}{10} - \frac{23\beta^2}{50} + \frac{8\beta}{25} + \frac{11}{25} &\geq 0,\\ -\frac{\beta^2}{2} + \frac{13\beta}{50} + \frac{27}{50} &\geq 0.\end{aligned}$$

Straightforward computations show that both inequalities hold for $\beta \in (0,1)$. Hence, for $\beta \in (0,1)$ the inequality $f(\beta) \geq 0$ holds. Namely, $\bar{\theta} \in \mathcal{T}$, thus $\mathcal{T} \neq \emptyset$ completing the analysis of the corrector step, which is described in Subsection 4.2.

The last step of the proof is derived as follows. Assume $\bar{\theta}$ is given, $\beta \in (0,1)$ and $n \geq 2$. Furthermore, at the beginning of the iteration we have $(\mathbf{x}^k, \mathbf{s}^k) \in \mathcal{D}_\varphi(\beta)$ with μ_k, and we compute the predictor solution $(\mathbf{x}^p, \mathbf{s}^p)$ with μ_p.

From (5.7) and assuming $n \geq 2$ we have

$$\begin{aligned}\mu_c &= \mu_c(\theta_c) \leq \mu_c(\bar{\theta})\\ &= \mu_c\left(\frac{\beta}{2((1+4\kappa)n+1)}\right) \leq \left(1 + \frac{\beta^2(1-\beta)(5((1+4\kappa)n+1)+\beta)}{10((1+4\kappa)n+1)^3}\right)\mu_p\end{aligned}$$

Since $\frac{5((1+4\kappa)n+1)+\beta}{10((1+4\kappa)n+1)} = \frac{1}{2} + \frac{\beta}{10((1+4\kappa)n+1)} \leq \frac{2}{3}$ we have

$$\mu_c \leq \left(1 + \frac{2\beta^2(1-\beta)}{3((1+4\kappa)n+1)^2}\right)\mu_p < \left(1 + \frac{2\beta(1-\beta)}{3((1+4\kappa)n+1)^2}\right)\mu_p. \tag{5.11}$$

Using (5.2) and (5.11) we obtain

$$\begin{aligned}\mu_c &\le \left(1-\frac{79\beta\sqrt{1-\beta}}{200((1+4\kappa)n+2)}\right)\left(1+\frac{2\beta(1-\beta)}{3((1+4\kappa)n+1)^2}\right)\mu\\ &\le \left(1-\frac{79\beta(1-\beta)}{200((1+4\kappa)n+2)}\right)\left(1+\frac{2\beta(1-\beta)}{3(1+4\kappa)n((1+4\kappa)n+2)}\right)\mu\\ &\le \left(1-\frac{79\beta(1-\beta)}{200((1+4\kappa)n+2)}+\frac{2\beta(1-\beta)}{3(1+4\kappa)n((1+4\kappa)n+2)}\right)\mu\\ &\le \left(1-\left(\frac{79}{200}-\frac{2}{3(1+4\kappa)n}\right)\frac{\beta(1-\beta)}{((1+4\kappa)n+2)}\right)\mu\\ &\le \left(1-\frac{(1-\beta)\beta}{20((1+4\kappa)n+2)}\right)\mu, \qquad (5.12)\end{aligned}$$

where the last inequality follows from the fact that $\frac{79}{200}-\frac{2}{3(1+4\kappa)n}\ge\frac{37}{600}>\frac{1}{20}$, where $n\ge 2$.

Furthermore, we denote by $\mu=\mu(\mathbf{x}^k,\mathbf{s}^k)=\mu_k$ and $\mu_{k+1}=\mu_c$. Hence, we obtained the desired result. □

The following corollary is a consequence of Theorem 5.1.

Corollary 5.2. Let $n\ge 2$ and $\beta\in(0,1)$. Then, Algorithm 1 produces a point $(\mathbf{x}^k,\mathbf{s}^k)\in\mathcal{N}^-_{\infty,\varphi}(1-\beta)$ with $\mathbf{x}^k\mathbf{s}^k\le\epsilon$ in at most $\mathcal{O}\left((1+\kappa)\,n\,\log\left(\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\epsilon}\right)\right)$ iterations.

It should be mentioned that Algorithm 1 depends on a given parameter $\kappa\ge\hat{\kappa}(M)$ because of the parameter γ given in (4.9). It may be difficult and expensive to find on upper bound for the handicap $\hat{\kappa}(M)$ in case of many applications, see [13,59,60]. That is why in the following section we present another variant of the PC IPA.

6. Extension of predictor-corrector interior-point algorithm for unknown handicap

We propose a new version of the PC IPA presented in Algorithm 1. If the algorithm fails to produce a point in $\mathcal{N}^-_{\infty,\varphi}(1-\beta)=\mathcal{D}_\varphi(\beta)$ with $\varphi(t)=\sqrt{t}$, then the current value of κ may be too small. Hence, we double the value of κ and restart Algorithm 1 from the last point produced in $\mathcal{D}_\varphi(\beta)$. In this way, we have to double the value of κ at most $\lceil\log_2\hat{\kappa}(M)\rceil$ times. This new version of the algorithm is presented in Algorithm 2.

Using Theorem 3.9 in [39], Theorem 5.1, Corollary 5.2 we obtain the following.

Theorem 6.1. Algorithm 2 produces a point $(\mathbf{x}^k,\mathbf{s}^k)\in\mathcal{N}^-_{\infty,\varphi}(1-\beta)$ with $\mathbf{x}^k\mathbf{s}^k\le\epsilon$ in at most $\mathcal{O}\left((1+\hat{\kappa}(M))\,n\,\log\left(\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\epsilon}\right)\right)$ iterations.

Algorithm 2: Predictor-corrector interior-point algorithm not depending on κ.

Input:
$(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta)$, $\beta \in (0,1)$;
Set $\kappa = 1$
Let $\mu_0 = \mu(\mathbf{x}^0, \mathbf{s}^0)$ and $k = 0$
$\varepsilon > 0$ precision value.
Output: $(\mathbf{x}^k, \mathbf{s}^k) : \mathbf{x}^{kT}\mathbf{s}^k \leq \varepsilon$
begin
- **while** $n\mu \geq \varepsilon$ **do**
  - (Predictor step);
  - $\mathbf{x} := \mathbf{x}^k$, $\mathbf{s} := \mathbf{s}^k$;
  - Step 1. Calculate affine direction from (4.4);
  - Step 2. Calculate the predictor steplength using (4.20);
  - Step 3. Calculate $(\mathbf{x}^p, \mathbf{s}^p)$;
  - **if** $\mu(\mathbf{x}^p, \mathbf{s}^p) = 0$ **then**
    - STOP; Optimal solution found;
  - **else**
    - **if** $(\mathbf{x}^p, \mathbf{s}^p) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta)$ **then**
      - $(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}) = (\mathbf{x}^p, \mathbf{s}^p)$, $\mu^{k+1} = \mu(\mathbf{x}^p, \mathbf{s}^p)$, $k = k+1$, RETURN;
    - **else**
      - (Corrector step);
      - Step 4. Calculate centering direction from (4.27);
      - Step 5. Calculate centering steplength using (4.29);
      - Step 6. Calculate $(\mathbf{x}^c, \mathbf{s}^c)$;
      - **if** $(\mathbf{x}^c, \mathbf{s}^c) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta)$ **then**
        - $(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}) = (\mathbf{x}^c, \mathbf{s}^c)$, $\mu^{k+1} = \mu(\mathbf{x}^c, \mathbf{s}^c)$, $k = k+1$, RETURN;
      - **else**
        - $\kappa = 2\kappa$; $(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}) = (\mathbf{x}^k, \mathbf{s}^k)$, $\mu^{k+1} = \mu(\mathbf{x}^k, \mathbf{s}^k)$, $k = k+1$, RETURN;
      - **end**
    - **end**
  - **end**
- **end**

end

Proof. Consider $\bar{\kappa}$ as the largest value of κ used in Algorithm 2. Then, we have $\bar{\kappa} < 2\hat{\kappa}(M)$. Now we consider that at iteration k of Algorithm 2 we have $\kappa < \hat{\kappa}(M)$. If $(\mathbf{x}^c, \mathbf{s}^c) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta)$, then $(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}) = (\mathbf{x}^c, \mathbf{s}^c)$. Using that Lemmas 4.1 and 4.2 hold for $\kappa = \hat{\kappa}(M)$ and the bound on the predictor step size depends on γ which is decreasing in κ, we obtain that

$$\mu_{k+1} \leq \left(1 - \frac{(1-\beta)\beta}{20((1+4\hat{\kappa}(M))n+2)}\right)\mu_k \leq \left(1 - \frac{(1-\beta)\beta}{20((1+8\hat{\kappa}(M))n+2)}\right)\mu_k.$$

Furthermore, if $\kappa \geq \hat{\kappa}(M)$, then the corrector step is never rejected. Hence, using Theorem 5.1 and the fact that $\kappa \leq \bar{\kappa} < 2\hat{\kappa}(M)$, we obtain

$$\mu_{k+1} \leq \left(1 - \frac{(1-\beta)\beta}{20((1+4\kappa)n+2)}\right)\mu_k \leq \left(1 - \frac{(1-\beta)\beta}{20((1+8\hat{\kappa}(M))n+2)}\right)\mu_k.$$

Since there can be at most $\log_2(\bar{\kappa})$ rejections we obtain the final result. $\square$

7. Numerical results

We implemented a variant of the proposed PC IPA in the C++ programming language. The computations were performed on a desktop computer with Intel quad-core 2.6 GHz processor and 16 GB RAM. Due to the fact that in many cases we do not have information about the κ, we used Algorithm 2 in our implementation. We set the value of $\beta = 0.1$ and $\epsilon = 10^{-5}$.

It is important to mention that several implementations related to IPAs from the literature differ from the theoretical versions of the proposed IPAs. The implemented version of our PC IPA is very close to the theoretical version of the PC IPA. In the predictor step we used the same steplength as in (4.20) and in the corrector step we define the corrector steplength by giving an approximate solution of the optimization problem (4.29).

Moreover, it should be mentioned that most of the numerical results related to $P_*(\kappa)$-LCPs are obtained for problems where the value of κ is zero, that lead to LO problems. Gurtuna et al. [37] and Asadi et al. [61] provided numerical results related to $P_*(\kappa)$-LCPs having positive handicap, by considering 2×2 or 3×3 matrices. They also analysed block diagonal matrices formed by the aformentioned ones. Darvay et al. [27] presented numerical results where they solved $P_*(\kappa)$-problems with matrices having positive κ parameters generated by Illés and Morapitiye [62].

However, in this paper we considered the special matrix proposed by Csizmadia:

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix}. \tag{7.1}$$

E.-Nagy proved that $\hat{\kappa}(M) = 2^{2n-8} - 0.25$, see [60].

We generated the test problems in the following way: $\mathbf{q} := -M\mathbf{e} + \mathbf{e}$. We considered $\mathbf{x}^0 = \mathbf{e}$ and $\mathbf{s}^0 = \mathbf{e}$ as starting points for our PC IPA.

In our computational study we compared our algorithm to the one of Potra and Liu [39] working in $\mathcal{D}_\varphi(0.1)$ with $\varphi(t) = t$. The obtained results are summarized in Table 1.

In spite of the fact that the analysis of the algorithm refers to the case when $\varphi(t) = \sqrt{t}$, we also tested the PC IPA using the generalized wide neighbourhood $\mathcal{D}_\varphi(\beta)$ with $\varphi(t) = t - \sqrt{t}$. We also compared this version of the algorithm to the short-step PC IPA presented in [27] which works in different type of neighbourhood and uses $\varphi(t) = t - \sqrt{t}$ in the AET technique. The results are given in Table 2. As it was expected, the PC IPAs working in wide neighbourhood gave better results than the short-step PC IPA proposed in [27].

It seems that the practical iteration complexity is significantly better than the theoretical worst case guarantee for the special class of LCPs with the matrix M introduced

Table 1
Numerical results using Algorithm 2 in case of $\mathcal{D}_\varphi(0.1)$ with matrix given in (7.1).

n	$\varphi(t) = t$	$\varphi(t) = \sqrt{t}$
	Nr. of Iter.	Nr. of Iter.
10	8	7
20	10	9
50	16	15
100	25	24
200	47	43
300	66	63
400	87	82

Table 2
Numerical results using Algorithm 2 in case of $\mathcal{D}_\varphi(\beta)$ and the PC IPA from [27] with matrix given in (7.1).

n	PC IPA using $\mathcal{D}_\varphi(\beta)$ with $\varphi(t) = t - \sqrt{t}$	PC IPA from [27] using $\varphi(t) = t - \sqrt{t}$
	Nr. of Iter.	Nr. of Iter.
10	21	53
20	20	91
100	40	97
200	61	112

by Zs. Csizmadia. Seemingly, in practice, the iteration number does not have strong relation with the value of κ. This fact needs further analysis.

The obtained numerical results show that these algorithms have better iterations number than it is predicted by their complexity results. Hence, it would be worth trying to prove better theoretical complexity results in case of LPCs with these special matrices. In the following section some concluding remarks are presented.

8. Conclusions and further research

In this paper we proposed a new PC IPA for solving $P_*(\kappa)$-LCPs. The proposed IPA uses new search direction and works in the generalized wide neighbourhood $\mathcal{D}_\varphi(\beta)$ with $\varphi(t) = \sqrt{t}$. However, the PC IPA depends on the parameter κ, which could usually be computed by an algorithm that has no polynomial complexity, see [59]. Therefore, we proposed another variant of the PC IPA, where we initially set $\kappa = 1$ and used Algorithm 1 for this value of κ. If at a certain iteration the algorithm fails to produce a point in $\mathcal{D}_\varphi(\beta)$, we double the value of κ and restart Algorithm 1 from the last point produced in $\mathcal{D}_\varphi(\beta)$. Following the results of Potra and Liu [39], our algorithm keeps the property that the predictor and corrector steplengths can be computed as a solution of some optimization problems. In this computation of the steplengths we do not need the apriori knowledge of the handicap of the problem's matrix. For simplification purposes, during the complexity analysis of the algorithm we used steplengths depending on κ satisfying feasibility property of the above mentioned optimization problems. We proved that the PC IPA has $\mathcal{O}\left((1+\kappa)n \log\left(\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\epsilon}\right)\right)$ iteration complexity. Because of the used search direction the complexity analysis of the PC IPA became more complicated compared to the analysis given in [39]. To overcome this, we restructurated the proofs in order to be easily followable. We also provided numerical results where we compared our PC IPA to other ones that use different search directions or neighbourhoods. We tested the PC IPA working in the generalized wide neighbourhood $\mathcal{D}_\varphi(\beta)$ with $\varphi(t) = t - \sqrt{t}$. Lemma 3.1 shows that the methods used in the complexity analysis presented in this paper with $\varphi(t) = \sqrt{t}$ might not work in this form in case of $\varphi(t) = t - \sqrt{t}$. However, the obtained numerical results for this variant of PC IPA show that it would be worth analysing the theoretical approach for this type of algorithm, as well.

As further research plans, it would be important to understand for which θ, β, γ values we can obtain polynomial complexity. Usually in case of IPAs the complexity analysis of the algorithms is proven only for a given value of the parameters, although we know that there exists a whole set of parameters for which the algorithm is well defined and usually a smaller one for which the complexity analysis works as well. The determination of the whole set is not trivial, but the choice of a subset could be very interesting, see [46,47,63]. Furthermore, it would be interesing to extend the algorithm in a similar way that Illés et al. did in [64]. For this reason, it would be good to collect as many general LCP test problems as possible in order to make the algorithms developed for general LCPs testable in practice, too.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The authors would like to thank the anonymous referees for the valuable suggestions that helped the improvement of the paper. Petra Renáta Rigó is thankful for the support provided by the Ministry of Culture and Innovation of Hungary from the National Research, Development and Innovation Fund, financed under the PD_22 funding scheme, project no. 142154. The research of P.R. Rigó has been supported by the ÚNKP-22-4 New National Excellence Program of the Ministry for Culture and Innovation from the source of the National Research, Development and Innovation Fund. The authors would like to thank Zsolt Darvay for his useful comments and suggestions on earlier version of the paper.

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