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Sensitivity of fair prices in assignment markets☆

Tamás Solymosi

Department of Operations Research and Actuarial Sciences, Corvinus Center for Operations Research, Corvinus University of Budapest, H-1093 Budapest, Fővám tér 8, Hungary

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ABSTRACT

It is well known that in assignment markets competitive prices always exist, but no price mechanism is strategy-proof for all agents. We investigate the extent a single agent can influence three special competitive price vectors by misreporting his/her reservation values. We provide an exact formula how the minimum, the maximum, and the fair competitive price vectors change, and show that at the fair prices no agent can gain more than half of the deviation from the true values. We also derive the analogous results for the corresponding core payoffs of the associated assignment game via graph-theoretic characterizations of the two side-optimal core payoffs.

1. Introduction

We consider the two-sided matching markets introduced by Koopmans and Beckmann (1957) with two types of agents and two types of goods: one is indivisible (e.g. houses, locations, positions), the other is perfectly divisible (e.g. money). In these markets each agent is either a seller or a buyer; each seller owns one unit of the indivisible good (e.g. has one house to sell) and no buyer demands more than one unit of the indivisible good (e.g. wants to buy at most one house). Due to these unit supplies and unit demands, the indivisible goods are exchanged in exclusive seller-buyer partnerships. The perfectly divisible good, however, can be freely reallocated among the agents, so "side-payments" to third parties are allowed. Finally, individual utilities are assumed to be transferable, one unit of the divisible good is valued the same by all agents, so comparison and aggregation of individual utilities are meaningful.

Koopmans and Beckmann (1957) formulated the optimization problem, called linear assignment problem, to find the collectively most efficient allocation of the indivisible goods, and investigated the dual problem of finding prices which provide suitable incentives for the autonomous profit-maximizing agents in a market mechanism to arrive at the collectively optimal market outcome. They proved the existence of such competitive equilibrium prices.

Shapley and Shubik (1972), extending earlier work by Shapley (1955), investigated assignment markets using cooperative game theory. They proved that the core of the assignment game associated with any assignment market is not empty, moreover, it coincides with the set of competitive equilibrium payoffs vectors. Shapley and Shubik (1972) also investigated the structure of the core and showed that it has a lattice structure and two special extreme elements: one is simultaneously the best for all buyers (and the worst for all sellers), the other is simultaneously the best for all sellers (and the worst for all buyers). The average of these two side-optimal core payoffs (corresponding to the average of the minimum and the maximum competitive prices) was recommended by Thompson (1981) as the "fair" payoffs favoring neither side. It was shown to coincide with the tau-value, a point-valued solution to cooperative games introduced by Tijs (1981), of the assignment game by Núñez and Rafels (2002). They also showed that the tau-value in assignment games is pairwise monotonic, meaning that if the pairwise profit of a seller-buyer pair is increased, but all other pairwise profits remain unchanged, the payoff of neither this seller, nor this buyer decreases at the tau-value payoff. This assignment game specific monotonicity result was the motivation for this study.

Our main question is what happens to the fair (tau) payoffs if *all* pairwise profits of an agent change due to some unilateral change in that agent's market valuations, but all other pairwise profits remain unchanged. Exactly this happens if we consider a market situation where the agents have reservation values for the possible partnerships, and one agent changes his/her reservation value for *all* of his/her possible partnerships, but all other agents keep their original valuations. We focus on the uniform case when the change is the same for *all* possible partnerships of the particular agent. For example, in a housing market a seller "typically" sets a single reservation price for his house irrespective of who the buyer would be. On the other hand, the

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E-mail address: tamas.solymosi@uni-corvinus.hu.

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graphical characterization of the side-optimal core payoffs we use to prove our main results can also be used to investigate the effect of non-uniform changes in the reservation values of the agent, like in the properties object-valuation antimonotonicity and buyer-valuation monotonicity discussed by van den Brink et al. (2021).

Our main question and results have relevance also to the manipulability of the market mechanisms which yield competitive equilibrium prices with respect to the reported valuations by the agents. We only highlight some fundamental results on the incentives of the agents in such competitive assignment market mechanisms. Roth and Sotomayor (1990) (Theorem 7.3) showed that no stable matching mechanism exists for which stating the true reservation value is a dominant strategy for every agent. This impossibility result basically rests on the opposite interest of sellers versus buyers over the set of stable payoffs which is typically not a singleton. Pérez-Castrillo and Sotomayor (2013, 2017) strengthened and extended this insight and proved that if the assignment market (defined by the true valuations of the agents) has more than one vector of competitive prices and a competitive equilibrium allocation rule does not yield a buyer-optimal (respectively, seller-optimal) competitive equilibrium for this market, then any buyer (respectively, seller) who is not receiving his (her) optimal competitive equilibrium payoff for this market can profitably misrepresent his (her) valuations, assuming the others tell the truth.

In contrast to this general manipulability theorem, Demange (1982) and Leonard (1983) proved that if the competitive matching mechanism selects the minimum competitive prices then truthful reporting is a dominant strategy for every buyer. This partially positive result makes it possible to design a multi-object sealed-bid second price auction in which no buyer has an incentive to misreport his reservation values. Finally, we mention that Demange et al. (1986) constructively proved that the minimum competitive prices can be achieved by ascending multi-object auctions. Roth and Sotomayor (1990) (Chapters 7 and 8) and the survey by Núñez and Rafels (2015) provide further discussions on the mechanism design aspects of assignment markets.

The aforementioned results tell us that only mechanisms yielding the side-optimal competitive equilibrium prices can be partially strategy-proof, any other mechanism which yields competitive equilibrium prices are manipulable. We investigate the (at least to best of our knowledge) unexplored question of *how much* an agent can gain by not telling the truth in a competitive equilibrium mechanism. We focus on the sensitivity of the minimum, the maximum, and the "fair" competitive prices with respect to unilateral changes in the reservation values of a single agent.

We prove for all three aforementioned competitive price rules that the payoff of an agent cannot decrease if he/she changes his/her reservation value in the "natural way" (i.e. sellers increase, buyers decrease), but all other reservation values are kept unchanged. In fact, the same holds for all agents on the side of the misreporting agent. More importantly, we provide an *exact formula* how the price concerning the misreporting agent changes and establish *sharp upper bounds* for the extent the payoff of any agent of the same type can increase. In particular, we show that if one seller (buyer) reports a higher (uniformly lower) reservation value(s) than the true one(s), but all other agents report honestly to the mechanism, each fair competitive price can only increase (decrease), but by no more than half of the deviation from the true value. Our proofs rely on a graph-theoretic characterization of the side-optimal competitive payoffs, that can also be used for their efficient computation.

Given the coincidence of the set of competitive payoff vectors in the market with the core of the associated assignment game induced by the matrix of pairwise profits (Shapley and Shubik, 1972), our results apply for the two side-optimal core vectors and the tau-value. Since changing the reservation value of a single agent in the market causes uniform change (subject to keeping nonnegativity) in all entries of a row or a column in the pairwise profit matrix, our results provide a remarkably precise sensitivity analysis for both side-optimal core vectors and for the tau-value in the unusual case when not just one, but several model parameters are changed in a uniform way.

The rest of the paper is organized as follows. In the next section we formally introduce assignment markets and games, and review the early fundamental results on the set of competitive equilibrium and core payoffs. In Section 3, we use a 2-seller, 2-buyer assignment market to preview our results which are formally proved in two subsequent sections. First, in Section 4, we present a graphical characterization of the two side-optimal payoff vectors. Based on that, we establish our results on the sensitivity of the two side-optimal and the fair payoff vectors in Section 5. We conclude in Section 6.

2. Assignment markets and stable outcomes

In an *assignment market* there are two types of agents, a set of indivisible objects and money. It is assumed that agents' utilities are identified with money. We call the two types of agents sellers and buyers, and denote their disjoint sets by I and J, respectively. Each seller owns one indivisible object which are similar in kind but not identical (e.g. houses, cars, jobs). Seller $i \in I$ will not sell his object for less than s_i. Each buyer wants one of these objects, buyer j will not pay for the object owned by seller i more than t_{ij}. An assignment market is given by the two disjoint sets I and J, by the vector $[s_i]_{i\in I}$ containing the reservation values of the sellers, and the matrix $[t_{ij}]_{i\in I, j\in J}$ containing columnwise the valuations of the buyers on the objects.

An *outcome* of the market will specify an allocation of the objects (to buyers, if sold, or to sellers, if unsold) and the monetary transfers among the agents. Because of the indivisibility of the objects and the unit supplies and demands of the agents, the objects can only be traded in exclusive bilateral partnerships. Following Roth and Sotomayor (1990) (chapter 8), we call a binary matrix $X = [x_{ij}]_{i\in I, j\in J}$ an *assignment* of sellers/objects to buyers, if it satisfies

$$\begin{aligned} \sum_{j\in J} x_{ij} &\leq 1 && \text{for all } i \in I \\ \sum_{i\in I} x_{ij} &\leq 1 && \text{for all } j \in J \\ x_{ij} &\in \{0,1\} && \text{for all } i \in I, j \in J. \end{aligned}$$

Clearly, the positive components of an assignment matrix X describe a matching between I and J, and vice versa, any matching between I and J can be described by an assignment matrix X.

A market outcome also specifies the monetary transfers among the agents. If buyer j buys the object from seller i at a price p_i, and if no other monetary transfer are made to or received from third parties, the profit of buyer j equals $t_{ij} - p_i$ and the profit of seller i equals $p_i - s_i$. The potential total profit of the exclusive partnership between i and j is then $a_{ij} = \max\{0, t_{ij} - s_i\}$, independently of the actual price p_i. We call a payoff vector $\big((u_i)_{i\in I}\,;\,(v_j)_{j\in J}\big)$ *feasible for assignment* X, if

$$\sum_{i\in I} u_i + \sum_{j\in J} v_j \leq \sum_{i\in I, j\in J} a_{ij} x_{ij}$$

holds. We emphasize that although the objects are traded in exclusive bilateral partnerships, side-payments to third parties are allowed. By a *feasible outcome* of an assignment market

$\left(I, J, [s_i]_{i\in I}, [t_{ij}]_{i\in I, j\in J}\right)$ we mean a pair $\left(X, (u; v)\right)$ consisting of an assignment X and a feasible payoff vector $(u; v)$ for X.

A feasible outcome $\left(X, (u; v)\right)$ is called *stable*, if its payoff vector is *acceptable* for each agent and for each seller-buyer pair:

- $u_i \geq 0$, $v_j \geq 0$ for all $i \in I, j \in J$,
- $u_i + v_j \geq a_{ij}$ for all $i \in I, j \in J$.

One quickly observes that the individual and pairwise lower bounds on the payoffs needed for stability and the collective upper bound required by feasibility can only be consistent if certain relations hold.

Proposition 1. *An acceptable payoff vector $(u; v)$ is feasible for assignment X if and only if the following* complementarity *conditions are satisfied:*

- $u_i = 0$ *for each unmatched seller i, (i.e. $\sum_{j\in J} x_{ij} < 1$),*
- $v_j = 0$ *for each unmatched buyer j, (i.e. $\sum_{i\in I} x_{ij} < 1$),*
- $u_i + v_j = a_{ij}$ *for each matched pair i, j, (i.e. $x_{ij} = 1 > 0$),*

The proof is straightforward, see e.g. Roth and Sotomayor (1990) (Lemma 8.5). It is also easily seen that the stability of a market outcome can be characterized by the following two optimality conditions.

Proposition 2. *A feasible outcome $\left(X, (u; v)\right)$ is stable, if and only if*

- *X maximizes the total value $\sum_{ij} a_{ij}x_{ij}$ over all assignments, and*
- *$(u; v)$ minimizes the total payoff $\sum_i u_i + \sum_j v_j$ over all acceptable payoffs.*

It follows from Propositions 1 and 2 that the existence of a stable outcome in an assignment market is equivalent to whether both related optimization problems have an optimal solution and, by complementarity, the same optimum value. Since there are finite many assignments, there is always an assignment of maximum value. The question remains whether there exists an acceptable payoff vector such that its total equals that maximum assignment value. As noticed by Biró (2007), this question is affirmatively answered for any assignment market induced by a nonnegative pairwise profit matrix by the duality theorem of Egerváry (1931), stated below in our terminology.

Theorem 1 (*Egerváry, 1931*)**.** *For any nonnegative matrix $A = \left[a_{ij}\right]$,*

$$\max_{X \text{assignment}} \sum_{ij} a_{ij}x_{ij} = \min_{(u;v)\text{acceptable}} \sum_i u_i + \sum_j v_j .$$

Having the existence of stable outcomes guaranteed for any assignment market, from the acceptability and complementarity conditions one could easily obtain all the well-known structural results on the set of stable payoff vectors which we summarize next by reviewing the early fundamental results on this model.

The linear assignment problem of finding a maximum value assignment and the question of how it can be sustained by a price system in a decentralized economic setting was formulated and investigated by Koopmans and Beckmann (1957). They derived the existence of a stable outcome from the Minkowski–Farkas lemma for linear inequalities applied to the equivalent linear programming problem:

$$\begin{array}{rcll} \sum_{ij} a_{ij}x_{ij} & \rightarrow & \max & \\ \sum_{j\in J} x_{ij} & \leq & 1 & \text{for all } i \in I \\ \sum_{i\in I} x_{ij} & \leq & 1 & \text{for all } j \in J \\ x_{ij} & \geq & 0 & \text{for all } i \in I, j \in J. \end{array} \tag{1}$$

Theorem 2 (*Koopmans, Beckmann, 1957*)**.** *For any square matrix $A = \left[a_{ij}\right]$, if the identity assignment is of maximum value then there exists a payoff vector $(u_i; v_j)_{i,j}$ such that the following linear conditions hold:*

$$u_i + v_i = a_{ii} \text{ for all } i \quad ; \quad u_i + v_j \geq a_{ij} \text{ for all } i, j. \tag{2}$$

Conversely, if there exists a payoff vector $(u_i; v_j)_{i,j}$ satisfying (2) then the identity assignment is of maximum value.

Moreover, if all matrix entries are nonnegative (positive), also a nonnegative (positive) such payoff vector exists.

Koopmans and Beckmann (1957), in their informal style, discuss why the existence of such payoff vector(s) implies that "an optimal assignment can be sustained by profit-maximizing agents in a decentralized market mechanism operating through profit-maximizing responses" (we refer to their paper for details). Gale (1960) offers a more formal treatment. He shows that for any optimal solution $(u_i; v_j)_{i,j}$ of the following LP, that is dual to the assignment LP (1),

$$\begin{array}{ccccccl} \sum_{i\in I} u_i & + & \sum_{j\in J} v_j & \rightarrow & \min & \\ u_i & + & v_j & \geq & a_{ij} & \text{for all } i \in I, j \in J \\ u_i & & & \geq & 0 & \text{for all } i \in I \\ & & v_j & \geq & 0 & \text{for all } j \in J, \end{array} \tag{3}$$

the price vector $(p_i = s_i + u_i)_{i\in I}$ is a *competitive equilibrium price* vector in the assignment market $\left(I, J, [s_i]_{i\in I}, [t_{ij}]_{i\in I, j\in J}\right)$, meaning that there exists an assignment μ such that the following conditions hold:

$$\begin{array}{rcl} p_i & \geq & s_i \quad \text{for all } i \in I \\ p_i & = & s_i \quad \text{if } i \in I \text{ is unmatched in } \mu \\ \mu(j) & \in & \left\{ i \in I : t_{ij} - p_i = \max_{k\in I\cup\{i_0\}} \{t_{kj} - p_k\} \right\} \text{ for all } j \in J, \end{array}$$

where i_0 is a "null object/seller" representing the status "unmatched" for possibly several buyers. The value of this option is zero to all buyers, while its price is always zero.

Koopmans and Beckmann (1957) also "register a few straightforward yet interesting implications" of the inequalities (2) for the connections between the payoff and the profitability differences between buyers (sellers) paired with the same seller (buyer). Particularly interesting for this paper are the following ones:

Corollary 1. *If the identity assignment is optimal, the following inequalities hold for any payoff vector $(u_i; v_j)_{i,j}$ satisfying conditions (2) and for any i, j, k:*

$$a_{ki} - a_{ii} \leq u_k - u_i \quad ; \quad a_{ij} - a_{ii} \leq v_j - v_i .$$

The first type of inequalities indeed readily come from eliminating the payoff v_i of buyer i, by subtracting the complementarity equation $u_i + v_i = a_{ii}$ from the acceptability inequality $u_k + v_i \geq a_{ki}$ for the optimally-not-matched (off-diagonal) pair of buyer i and seller $k \neq i$ in (2). The second type of inequalities are obtained analogously.

The third set of fundamental results is due to Shapley and Shubik (1972). Following up the early work of Shapley (1955), the authors associated a cooperative TU game to an assignment market in two steps: first, starting from an assignment market $\left(I, J, [s_i]_{i\in I}, [t_{ij}]_{i\in I, j\in J}\right)$, the pairwise profit matrix $A = \left[a_{ij} \geq 0\right]_{i\in I, j\in J}$ where $a_{ij} = \max\{t_{ij} - s_i, 0\}$ is determined, then second, a related coalitional game, called *assignment game*, $(I \cup J, w_A)$ is defined from matrix A on player set $N = I \cup J$ by the coalitional function

$$w_A(S) = \max_{\mu\in\mathcal{M}(S\cap I, S\cap J)} \sum_{(i,j)\in\mu} a_{ij} \qquad \text{for all } S \subseteq N,$$

where $\mathcal{M}(S \cap I, S \cap J)$ denotes the set of seller-buyer matchings in S.

This framework differs from the setting of Koopmans and Beckmann (1957) in two major aspects. First, Shapley and Shubik (1972) start from "more primitive" market data, namely the individual reservation values of the agents, and derive the potential proceeds of pairwise cooperation from these information. On the other hand, they use the more abstract coalitional game model, which allows richer forms of cooperation in larger groups, to analyze the incentives of the agents and the prospects of reaching a stable outcome. The possibility of a different number of sellers and buyers and the assumed nonnegativity of the pairwise profits are negligible differences in the two settings. In fact, due to the nonnegativity of A in the assignment game model, we can assume, without loss of generality, that any optimal matching is a complete matching for the "short side" of the set of players, and by possibly adding "null agents" to the "short side", we can get an equivalent model with equal number of sellers and buyers.

Shapley and Shubik (1972) investigate the core, the major solution concept for coalitional games which formalizes stability of payoff allocations in terms of efficiency for the grand coalition and acceptability by all coalitions. Since in assignment game $(I \cup J, w_A)$, we clearly have

- $w_A(k) = 0$ for all $k \in I \cup J$, and
- $w_A(ij) = a_{ij}$ for all $i \in I, j \in J$,

the value of any other coalition is equal to the value of one of its partitions in single-player and mixed-pair sub-coalitions. Thus, the single-player and mixed-pair coalitions suffice to describe the core of an assignment game, all other coalitions are inessential. Since $w_A(I \cup J) = \sum_{(i,j)\in\mu_A} w_A(ij)$ with some matching $\mu_A \in \mathcal{M}(I, J)$ of maximum value in A, the efficiency condition for the grand coalition $w_A(I \cup J) = \sum_i u_i + \sum_j v_j$ can be replaced by the set of pairwise efficiency conditions $w_A(ij) = u_i + v_j$ for all optimally-matched pairs $(i, j) \in \mu_A$. Therefore, for nonnegative square matrix $A = \left[a_{ij} \geq 0\right]$, the core of the associated assignment game is precisely the set of nonnegative payoff vectors $(u_i \geq 0;\ v_j \geq 0)_{i,j}$ satisfying the linear conditions in (2).

We summarize the key results on the core of assignment games in the following

Theorem 3 (*Shapley and Shubik, 1972*)**.** *In assignment game $(I \cup J, w_A)$ induced by nonnegative matrix $A_{I\times J}$,*

- *the core is not empty, because it coincides with the nonempty set of optimal solutions to the dual assignment LP (3), and also with the set of payoff vectors attained at competitive equilibrium prices,*
- *the core has a seller-optimal extreme point where each seller receives his core maximum payoff and each buyer receives her core minimum payoff, and similarly, the core has a buyer-optimal extreme point where each buyer receives her core maximum payoff and each seller receives his core minimum payoff.*

Although Shapley and Shubik (1972) have not used the concept of stable outcomes in the market, it is clear that any optimal assignment X for $A_{I\times J}$ combined with any core vector (u, v) gives a stable outcome $(X, (u, v))$ in any assignment market with pairwise profit matrix $A_{I\times J}$, and vice versa, any stable outcome $(X, (u, v))$ for an assignment market with pairwise profit matrix $A_{I\times J}$ is a pair of primal–dual optimal solutions to the corresponding assignment LP with objective coefficient matrix $A_{I\times J}$. Henceforth, we can equivalently use the terminology for the two models and analyze the core of the assignment game and derive results for the underlying assignment market, and vice versa.

3. Preview: two-seller, two-buyer markets

In this section, we use a series of closely related markets with two sellers and two-buyers to illustrate the model and the structure of the set of stable payoff vectors; then to motivate our questions on the sensitivity of the buyer-optimal, the seller-optimal, and the fair (the average of the previous two) competitive equilibrium prices with respect to unilateral changes in the reservation values; and finally, to foreshadow the general results. The various markets in the example will also be used to prove the sharpness of the payoff changes in our main theorem.

Example 1. We consider assignment markets with two sellers $I = \{1, 2\}$ and two buyers $J = \{1', 2'\}$. Each seller i puts a reservation price s_i for his object that applies for both buyers, but a buyer j could set a different willingness to pay value t_{ij} for each of the heterogeneous objects owned by the sellers. From these reservation values we derive the nonnegative pairwise profits $a_{ij} = \max\{0, t_{ij} - s_i\}$.

i	s_i
1	20
2	22

t_{ij}	$1'$	$2'$
1	26	22
2	26	25

$\Longrightarrow$

a_{ij}	$1'$	$2'$
1	[6]	2
2	4	[3]

Since the diagonal entries form the (unique) optimal assignment of value $a_{11}+a_{22} = 9$, the set of stable payoffs (u_i for sellers and v_j for buyers) is given by the following system of linear constraints (on the left below), that, due to the complementarity equations

$$v_1 = 6 - u_1\,, \quad v_2 = 3 - u_2\,,$$

can be equivalently expressed solely in terms of the sellers' payoffs (on the right below).

	$v_1 \geq 0$	$v_2 \geq 0$
$0 \leq u_1$	[$u_1 + v_1 = 6$]	$u_1 + v_2 \geq 2$
$0 \leq u_2$	$u_2 + v_1 \geq 4$	[$u_2 + v_2 = 3$]

	$u_1 \leq 6$	$u_2 \leq 3$
$0 \leq u_1$	.	$u_1 - u_2 \geq -1$
$0 \leq u_2$	$u_2 - u_1 \geq -2$	.

As remarked after Theorem 3, the set of stable payoff vectors coincides with the core of the assignment game induced by the pairwise profit matrix. The pairs of competitive equilibrium prices (p_1, p_2), which trigger stable outcomes of the market, correspond to the pairs of sellers' core payoffs (u_1, u_2) by the following bijective relations:

$$p_1 = s_1 + u_1\,, \quad p_2 = s_2 + u_2\,,$$

where $s_1 = 20$, $s_2 = 22$ are the reservation values of objects 1, 2, respectively. In order to emphasize that the set of competitive equilibrium prices is an additively shifted copy of the core (expressed in terms of sellers' payoffs), we set the origins of the coordinate axes in the following pictures not to 0, but to the actual reservation values, and the scales along the axes indicate the core payoffs.

(Market A) From reservation values

(A)	$t_{.1}$	$t_{.2}$
$s_1 = 20+$	[6]	2
$s_2 = 22+$	4	[3]

we get the set of competitive prices/core payoffs pictured in Fig. 1.

The buyer-optimal core payoffs of the sellers are $(u_1 = 0;\ u_2 = 0)$, the minimum competitive prices are $(p_1 = 20{+}0;\ p_2 = 22{+}0)$. Similarly, the seller-optimal core payoffs are $(u_1 = 5;\ u_2 = 3)$, so the maximum competitive prices are $(p_1 = 20 + 5;\ p_2 =$

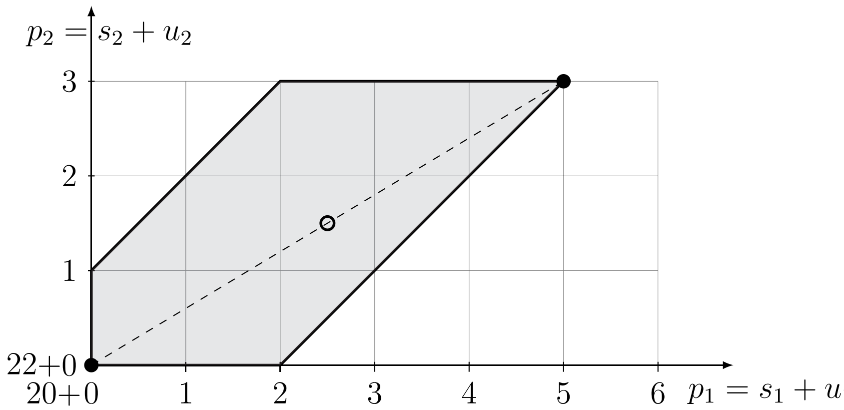


Fig. 1. Competitive prices under the true valuations.

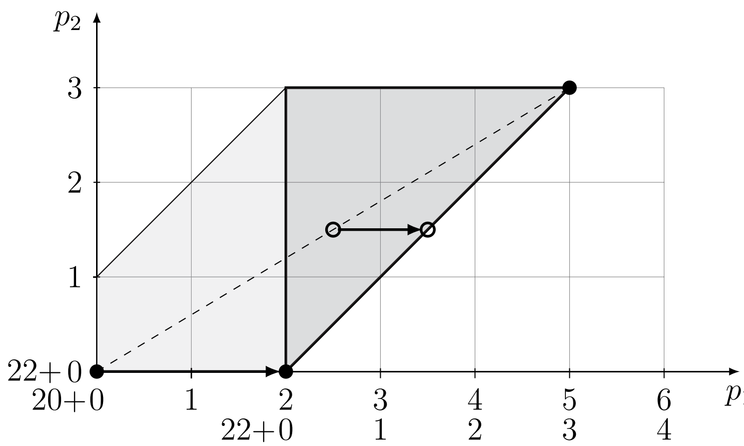


Fig. 2. Price shifts when $s_1 = 20 + 2$ is reported.

22 + 3). The fair competitive prices are ($p_1 = 22.5$; $p_2 = 23.5$) corresponding to the midpont ($u_1 = 2.5$; $u_2 = 1.5$) of the longest chord of the core between the two side-optimal corners. For simplicity, we call it the *tau point* of the core.

We investigate what is the effect on these three special stable payoffs of a "sufficiently small" change in a single reservation value. We might think of an allocation mechanism which takes the reservation values as inputs and produces a stable outcome of the market, typically consisting of an unique optimal assignment of sellers to buyers and a set of competitive prices. If the mechanism is known to always output a particular one from the set of competitive price vectors compatible with the reported reservation prices, the question of strategic manipulation of the mechanism arises.

Naturally, each seller would like to get a higher price for his object, to obtain a higher profit. Let us see what happens in our example, if seller 1 reports a higher reservation price. Suppose that seller 1 increases his reservation price by 2 units to $s_1 = 20 + 2$. Since he makes at least 2 units profit with any of the buyers, the pairwise profits calculated from the new reservation values decrease by 2 for the pairs containing seller 1, and remain the same otherwise. Thus, the value of any assignment of the sellers (objects) to buyers decreases by 2, hence the diagonal assignment remains the optimal one.

(Market B) From the new reported reservation values

(B)	$t_{.1}$	$t_{.2}$
$s_1 = 22+$	$\boxed{4}$	0
$s_2 = 22+$	4	$\boxed{3}$

we get the set of competitive prices/core payoffs pictured in Fig. 2.

The minimum competitive price vector shifted to ($p_1 = 22 + 0$; $p_2 = 22 + 0$), so seller 1 could cash in the whole 2 units by which he falsified his reservation value, provided all other agents behaved honestly. On the other hand, the maximum competitive price vector remained ($p_1 = 25$; $p_2 = 25$), seller 1 could not influence that. As a result, the fair competitive prices shifted to ($p_1 = 22 + 1.5$; $p_2 = 22 + 1.5$), so by misreporting, seller 1 could only gain half of the 2 units by which he unilaterally increased his reservation value.

Notice that, as the pairwise profits decreased, the core shrinked (to the darker shaded triangle), and the u_1 profit scale has changed (to the one below the original scale). In terms of these new values, the buyer-optimal corner of the core remained ($u_1 = 0$; $u_2 = 0$), but the seller-optimal corner became ($u_1 = 3$; $u_2 = 3$), thus the tau point is now ($u_1 = 1.5$; $u_2 = 1.5$). Therefore, seller 1 seemingly did not profit anything if the minimum competitive prices are determined, and even lost 2 units of his profit at the maximum competitive prices, hence lost 1 unit when the fair competitive prices are applied. Compared to his true reservation values, however, he actually increased his profit by 2 units, kept his profit, and increased his profit by 1 unit at the minimum, the maximum, and the fair competitive prices, respectively, due to the fact, that he already "secured" 2 units profit by misreporting.

In this situation, only seller 1 benefited from his unilateral deviation from true reporting, at the expense of his optimally matched partner, buyer 1'. The determined price for object 2 remained the same for all three special competitive price rules we discuss, hence the profits of seller 2 and buyer 2' stayed the same. It is clear from the picture, however, that this will change

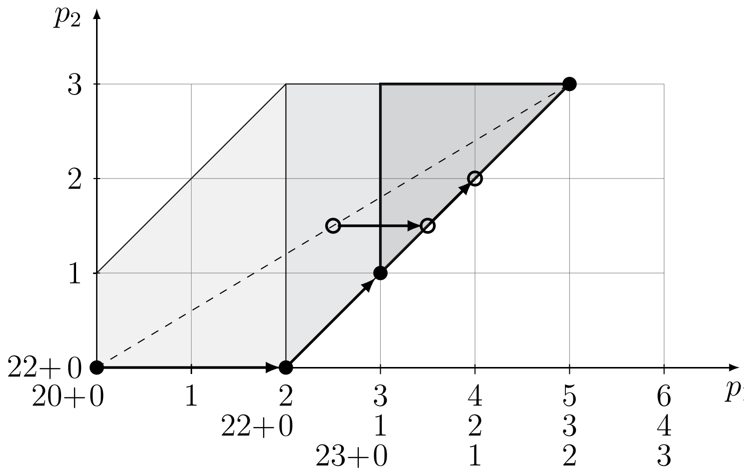


Fig. 3. Price shifts when $s_1 = 20 + 2 + 1$ is reported.

if seller 1 further increases his reservation value, it will influence other agents' profit, too.

Suppose that seller 1 further increases his reservation value by 1 unit to $s_1 = 20 + 3$. Now only the pairwise profit which he can make with buyer 1' decreases by 1, but the profitability of his partnership with buyer 2' cannot decrease as it was already 0. The unique optimal assignment, however, still remains the diagonal one, although the difference in the values of the best and the second best assignments decreased by 1, from $3 = (4+3)-(0+4)$ to $2 = (3 + 3) - (0 + 4)$.

(Market C) From the new reported reservation values

(C)	$t_{.1}$	$t_{.2}$
$s_1 = 23+$	3	0
$s_2 = 22+$	4	3

we get the set of competitive prices/core payoffs pictured in Fig. 3.

The new minimum competitive price vector is $(p_1 = 23 + 0; p_2 = 22 + 1)$. Notice that, unlike in the previous case, seller 1's misreporting also helped seller 2 as his minimum competitive price also increased by 1 unit to $p_2 = 23$. Since the maximum competitive price vector remained $(p_1 = 25; p_2 = 25)$, the new fair competitive prices are $(p_1 = 23 + 1; p_2 = 22 + 2)$, the unilateral 1 unit increase in s_1 caused the same 1/2 units increase in both fair competitive prices. In terms of the core payoffs, the nominal changes are different. Seller 1's core payoff remained zero at the buyer-optimal corner, decreased by 1 unit at the seller-optimal corner, and decreased by 1/2 units at the tau point. To get the changes in his net profit, however, the 1 unit he "secured" by misreporting must be added. On the other hand, seller 2's core payoff increased by 1 unit at the buyer-optimal corner, remained the same at the seller-optimal corner, and increased by 1/2 units at the tau point. In his case, these are precisely the net profit changes. Naturally, the changes in the profits of the buyers are precisely the opposite of the changes in the profits of their optimally matched partners.

It is clear from the picture, that for each agent we would observe the same change in his/her profit, if seller 1 further increased his reservation price by 1 unit to $s_1 = 20+4$, or by 1+1 units to $s_1 = 20 + 5$, but all other agents continued to behave honestly. In the latter case, the set of competitive prices shrinks to the singleton containing only the maximum competitive price vector $(p_1 = 25; p_2 = 25)$.

(Markets D, E) Now let us summarize how **seller 2** can influence the three special competitive prices we consider, provided the other agents report truthfully. The range for possible increase in s_2 without pricing seller 2 out from the market consists of two segments in which the price changes are linear: from 0 to 1, then from 1 to 3 (the maximum core payoff to seller 2). Along the sequence of reported reservation values

(A)	$t_{.1}$	$t_{.2}$
$s_1 = 20+$	6	2
$s_2 = 22+$	4	3

(D)	$t_{.1}$	$t_{.2}$
$s_1 = 20+$	6	2
$s_2 = 23+$	3	2

(E)	$t_{.1}$	$t_{.2}$
$s_1 = 20+$	6	2
$s_2 = 25+$	1	0

we get the accordingly shrinking set of competitive prices/core payoffs pictured in Fig. 4.

(Markets F, G, H) Now let us see how **buyer 1** can influence the three special competitive price vectors we consider, provided the other agents report truthfully. Suppose she misreports both of her reservation prices that are decreased by the same amount. It can range from 0 to 6 (the maximum core payoff to buyer 1') without her losing the possibility to buy something. Notice that, unlike for the other three agents, the minimum core payoff to buyer 1' is positive, namely $1 = 26 - 25$. It follows from $v_2 \geq 0$ and inequality $v_1 - v_2 \geq a_{21} - a_{22} = 4-3 = 1$ stated in Corollary 1, that in turn comes by subtracting the complementarity equality $u_2 + v_2 = a_{22}$ from the stability inequality $u_2 + v_1 \geq a_{21}$. Thus, if buyer 1' misreports her reservation value downward by not more than 1 unit, nothing happens to the set of competitive prices, so she cannot even decrease the highest price $p_1 = 25$ of object 1 she is going to obtain. The range when buyer 1' can influence the competitive prices consists of two segments in which the price changes are linear: from 1 to 4, then from 4 to 6. Along the sequence of reported reservation values

(A)‖|(F)	$t_{.1}$	$t_{.2}$	$t_{.1}$	$t_{.2}$
$s_1 = 20+$	6	2	5	2
$s_2 = 22+$	4	3	3	3

(G)	$t_{.1}$	$t_{.2}$
$s_1 = 20+$	2	2
$s_2 = 22+$	0	3

(H)	$t_{.1}$	$t_{.2}$
$s_1 = 20+$	0	2
$s_2 = 22+$	0	3

we get the accordingly shrinking set of competitive prices/core payoffs pictured in Fig. 5.

Notice that in market (G), similarly to markets (B) and (D), the optimally matched partner (seller 1) of the misreporting agent

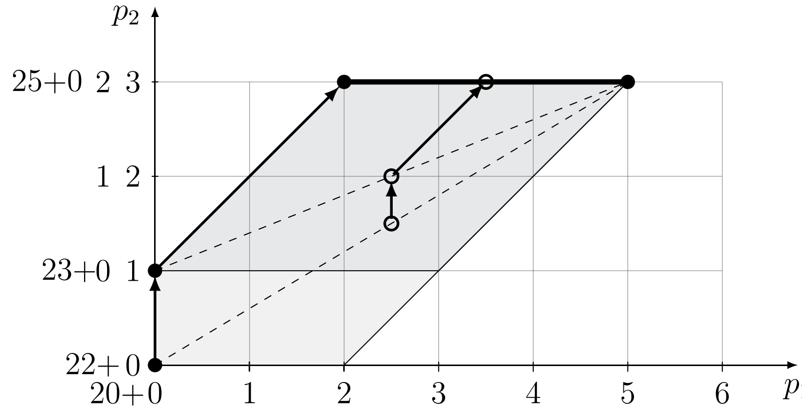


Fig. 4. Price shifts when $s_2 = 22 + 1 + 2$ is reported.

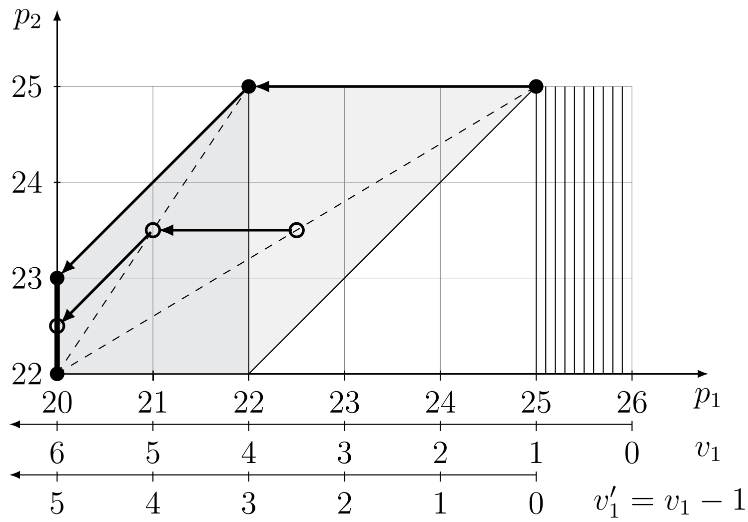


Fig. 5. Price shifts when $t'_{\cdot 1} = t_{\cdot 1} - 1 - 3 - 2$ is reported.

(buyer 1′) has two equally profitable partnerships (seller 1 could make 2 units profit with buyer 2′ too). This indifference causes the same change in the price of the object in the alternative trading pair (p_2 decreases precisely as p_1 does) as long as the optimal assignment of objects/sellers and buyers does not change.

4. Extreme core payoffs for sellers

In this section we give a graph-theoretic characterization of the minimum and the maximum competitive price vectors. It will be our key tool in studying the sensitivity of these two extreme points of the set of competitive prices in the next section. Since we use the same directed graph for all assignment markets with the same matrix of pairwise profits, it will be more convenient to use the equivalent game setting and characterize the two side-optimal core payoff vectors of the associated assignment game. As the section title shows, we opted to describe the core in terms of the sellers' payoffs.

The strong connection between the minimum/maximum competitive equilibrium price vectors in two-sided matching markets and shortest path optimization problems in appropriately defined arc-weighted directed graphs are explored in several papers, see e.g. Mishra and Talman (2010) for the transferable utility case (our setting), and Caplin and Leahy (2014) for the non-transferable utility model discussed in Demange and Gale (1985). Closest to our (self-contained) presentation, however, is the technique used in the assignment nucleolus algorithm (Solymosi and Raghavan, 1994) to find the direction where e.g. the buyer-optimal vertex of the core moves when the pairwise acceptability constraints are strengthened. In fact, our graph-theoretic reformulation of finding the minimum competitive equilibrium price vector is basically the optimization problem to be solved in the first iteration of that assignment nucleolus algorithm.

If there are more sellers (buyers) than buyers (sellers) we can add dummy buyers (sellers) to equalize the number of agents on the two sides of the market. Partnership with a dummy agent is worthless, so in such cases we make the valuation matrices and the induced pairwise profit matrix a square matrix by adding zero columns or rows. In the corresponding assignment game model this means introducing null players of the appropriate type. It is well known that the core satisfies the null player property, meaning that at any core allocation of the augmented assignment game, all null players receive zero payoff.

In order to obtain a unified notation, we introduce a fictitious row player and a fictitious column player, and consider a single-player coalition as a fictitious mixed-pair coalition consisting of the 'real' player and the fictitious one of the other type. Moreover, we identify the mixed-pair coalitions with the ordered pairs of the two players, always the row player written first. More formally, (i, j) denotes the 'real' mixed-pair coalition $\{i, j\}$, $i \in I$, $j \in J$; we write $(i, 0)$ for single-player coalition $\{i\}$, $i \in I$, and $(0, j)$ for $\{j\}$, $j \in J$; finally, $(0, 0)$ denotes the coalition of the fictitious row and column players. To obtain a unified treatment of these coalitions in the description of the core, we augment the original (square) profit matrix with entries $a_{i0} = 0$ for all $i \in I$, also $a_{0j} = 0$ for all $j \in J$, finally $a_{00} = 0$. Since the type of the players is determined by their positions in the ordered pairs, it will be convenient to use a common set $M_0 = \{0, 1, 2, \ldots, m\}$ of indices, where $m = |I| = |J|$. The set of indices for the 'real' row or column players is $M = \{1, 2, \ldots, m\}$.

Given an augmented (square) pairwise profit matrix $A_{M_0 \times M_0}$, we build a complete, simple digraph $G_u(A)$, called the *reference graph for sellers*, with node set M_0 and arcs (i, k), $i, k \in M_0$, $i \neq k$, of length

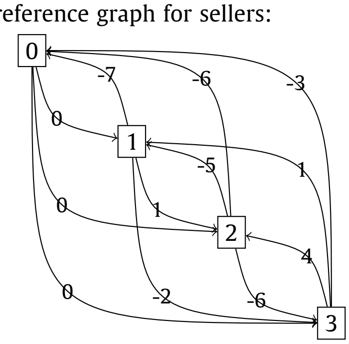
$$d_{ik} := \left\{ \begin{array}{lll} a_{ki} - a_{ii} & \text{for} & i, k \in M \\ 0 - a_{ii} & \text{for} & i \in M, k = 0 \\ 0 - 0 & \text{for} & i = 0, k \in M \end{array} \right\}.$$

For illustration, below we show a 3×3 pairwise profit matrix, its augmented 4×4 version with the fictitious row and column players, and the derived matrix of arc lengths which induces a complete, simple digraph on node set $M_0 = \{0, 1, 2, 3\}$ called the sellers' reference graph:

$$\begin{array}{|ccc|} \boxed{7} & 1 & 4 \\ 8 & \boxed{6} & 7 \\ 5 & 0 & \boxed{3} \end{array} \qquad \text{augmented:} \quad \begin{array}{|c|ccc|} \boxed{0} & 0 & 0 & 0 \\ \hline 0 & \boxed{7} & 1 & 4 \\ 0 & 8 & \boxed{6} & 7 \\ 0 & 5 & 0 & \boxed{3} \end{array}$$

$$\text{arc lengths:} \quad \begin{array}{|c|ccc|} . & -7 & -6 & -3 \\ \hline 0 & . & -5 & +1 \\ 0 & +1 & . & +4 \\ 0 & -2 & -6 & . \end{array}$$

reference graph for sellers:



In the complete and simple (i.e. without loop arcs and parallel arcs) digraph $G_u(A)$ there is a unique arc from any node to any different node, hence any sequence of nodes defines a *walk* from the node listed first to the node listed last. The walk is said to be *closed* if the first and last nodes in the sequence are the same. If only the first and last nodes are the same in the sequence, the closed walk is called a *cycle*. If no node appears twice in the sequence, the walk is called a *path* from the node listed first to the node listed last. Notice that any path and any cycle contains at least two different nodes. The *length* of a walk/path/cycle is the sum of the lengths of the arcs in the walk/path/cycle.

A feature of the associated reference digraph that is crucial for the efficient computation of the side-optimal core payoff vectors is explained next.

Proposition 3. *The digraph $G_u(A)$ associated to matrix A contains no cycle of positive length if and only if the diagonal assignment is of maximum value in A.*

Proof. Sufficiency. Let the diagonal assignment be of maximum value in A, but suppose that a cycle $(i_1, i_2), \ldots, (i_k, i_1)$ in $G_u(A)$ has positive length. Then, with $i_{k+1} = i_1$, we get $\sum_{h=1}^{k} d_{i_h i_{h+1}} = \sum_{h=1}^{k} a_{i_h i_{h+1}} - \sum_{h=1}^{k} a_{i_h i_h} > 0$, a contradiction to the optimality of the diagonal assignment.

Necessity. Take any bijection $\pi : M_0 \to M_0$. Let $M_0 = M_1 \cup \cdots \cup M_p$, $p \geq 1$, be the finest partition of M_0 induced by π, that is $M_r \subseteq M_0$ is a smallest (for inclusion) subset of M_0 for which $\pi(M_r) = M_r$ holds for every $r = 1, \ldots, p$. For any $r \in \{1, \ldots, p\}$, if $|M_r| = k \geq 2$ then it can be written as $M_r = \{i = \pi^0(i), \pi(i), \ldots, \pi^{k-1}(i)\}$ with any $i \in M_r$, and it defines the cycle $(i, \pi(i)), (\pi(i), \pi^2(i)), \ldots, (\pi^{k-1}(i), \pi^k(i_1) = i)$ in $G_u(A)$. The length of any cycle is assumed to be nonpositive, so $\sum_{h=0}^{k-1} d_{\pi^h(i)\pi^{h+1}(i)} = \sum_{h=0}^{k-1} a_{\pi^{h+1}(i)\pi^h(i)} - \sum_{h=0}^{k-1} a_{\pi^h(i)\pi^h(i)} \leq 0$. Summing up these inequalities for all $r \in \{1, \ldots, p\}$ with $|M_r| \geq 2$ and the equalities $a_{\pi(i)i} - a_{ii} = 0$ for any singleton $M_r = \{i\}$ gives $\sum_{i \in M_0} a_{\pi(i)i} - \sum_{i \in M_0} a_{ii} \leq 0$. Thus, the diagonal assignment is of maximum value in matrix A. $\square$

Notice that the digraph $G_u(A)$ contains a cycle of zero length if and only if the diagonal is not the only maximum-value assignment in A.

In the sequel we assume that the rows and columns of the augmented (square) profit matrix are arranged such that the diagonal assignment $\{(i, i) : i \in M_0\}$ is of maximum value, i.e. $w_A(I \cup J) = \sum_{i=1}^{m} a_{ii}$, because, by definition, $a_{00} = 0$. Equivalently, the reference digraph for sellers $G_u(A)$ contains no cycle of positive length.

Leaving out a cycle (i.e. a subsequence between consecutive repetitions of some node) from a walk can only increase the length of the shortend walk. Thus, for any walk from a node i to a distinct node $j \neq i$ there exists an at least as long path from i to j. As the number of $i \rightsquigarrow j$ paths are finite, for any two distinct nodes $i, j \in M_0$ there exists a longest $i \rightsquigarrow j$ path (or paths) and the length of this longest path(s) is an upper bound for the length of any walk from i to j. Notice that the length of any closed walk is also nonpositive, as its set of arcs is the union of the arc sets of the cycles the closed walk is composed of.

It follows from the above discussion that for any node $k \in M$, the following numbers are well defined.

- $\alpha_k :=$ maximum length of $0 \rightsquigarrow k$ paths,
- $\beta_k :=$ maximum length of $k \rightsquigarrow 0$ paths.

Proposition 4. *For any $k \in M$, the following inequalities hold:*

$$0 \leq \alpha_k \leq a_{kk}, \qquad -a_{kk} \leq \beta_k \leq 0, \qquad \alpha_k + \beta_k \leq 0.$$

Moreover, the lower bounds are always attained, i.e. $\alpha_i = 0$ for some $i \in M$ and $\beta_j = -a_{jj}$ for some $j \in M$.

Proof. Since arc $0 \to k$ of length $d_{0k} = 0$ is one of the $0 \rightsquigarrow k$ paths, $\alpha_k \geq 0$ follows for any $k \in M$. Arc $0 \to k$ followed by a longest $k \rightsquigarrow 0$ path forms a cycle of length $d_{0k} + \beta_k = 0 + \beta_k \leq 0$, thus $\beta_k \leq 0$ for any $k \in M$.

Similarly, arc $k \to 0$ of length $d_{k0} = -a_{kk}$ is one of the $k \rightsquigarrow 0$ paths, thus $-a_{kk} \leq \beta_k$ for any $k \in M$. Arc $k \to 0$ followed by a longest $0 \rightsquigarrow k$ path forms a cycle of length $d_{k0} + \alpha_k = -a_{kk} + \alpha_k \leq 0$, thus $\alpha_k \leq a_{kk}$ for any $k \in M$.

Inequality $\alpha_k + \beta_k \leq 0$ comes from the nonpositive length of the closed walk composed of a longest $0 \rightsquigarrow k$ path and a longest $k \rightsquigarrow 0$ path.

To see that the two lower bounds are sharp, take an arbitrary $k \in M$. The first arc, say $0 \to i$, of a longest $0 \rightsquigarrow k$ path, and the last arc, say $j \to 0$, of a longest $k \rightsquigarrow 0$ path are themselves longest $0 \rightsquigarrow i$ and $j \rightsquigarrow 0$ paths, respectively, for any subpath of a longest path must be a longest path from the first node to the last node of the subpath. Thus, $\alpha_i = d_{0i} = 0$ for some $i \in M$ and $\beta_j = d_{j0} = -a_{jj}$ for some $j \in M$. $\square$

Next we establish the link between the lengths of certain longest reference paths and the minimum and maximum core payoffs to the sellers.

Lemma 1. *For any seller $k \in M$,*

- $\alpha_k = \underline{u}_k$ *(= the minimum core payoff to seller $k \neq 0$),*
- $-\beta_k = \overline{u}_k$ *(= the maximum core payoff to seller $k \neq 0$).*

Moreover, $\underline{u}_i = 0$ ($\overline{v}_i = a_{ii}$) for some $i \in M$; and $\overline{u}_j = a_{jj}$ ($\underline{v}_j = 0$) for some $j \in M$.

Proof. First, we show that $\alpha_k \leq \underline{u}_k$. Since, by our assumption, the diagonal (identity) assignment is maximal, all core payoff vectors $(u; v)$ satisfy $u_k + v_i \geq a_{ki}$ and $u_i + v_i = a_{ii}$ for any $i \neq k$. By subtracting the second from the first, we get $u_k - u_i \geq a_{ki} - a_{ii} = d_{ik}$ for all core payoffs to sellers $i \neq k$. In particular, with $i = 0$, $u_k \geq 0$ for any seller k. Applied to the buyer-optimal core payoff vector $(\underline{u}, \overline{v})$, we get that $\underline{u}_k - \underline{u}_i \geq a_{ki} - a_{ii} = d_{ik}$ for any pair $i \neq k$ of sellers. Take a longest $0 \rightsquigarrow k$ path $(0, i_1), (i_1, i_2), \ldots, (i_h, k)$. We have $\alpha_k = d_{0i_1} + d_{i_1 i_2} + \cdots + d_{i_h k} \leq (\underline{u}_{i_1} - \underline{u}_0) + (\underline{u}_{i_2} - \underline{u}_{i_1}) + \cdots + (\underline{u}_k - \underline{u}_{i_h}) = -\underline{u}_0 + \underline{u}_k = \underline{u}_k$ by recalling that the payoff to the fictitious row player is $u_0 = 0$ in any core allocation.

Second, we show that $\alpha_k \geq \underline{u}_k$. To this end, we define the payoff vector $(u; v)$ by $u_i = \alpha_i$ and $v_i = a_{ii} - \alpha_i$ for any $i \in M$. The inequality $\alpha_k \geq \underline{u}_k$ readily follows if we prove that $(u; v)$ is in the core. The $u_i \geq 0$ and $v_i \geq 0$ ($i \in M$) core conditions come from the inequalities $0 \leq \alpha_i \leq a_{ii}$ of Proposition 4. By definition, $u_i + v_i = a_{ii}$ for any $i \in M$. The remaining $u_i + v_j \geq a_{ij}$ ($i \neq j$) core conditions also hold, because the composition of a longest $0 \rightsquigarrow j$ path and the arc $j \to i$ is a walk from 0 to i of length $\alpha_j + d_{ji}$ that cannot be greater than the length α_i of a longest $0 \rightsquigarrow i$ path, thus $\alpha_i - \alpha_j \geq d_{ji} = a_{ij} - a_{jj}$, implying the claimed $u_i + v_j \geq a_{ij}$.

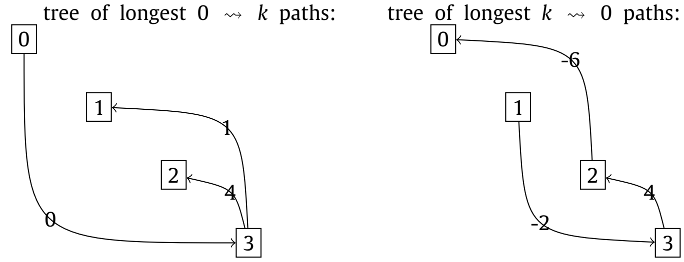
Third, we show that $\beta_k \leq -\overline{u}_k$. Similarly to our first argument, we take the seller-optimal core payoff vector $(\overline{u}, \underline{v})$, and get that $\overline{u}_k - \overline{u}_i \geq a_{ki} - a_{ii} = d_{ik}$ for any pair $i \neq k$ of sellers. Applied to a longest $k \rightsquigarrow 0$ path $(k, i_1), (i_1, i_2), \ldots, (i_h, 0)$, we have $\beta_k = d_{ki_1} + d_{i_1 i_2} + \cdots + d_{i_h 0} \leq (\overline{u}_{i_1} - \overline{u}_k) + (\overline{u}_{i_2} - \overline{u}_{i_1}) + \cdots + (\overline{u}_0 - \overline{u}_{i_h}) = -\overline{u}_k + \overline{u}_0 = -\overline{u}_k$ by recalling that $u_0 = 0$ in any core allocation.

Fourth, we show that $\beta_k \geq -\overline{u}_k$. Similarly to our second argument, we define the payoff vector $(u; v)$ by $u_i = -\beta_i$ and $v_i = a_{ii} + \beta_i$ for any $i \in M$. The inequality $\beta_k \geq -\overline{u}_k$ readily follows if we prove that $(u; v)$ is in the core. The $u_i \geq 0$ and $v_i \geq 0$ ($i \in M$) core conditions come from the inequalities $-a_{ii} \leq \beta_i \leq 0$ of Proposition 4. By definition, $u_i + v_i = a_{ii}$ for any $i \in M$. The remaining $u_i + v_j \geq a_{ij}$ ($i \neq j$) core conditions also hold, because the composition of the arc $j \to i$ and a longest $i \rightsquigarrow 0$ path is a walk from j to 0 of length $d_{ji} + \beta_i$ that cannot be greater than the length β_j of a longest $j \rightsquigarrow 0$ path, thus $\beta_j - \beta_i \geq d_{ji} = a_{ij} - a_{jj}$, implying the claimed $u_i + v_j \geq a_{ij}$.

To see that both theoretical bounds on the core payoffs to sellers are attained for some sellers, recall from Proposition 4 that always exists some $i \in M$ with $\alpha_i = 0$, and some $j \in M$ with $-\beta_j = a_{jj}$. The analogous statements for the buyers come from the complementarity equations for the optimally-matched pairs. □

For illustration, let us revisit our 3×3 example. Here all longest paths from node 0 to the other nodes are unique, hence their arcs form a directed tree, called the tree of longest paths, rooted in node 0 (on the left below). Similarly, all longest paths to node 0 from each of the other nodes are unique, hence their arcs form a reversely directed tree whose sink is node 0 (on the right below). From the lengths of all these longest paths we obtain the minimum and the maximum core payoffs to each seller, and from the pairwise efficiency equations, the maximum and the minimum core payoffs to their optimally-matched buyers.

tree of longest $0 \rightsquigarrow k$ paths: tree of longest $k \rightsquigarrow 0$ paths:



Core min $\underline{u} = (1, 4, 0) = \alpha$ Core max $\overline{u} = (4, 6, 2) = -\beta$

The buyer-optimal corner of the core is $(\underline{u}; \overline{v}) = (1, 4, 0; 6, 2, 3)$, while the seller-optimal corner is $(\overline{u}; \underline{v}) = (4, 6, 2; 3, 0, 1)$. In this example, only seller 3 and buyer 2 receive the theoretical minimum 0 payoff in the core, thus their respective partners get the full profit of what they can achieve together.

We remark that the vectors $(\alpha_k)_{k \in M}$ and $(\beta_k)_{k \in M}$ of maximum path lengths from and to node 0 can be computed in polynomial time. Among the various available approaches which can be used (with more or less the same computational efficiency), we only mention two methods (without giving any algorithmic details) which we find particularly close to our discussion. Namely the method of

- Floyd (1962) and Warshall (1962) which determines the longest paths between all pairs of nodes in an arc-weighted directed graph;
- Solymosi and Raghavan (1994) which determines the path traversed by the minimum corner of ε-core boxes as ε increases, starting at the minimum corner of the imputation box (when, typically, $\varepsilon < 0$), passing through the minimum corner of the core (when $\varepsilon = 0$), all the way to nucleolus point (when, typically, $\varepsilon > 0$).

5. Sensitivity of the side-optimal and the fair payoffs

Thompson (1981) suggested to use the *fair prices*, the simple average of the seller-optimal prices and the buyer-optimal prices. Núñez and Rafels (2002) proved that the tau-value, a point-valued solution concept for coalitional games introduced by Tijs (1981), of the corresponding assignment game is precisely the vector of payoffs to the agents obtained when the objects are sold at the fair prices. We denote the fair (tau) payoffs to sellers and buyers by $\widetilde{u}$ and $\widetilde{v}$, respectively, that is, $\widetilde{u}_i = \frac{\underline{u}_i + \overline{u}_i}{2}$ and $\widetilde{v}_i = \frac{\overline{v}_i + \underline{v}_i}{2}$ for $i \in M_0$. Obviously, $\widetilde{u}_0 = \widetilde{v}_0 = 0$.

We show that every seller can unilaterally manipulate to his advantage the fair prices by increasing his reservation value. On the other hand, the amount a seller can gain by misreporting his reservation value cannot exceed the half of the amount of manipulation (i.e. the difference between the reported and the true reservation values), provided that amount is not more than the maximum core payoff to this seller.

We denote by $x^+ = \max\{x, 0\}$ the positive part of $x \in \mathbb{R}$, and by $x \wedge y = \min\{x, y\}$ the minimum of $x, y \in \mathbb{R}$.

Theorem 4. *Let seller $k \in M$ increase his reservation value by $0 \leq c \leq \overline{u}_k$, but let all other agents keep their original valuations, so the new pairwise profits (including those with the fictitious players) become*

$$a'_{kj} = (a_{kj} - c)^+ \text{ for all } j \in M_0, \qquad a'_{ij} = a_{ij} \text{ for all } i, j \in M_0, i \neq k.$$

Then

1. *the optimal assignment of sellers and buyers remains optimal in the new setting;*

2. $\underline{u}'_k = (\underline{u}_k - c)^+ = \underline{u}_k - (c \wedge \underline{u}_k)$ *for the minimum core payoff to seller k, and* $\underline{u}_i \leq \underline{u}'_i \leq \underline{u}_i + c$ *for the minimum core payoff to any other seller* $i \in M$, $i \neq k$;
3. $\overline{u}'_k = \overline{u}_k - c$ *for the maximum core payoff to seller k, and* $\overline{u}'_i = \overline{u}_i$ *for the maximum core payoff to any other seller* $i \in M$, $i \neq k$;
4. *Consequently,* $\widetilde{u}'_k = \widetilde{u}_k - \dfrac{c + (c \wedge \underline{u}_k)}{2}$ *for the fair payoff to seller k, and* $\widetilde{u}_i \leq \widetilde{u}'_i \leq \widetilde{u}_i + \frac{c}{2}$ *for the fair payoff to any other seller* $i \in M$, $i \neq k$.

Moreover, all bounds are sharp in Claims 2 and 4.

Proof. First of all, let us see how the arc lengths in the corresponding reference digraph change if seller k increases his reservation value by $0 \leq c \leq \overline{u}_k$, but all other valuations remain the same. In the new pairwise profit matrix, $a'_{kj} = (a_{kj} - c)^+ = a_{kj} - (a_{kj} \wedge c)$ for all $j \in M_0$, in particular, $a'_{k0} = 0$ and $a'_{kk} = a_{kk} - c$ (since $0 \leq c \leq \overline{u}_k \leq a_{kk}$); and $a'_{ij} = a_{ij}$ for all $i, j \in M_0$ $i \neq k$. We get for the lengths of the

- outgoing arcs from node k: $d'_{kj} = d_{kj} + c$ for all $j \in M_0, j \neq k$, because of $a'_{jk} = a_{jk}$ for all $j \in M_0, j \neq k$, and $a'_{kk} = a_{kk} - c$;
- incoming arcs to node k: $d_{jk} \geq d'_{jk} = d_{jk} - (a_{kj} \wedge c) \geq d_{jk} - c$ for all $j \in M_0, j \neq k$, because of $a_{kj} \geq a'_{kj} = a_{kj} - (a_{kj} \wedge c) \geq a_{kj} - c$ and $a'_{jj} = a_{jj}$ for all $j \in M_0, j \neq k$;
- any other arc: $d'_{ji} = d_{ji}$ for all $i \neq j \in M_0$, $i, j \neq k$, because of $a'_{ij} = a_{ij}$ and $a'_{ii} = a_{ii}$ for all $i, j \in M_0$, $i \neq j$.

To see Claim 1, recall from Proposition 3 that the optimality of the original (diagonal) assignment is equivalent to the non-existence of a cycle of positive length in the corresponding reference digraph. If seller $k \in M$ increases his reservation value by $0 \leq c \leq \overline{u}_k$, only the lengths of the outgoing and the incoming arcs incident to node k change, hence only the lengths of the cycles which contain node k can change. Take such a cycle that it is composed of the arc $k \to j$, some path $j \rightsquigarrow i$, and the arc $i \to k$ with some $k \neq i, j \in M$ (maybe with $i = j$). The length of this cycle is $(a_{jk} - a_{kk}) + \ell(j \rightsquigarrow i) + (a_{ki} - a_{ii})$, that changes precisely as the difference $a_{ki} - a_{kk}$ does, because only these two terms are in the kth row of the matrix. In case of $c \leq a_{ki}$, we get $a'_{ki} - a'_{kk} = (a_{ki} - c) - (a_{kk} - c) = a_{ki} - a_{kk}$, so the length of the cycle remains the same nonpositive value. If $c \geq a_{ki}$, we get $a'_{ki} - a'_{kk} = 0 - (a_{kk} - c) = a_{ki} - a_{kk} + (c - a_{ki})$, so the length of the cycle increases by $c - a_{ki}$. However, for the length of this cycle we have $\ell(k \to j \rightsquigarrow i \to k) = \ell(k \to j \rightsquigarrow i \to 0) + a_{ki} \leq \beta_k + a_{ki} \leq -c + a_{ki}$, since the sequence $k \to j \rightsquigarrow i \to 0$ is an $k \rightsquigarrow 0$ path and $\beta_k = -\overline{u}_k \leq -c$. Thus, even the increased length of our cycle remains nonpositive in the new setting, $\ell'(k \to j \rightsquigarrow i \to k) = \ell(k \to j \rightsquigarrow i \to k) + c - a_{ki} \leq -c + a_{ki} + c - a_{ki} = 0$.

In Claim 2, equation $\underline{u}'_k = \underline{u}_k - (c \wedge \underline{u}_k)$ for the misrepresenting seller can be seen as follows. In case of $\underline{u}_k = 0$, arc $0 \to k$ is a longest $0 \rightsquigarrow k$ path whose length remains 0 also in the new setting. Since the length of any other incoming arc to node k can only decrease, the nonpositive length of any other $0 \rightsquigarrow k$ path can only decrease. Thus, arc $0 \to k$ remains a longest path also in the new setting, implying $\underline{u}'_k = 0 = 0 - (c \wedge 0)$ for $c \geq 0$. On the other hand, in case of $\underline{u}_k > 0$, any longest $0 \rightsquigarrow k$ path must be of the form $0 \rightsquigarrow i \to k$ consisting of a longest $0 \rightsquigarrow i$ path to some node $i \neq k$, $i \in M$ and the arc $i \to k$. Thus, we have $\underline{u}_k = \underline{u}_i - a_{ii} + a_{ki}$, implying $\underline{u}_k \leq a_{ki}$ because of $\underline{u}_i \leq a_{ii}$. If $c \leq \underline{u}_k$, the length of our path $0 \rightsquigarrow i \to k$ decreases by $c \wedge a_{ki} = c$ down to $\underline{u}_k - c \geq 0$. To see that this is the length of the longest $0 \rightsquigarrow k$ path(s) in the new setting, take any path of the form $0 \rightsquigarrow j \to k$ with some node $j \neq i, k$, $j \in M$. Its length is at most $\underline{u}_j - a_{jj} + a_{kj}$ originally, and decreases by $c \wedge a_{kj}$. Since $\underline{u}_j \leq a_{jj}$, the decreased length can only be positive if $a_{kj} - (c \wedge a_{kj})$ is positive, so $a_{kj} > c$, implying that in the new setting the length of the path $0 \rightsquigarrow j \to k$ is at most $\underline{u}_j - a_{jj} + a_{kj} - c \leq \underline{u}_k - c$. Therefore, we get $\underline{u}'_k = \underline{u}_k - c = \underline{u}_k - (c \wedge \underline{u}_k)$ also in case of $\underline{u}_k > 0$ and $c \leq \underline{u}_k$. The previous argument also tells that, even if originally positive, the length of any path of the form $0 \rightsquigarrow j \to k$ with some node $j \neq k$, $j \in M$, becomes nonpositive if $c \geq \underline{u}_k$, implying $\underline{u}'_k = 0 = \underline{u}_k - (c \wedge \underline{u}_k)$ also in this subcase, and making the proof of the equation for the misrepresenting seller complete.

In Claim 2, the inequalities $\underline{u}_i \leq \underline{u}'_i \leq \underline{u}_i + c$ for any other seller $i \in M$, $i \neq k$ follow from Lemma 1 once we show that the maximum length α_i of the $0 \rightsquigarrow i$ paths can only increase, but by no more than c. To this end, take a $0 \rightsquigarrow i$ path arbitrarily. If it contains no arc incident to node k, its length remains unchanged. Otherwise, the path contains exactly one outgoing arc from node k whose length increases by c, exactly one incoming arc to node k whose length decreases by at most c, and some other arcs whose lengths remain unchanged, so the length of the path can only increase by at most c. In any case, the length of a $0 \rightsquigarrow i$ path can only increase, but by no more than c, thus, $\alpha_i \leq \alpha'_i \leq \alpha_i + c$.

Claim 3 also follows from Lemma 1 once we show that the maximum length β_k of the $k \rightsquigarrow 0$ paths increases by exactly c, however, for $i \neq k$, the maximum length β_i of the $i \rightsquigarrow 0$ paths does not change. Since any $k \rightsquigarrow 0$ path consists of exactly one outgoing arc from node k whose length increases by c and some other arcs whose lengths remain unchanged, we readily get $\beta'_k = \beta_k + c$. For any other node $i \neq k$, the $i \rightsquigarrow 0$ paths are of two distinct types: those which do not go through node k and those which contain node k. The maximum length of the $i \rightsquigarrow 0$ paths is clearly the maximum of the maximum lengths of the paths in these two classes. We show that for either type of paths the maximum length remains unchanged, implying $\beta'_i = \beta_i$ for $i \neq k$. In case an $i \rightsquigarrow 0$ path contains no arc incident to node k, its length is left unchanged, so does the maximum of these path lengths. In case an $i \rightsquigarrow 0$ path contains one incoming arc to node k (its length decreases by at most c) and one outgoing arc from k (its length increases by c), and some other arcs (with unchanged lengths), the path length can only increase, but by at most c. We argue that, even if it increases, the length of such a path remains at most β_i. To see this, take a path of the form $i \rightsquigarrow p \to k \rightsquigarrow 0$ with some $p \in M$, $p \neq k$, but node p might be node i. Its length is $\ell(i \rightsquigarrow p) - a_{pp} + a_{kp} + \ell(k \rightsquigarrow 0)$, that will change by $c - (c \wedge a_{kp})$. Thus, the length of this path does not change if $c \leq a_{kp}$. On the other hand, if $c \geq a_{kp}$, the length of this path increases to $\ell(i \rightsquigarrow p) - a_{pp} + a_{kp} + \ell(k \rightsquigarrow 0) + c - a_{kp}$ that is at most $\ell(i \rightsquigarrow p) - a_{pp}$, because of $\ell(k \rightsquigarrow 0) \leq \beta_k = -\overline{u}_k \leq -c$. Since $\ell(i \rightsquigarrow p) - a_{pp}$ is precisely the length of the path $i \rightsquigarrow p \to 0$ which does not visit node k, it is at most β_i in both settings. We get that the maximum length of either type of $i \rightsquigarrow 0$ paths does not decrease but does not exceed β_i either, implying $\beta'_i = \beta_i$.

The inequalities in Claim 4 come from averaging the corresponding inequalities in Claim 2 and equations in Claim 3.

Finally, we show the sharpness of the bounds in Claims 2 and 4 by the markets discussed in Example 1 (Section 3).

- Fig. 2 shows a situation where seller $k = 1$ increases his reservation value by $c = 2 \leq 5 = \overline{u}_1$. It exceeds his minimum core payoff $\underline{u}_k = 0$, and his minimum core payoff indeed remains $\underline{u}'_k = (0 - 2)^+ = 0$, illustrating the equation for the misreporting seller in Claim 2. Since, by Claim 3, his maximum core payoff decreases by $c = 2$ (indeed, $\overline{u}'_k = 3 = 5 - 2 = \overline{u}_k - c$), his fair payoff decreases to $\widetilde{u}'_k = \frac{3}{2} = \frac{5}{2} - \frac{2 + (2 \wedge 0)}{2}$, as prescribed by the equation for the misreporting seller in Claim 4. In the same situation, the honestly reporting other seller $i = 2$ notices no change in his minimum core payoff, $\underline{u}_i' = 0 = \underline{u}_i$, proving the sharpness of the corresponding lower bound in Claim 2. His maximum core payoff is also left unchanged, $\overline{u}_i' = 3 = \overline{u}_i$, as prescribed by Claim 3 in general. Thus, seller 2's fair payoff remains $\widetilde{u}_i' = \frac{3}{2} = \widetilde{u}_i$, which shows the sharpness of the lower bound in Claim 4 for an truthfully reporting seller.

- The second situation pictured in Fig. 3 shows the sharpness of two other bounds. Assume that $s_1 = 22$ is the true reservation value of seller $k = 1$ that he increases by $c = 1 \le 3 = \overline{u}_1$. The pattern of changes in his payoffs are exactly the same as in the previous case. The minimum core payoff of the honestly reporting other seller $i = 2$, however, increases by the maximum prescribed amount, $\underline{u}_i' = 1 = 0 + 1 = \underline{u}_i + c$, proving the sharpness of the corresponding upper bound in Claim 2. Since, as always, his maximum core payoff remains the same, $\overline{u}_i' = 3 = \overline{u}_i$, seller 2's fair payoff also increases by the maximum prescribed amount, $\widetilde{u}_i' = 2 = \frac{3}{2} + \frac{1}{2} = \widetilde{u}_i + \frac{c}{2}$, proving the sharpness of the upper bound in Claim 4 for the honestly reporting seller. □

We remark that the first situation pictured in Fig. 5 illustrates the other case for the misreporting player in Claims 2 and 4, namely when both the minimum and the fair core payoffs decrease by exactly c because c does not exceed the positive minimum core payoff. The misreporting player k is buyer 1 now, she decreases both of her true valuations by the same amount $c = 1$, inducing a shift in her core payoff scale (from v_1 to v_1' in the figure). Her minimum core payoff decreases by the prescribed maximum amount $c = 1$ from $\underline{v}_k = 1$ to $\underline{v}_k' = 0$, illustrating the equation in Claim 2. Since, by Claim 3, her maximum core payoff decreases by $c = 1$ (indeed, $\overline{v}_k' = 5 = 6 - 1 = \overline{v}_k - c$), the change in her fair payoffs, $\widetilde{v}_k' = \frac{5}{2} = \frac{7}{2} - 1 = \widetilde{v}_k - c$, illustrates the equation in Claim 4 for the misreporting player. Notice that in this situation all three core payoffs remained unchanged for the honestly reporting buyer $i = 2$.

By translating the set of sellers' core payoff vectors $(u_i)_{i \in M}$ in the assignment game to the set of competitive equilibrium price vectors $(p_i)_{i \in M}$ in the underlying market via the bijective relations $p_i = s_i + u_i$ for all sellers $i \in M$, where s_i $(i \in M)$ is the true reservation value of seller $i \in M$, we readily get the following corollary of Theorem 4. Notice that $p_i = s_i + u_i$ simplifies to $p_i = s_i + 0$ for any seller $i \in M$ who is optimally not matched to a "real" buyer, thus is matched to a "null" buyer in our "squared" model.

Let $\overline{p}_i$ denote the maximum, $\underline{p}_i$ denote the minimum, and $\widetilde{p}_i$ denote the fair competitive equilibrium price of object $i \in M$.

Corollary 2. *Let seller $k \in M$ increase his reservation value by $0 \le c \le \overline{p}_k - s_k$, but let all other agents keep their original valuations, so the new reservation values are $s_k' = s_k + c$, but $s_i' = s_i$ and $t_{ij}' = t_{ij}$ for all $i, j \in M$. Then*

1. *the optimal assignment of objects to buyers remains optimal in the new setting;*
2. *$\underline{p}_k' = \underline{p}_k + (s_k' - \underline{p}_k)^+$ for the new minimum competitive price $\underline{p}_k'$ of object k; $\underline{p}_i \le \underline{p}_i' \le \underline{p}_i + c$ for the new minimum competitive price $\underline{p}_i'$ of object i for all other $i \in M$, $i \neq k$;*
3. *$\overline{p}_i' = \overline{p}_i$ for the new maximum competitive price $\overline{p}_i'$ of object i for all $i \in M$;*
4. *Consequently, $\widetilde{p}_k' = \widetilde{p}_k + \frac{(s_k' - \underline{p}_k)^+}{2}$ for the new fair competitive price $\widetilde{p}_k'$ of object k; $\widetilde{p}_i \le \widetilde{p}_i' \le \widetilde{p}_i + \frac{c}{2}$ for the new fair competitive price $\widetilde{p}_i'$ of object i for all other $i \in M$, $i \neq k$.*

Moreover, all bounds are sharp in Claims 2 and 4.

Notice that under the assumptions of Corollary 2, the sharp price bounds in Claims 2 and 4 also apply for object k, that is, $\underline{p}_k \le \underline{p}_k' \le \underline{p}_k + c$ and $\widetilde{p}_k \le \widetilde{p}_k' \le \widetilde{p}_k + \frac{c}{2}$ hold.

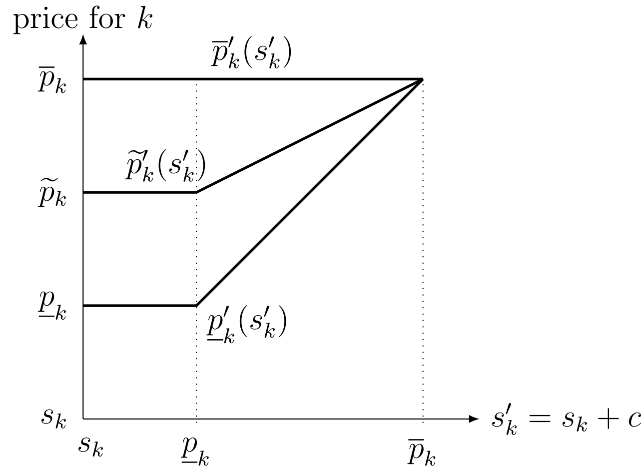


Fig. 6. The minimum, maximum, and fair price functions for seller k.

Fig. 6 illustrates (in terms of the true values) the exact functional relations between the reported reservation value s_k' of seller k and the minimum $\underline{p}_k'$, the maximum $\overline{p}_k'$, and the fair $\widetilde{p}_k'$ competitive prices the mechanism determines for object k (from the reported values).

For the truthfully reporting sellers the three price functions are also monotone nondecreasing and piecewise linear, but a general formula can be given only for the maximum competitive price functions: all of them are constant, just like for the misreporting seller (Corollary 2, Claim 3). From the above proofs it is not difficult to conclude that the minimum competitive price functions are built up from constant and 45 degree line pieces, just like for the misreporting seller, but various patterns can arise. For example, the minimum price $\underline{p}_i'(s_k')$ for seller $i \neq k$ could constantly remain $\underline{p}_i$ over the domain $s_k \le s_k' \le \overline{p}_k$, or could become constant after an increasing (1-sloped) segment. Furthermore, the highest minimum price determined from the reported valuations might stay below the maximum competitive price, that is $\underline{p}_i'(\overline{p}_k) < \overline{p}_i$ can happen, unlike for the misreporting seller. Naturally, the same observations can be made for the fair price function $\widetilde{p}_i'(s_k')$ for seller $i \neq k$ over $s_k \le s_k' \le \overline{p}_k$, with the difference that any 45-degree line segment must be replaced by a line segment of slope 1/2.

6. Concluding remarks

We considered assignment markets and investigated the sensitivity of the minimum, the maximum, and the fair competitive equilibrium prices with respect to changes in the valuation(s) of an agent. We found that, by increasing his reservation value, each seller can increase the minimum and the fair competitive equilibrium prices, but cannot influence the maximum competitive equilibrium prices. Analogously, by decreasing her reservation values by the same amount, each buyer can decrease the maximum and the fair competitive equilibrium prices, but cannot influence the minimum competitive equilibrium prices.

Moreover, we established sharp upper bounds for the extent these competitive equilibrium prices could be influenced by unilateral changes. The minimum competitive equilibrium prices could even increase by the same amount as the particular seller's reservation value. Analogously, the maximum competitive equilibrium prices could even decrease by the same amount as the particular buyer's reservation values. The fair competitive equilibrium prices, however, could only increase/decrease by at most half of the amount the particular seller/buyer increases/decreases his/her reservation values. All these sensitivity results are valid if the change in the agent's valuation does not exceed his/her maximum competitive payoff, consequently, the collectively optimal seller-buyer matching remains optimal.

We find this attenuative feature of the fair competitive prices very appealing. We conjecture that no other competitive price rule, that returns to each assignment market a unique competitive equilibrium price vector, could guarantee a smaller than 1/2 "depletion factor" in the worst case.

A natural "contender" of the fair price rule would be the nucleolus price rule, that returns to each assignment market the competitive equilibrium price vector obtained from the sellers' reservation values and the nucleolus payoffs in the associated assignment game. The nucleolus payoffs (hence the associated prices) can also be computed in strongly polynomial time from the market valuations (Solymosi and Raghavan, 1994), although slightly less efficiently than the minimum, the maximum, and the fair competitive equilibrium prices. The nucleolus, however, is in a certain sense the "innermost" point in the core, thus it could be expected being less sensitive than the tau-value that might lie on the boundary of the core, see e.g. Fig. 3. However, precisely the move from Market B to Market C in Example 1 shows that seller 1 could increase his nucleolus price by 2/3 units by unilaterally increasing his reservation value by 1 unit (from $s_1 = 22$ to $s_1 = 23$), whereas with the fair prices his gain is only 1/2 units. Indeed, the nucleolus prices in Market B are ($p_1 = 23 = 22 + 1$; $p_2 = 24 = 22 + 2$), these increase to ($p'_1 = 71/3 = 23 + 2/3$; $p'_2 = 73/3 = 22 + 7/3$) in Market C. We conjecture that for the nucleolus price rule the worst case "depletion factor" is $\frac{d}{d+1}$, where d is the dimension of the set of competitive equilibrium prices (equivalently, the dimension of the core of the associated assignment game). If indeed true, this instance would show that this worst case upper bound is sharp.

Data availability

No data was used for the research described in the article.

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