Operations Research Letters 51 (2023) 643–650

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Mixed-strategy equilibrium of the symmetric production in advance game: The missing case

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ARTICLE INFO

Article history:
Received 28 November 2022
Received in revised form 29 September 2023
Accepted 10 October 2023
Available online 17 October 2023

Keywords:
Price-quantity game
Bertrand-Edgeworth competition
Inventory

ABSTRACT

The mixed-strategy equilibrium of the symmetric production-in-advance type capacity-constrained Bertrand-Edgeworth duopoly game has not been derived analytically over the entire range of intermediate capacities in the literature. In this paper we derive for the missing region a symmetric mixed-strategy equilibrium analytically.

1. Introduction

We investigate a homogeneous good duopoly model introduced by Shubik [12] in which the firms set both their prices and quantities simultaneously. For this game [12] already found that it may not have an equilibrium in pure strategies. The existence of a mixed-strategy equilibrium (MSE) was established by Maskin [7] who called this game the production-in-advance game in which production takes place before sales are realized. In contrast, in the case of production to order, production takes place after prices are known.

From one point of view it is the most natural case that the firms have the freedom to set their own prices and quantities though not necessarily simultaneously. Markets of perishable goods are usually mentioned as examples of advance production in a market. Spot markets in general can also be regarded as production-in-advance markets. Phillips et al. [11] emphasized that there are also goods that can be traded both in a production-in-advance and in a production-to-order environment. For example, coal and electricity are sold in both types of environments. Based on Italian industry data Casaburi and Minerva [1] investigated the endogenous mode of production, that is whether firms choose to produce to order or in advance. They observed that production in advance occurs more frequently in homogenous industries than in differentiated ones, while production to order is more prevalent as product differentiation increases. In an experimental setting production in advance under the assumption of a non-atomistic buyer side has been investigated by Davis [3], Muren [9], and Orland and Selten [10].

In an earlier work Tasnádi [14] demonstrated for the case of identical capacities and constant unit costs that in equilibrium production-in-advance profits are equal to production-to-order profits, while prices are higher in the former case. Montez and Schutz [8] considered quantity as an unobservable inventory, hence though in their context the quantity decision precedes the price decision their game is equivalent to the production-in-advance game. Somogyi, Vergote and Virág [13] introduced capacity uncertainty into the model in order to explain the empirical observation that large firms set lower prices. Among others Hirata and Matsumura [5] analyzed the standard Bertrand price-setting game without capacity constraints.

Turning to the results on the MSE in closed form of the production-in-advance game, Levitan and Shubik [6] computed the MSE for the case of production in advance under linear demand and unlimited capacities. In the same framework Gertner [4] determined the MSE under more general conditions. Montez and Schutz [8] resolved limitations and corrected flaws of previous works. They calculated the MSE of the production-in-advance game for the case of large capacities.

Recently, Tasnádi [15] calculated a symmetric MSE for a large range of intermediate capacities. In this paper we address the missing region of intermediate capacities on which the MSE is far more complex and has to be determined successively in a finite number of steps.

The reasons for considering the symmetric setting are related to the mentioned experimental works [3] and [11], the technical dif-

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https://doi.org/10.1016/j.orl.2023.10.007

ficulty of the determination of an equilibrium and the complexity of the obtained expressions. Proposition 2 points out that for intermediate capacities the game has infinitely many asymmetric MSE besides the more plausible symmetric ones. Interestingly, firms can choose independently from any of these equilibrium strategies to obtain a MSE.

2. Preliminaries

This section contains the necessary assumptions, notations, and the required available results in the literature.

Assumption 1. The demand curve $D:\mathbb{R}_+\to\mathbb{R}_+$ is strictly decreasing on $[0,b]$, identically zero on $[b,\infty)$, continuous at b and twice continuously differentiable on $(0,b)$. Furthermore, the revenue function $pD(p)$ is strictly concave on $[0,b]$.

Let $a=D(0)$ and P be the inverse demand function. This means that the demand curve intersects the axes at a and b, respectively.

We consider the duopoly model in which both firms set their prices and quantities simultaneously.

Assumption 2. Firms 1 and 2 have identical positive unit costs $c\in(0,b)$ up to the same positive capacity constraint k. Each of them sets its price $p_1,p_2\in[0,b]$ and production quantity $q_1,q_2\in[0,k]$.

When referring to firms with A and B, our convention is that $A,B\in\{1,2\}$ and $A\neq B$.

Assumption 3. Incorporating the efficient rationing rule, the demand faced by firm A is given by

$$\Delta_A(p_1,q_1,p_2,q_2)=\begin{cases} D(p_A) & \text{if} \quad p_A<p_B,\\ \frac{q_A}{q_A+q_B}D(p_A) & \text{if} \quad p_A=p_B,\\ (D(p_A)-q_B)^+ & \text{if} \quad p_A>p_B,\end{cases}$$

where, as usual, $f^+(x)$ stands for $\max\{f(x),0\}$ for an arbitrary function $f:\mathbb{R}\to\mathbb{R}$. The interpretation of Assumption 3 is as follows: the low-price firm faces the entire demand, in case of ties firms split the demand in proportion to the firms' quantity decisions and the high-price firm faces the demand minus the quantity produced by the low-price firm. For more details on rationing rules we refer to [16]. Concerning price ties, our analysis remains valid for a large class of tie-breaking rules satisfying that a firm's demand is strictly increasing in its own quantity.

The firms' profits are given by

$$\pi_A((p_1,q_1),(p_2,q_2))=p_A\min\{\Delta_A(p_1,q_1,p_2,q_2),q_A\}-cq_A$$

for both $A\in\{1,2\}$, where we have taken into account that the firms are demand or capacity constrained.

From Dasgupta and Maskin [2, Theorem 6*] it follows that the symmetric production-in-advance game possesses a symmetric MSE. In the following, a mixed strategy μ_A is a probability measure defined on the σ-algebra of Borel measurable sets on $[0,b]\times[0,k]$, which can be restricted without loss of generality to $S=[c,b]\times[0,k]$. In equilibrium each firm is playing a best response. Such an equilibrium is denoted by (μ_1^*,μ_2^*). A MSE (μ_1^*,μ_2^*) can be calculated by the following two conditions:

$$\pi_1((p_1,q_1),\mu_2^*)\le\pi_1^*, \qquad \pi_2(\mu_1^*,(p_2,q_2))\le\pi_2^* \tag{1}$$

holds true for all $(p_1,q_1),(p_2,q_2)\in S$, and

$$\pi_1((p_1^*,q_1^*),\mu_2^*)=\pi_1^*, \qquad \pi_2(\mu_1^*,(p_2^*,q_2^*))=\pi_2^* \tag{2}$$

holds true μ_1^*-almost everywhere and μ_2^*-almost everywhere, where π_1^*,π_2^* stand for the equilibrium profits corresponding to (μ_1^*,μ_2^*). If a mixed strategy appears in the argument of the profit function π_A, we mean expected profits.

We define the market-clearing price p^* by

$$p^*=\begin{cases} D^{-1}(2k) & \text{if} \quad D(0)>2k\\ 0 & \text{if} \quad D(0)\le 2k.\end{cases}$$

The function $\pi^r(p)=(p-c)(D(p)-k)$ equals a firm's residual profit whenever its opponent sells k and $D(p)\ge k$. Let $\overline{p}=\arg\max_{p\in[c,b]}\pi^r(p)$ be the profit maximizing price on the residual demand curve and $\overline{\pi}=\pi^r(\overline{p})$ be the respective profit. Assumptions 1 and 2 assure that p^* and $\overline{p}$ are well defined. Furthermore, let $\underline{p}$ the price at which a firm is indifferent between selling its entire capacity and maximizing profits on the residual demand curve, i.e. $\underline{p}=c+\overline{\pi}/k$.

For the case of small capacities, i.e. $p^*\ge\overline{p}$, the game has a unique equilibrium in pure strategies in which the firms produce at their capacity limits and set the market-clearing price [14, Proposition 2]. The MSE for the case of large capacities, i.e. $D(c)\le k$, has been determined in [8]. Recently, we have determined a symmetric MSE on a subregion of intermediate capacities (i.e. $\overline{p}>\max\{p^*,c\}$).

Before recalling our recent proposition, we need to introduce several further notations. We shall denote by $\widehat{p}=\inf\{p\in[c,b]\mid \mu((p,b]\times[0,k])=0\}$ the highest possible price set by a firm when playing a strategy μ. Let $F(p)=\mu^*\left(\left[\underline{p},p\right]\times[0,k]\right)$ denote the cumulative distribution of equilibrium prices and let $G(q|p)$ be the conditional cumulative distribution function of equilibrium quantities given $p\in\left[\underline{p},\widehat{p}\right]$ in a symmetric equilibrium.

In the symmetric MSE at prices $p\in[c,\overline{p})\subset[c,b]$ firms set at most one quantity $s(p)\in[0,k]$ [14, Proposition 7]. At least in that price region the associated quantity was proven to be unique and equals k. Furthermore, in a subregion of intermediate capacities for any $p\in[\overline{p},\widehat{p}]\subset[c,b]$ there is a symmetric MSE in which the firms set at most one quantity $s(p)\in[0,k]$ at price p [15, Proposition 2]. Therefore, such a symmetric MSE can be given by the triple $(\widehat{p},s,F)$.

Proposition 1 *([15], Proposition 2). Let Assumptions 1-3 hold. If $\overline{p}>\max\{p^*,c\}$, then a symmetric MSE (μ^*,μ^*) of the production-in-advance game is given by the following equilibrium price distribution*

$$F(p)=\begin{cases} 0 & \text{if } 0\le p<\underline{p},\\ \frac{(p-c)k-\overline{\pi}}{p(2k-D(p))} & \text{if } \underline{p}\le p<\overline{p},\\ 1-\frac{c}{p} & \text{if } \overline{p}\le p<\widehat{p}, \text{ and}\\ 1 & \text{if } \widehat{p}\le p\le b\end{cases} \tag{3}$$

and by the 'supply' function $s(p)$ given by $s(p)=k$ for all $p\in\left[\underline{p},\overline{p}\right)$ and determined by

$$s(p)=D'(p)\left(\frac{p^2}{c}-p\right)+D(p)+\frac{\overline{\pi}}{c} \tag{4}$$

for all $p\in[\overline{p},\widehat{p}]$ if

$$\widehat{p}\le P(k), \tag{5}$$

where $\widehat{p}$ is the unique solution of $s(r)=D(r)/2$.

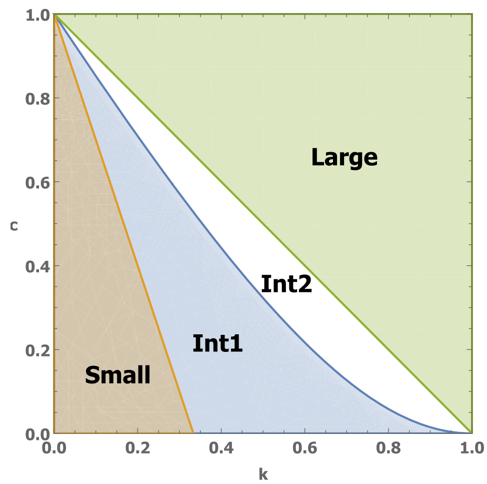

Fig. 1. Four different cases.

To illustrate the region of intermediate capacities covered by Proposition 1 we consider the demand curve $D(p) = 1 - p$. Hence, without loss of generality we can restrict ourselves to $c, k \in [0, 1]$. Note that for a given demand function only c and k are the parameters of the model, and therefore a pair of c and k determines the type of equilibrium. A higher c decreases the range of reasonable prices a firm can set, while a higher k increases the competition between firms and leads to lower prices at least in a stochastic sense. In particular, whether the price $p^* = P(2k)$ at which full capacity can be sold or the unit cost c is higher effects the equilibrium outcome.

Fig. 1 shows the four different cases we can have. The triangle labeled 'Large' depicts the case of large capacities ($c \geq 1 - k$), the triangle labeled 'Small' ($c \leq 1 - 3k$) depicts the case of small capacities, the shaded area labeled 'Int1' in the middle ($s(1-k) \leq k/2$) depicts the case of intermediate capacities covered by Proposition 1, and the white area labeled 'Int2' depicts the region of intermediate capacities for which this paper determines a MSE.

3. Supply functions

In this section we investigate the relationship between prices and quantities. We start with a simple observation.

Lemma 1. *Let Assumptions 1-3 and $\overline{p} > \max\{p^*, c\}$ hold. In a symmetric MSE for each price in case of price ties firms' total production exceeds market demand with probability zero.*

Proof. For any p we cannot have $\mu(\{p\}, (D(p)/2, \min\{k, D(p)\}]) > 0$, since otherwise undercutting (that is shifting a probability mass to a price slightly below p) would be beneficial. □

For a given symmetric mixed-equilibrium strategy μ and a $p \in [\overline{p}, \widehat{p}]$ let

$$q^*(p) = \sup\left\{q \in [0, k] \mid \mu\left(\left[\underline{p}, p\right] \times [0, q)\right) = 0\right\}$$

and note that

$$k = \inf\left\{q \in [0, k] \mid \mu\left(\left[\underline{p}, p\right] \times (q, k]\right) = 0\right\}$$

by Proposition 1. Then $q^*(p)$ and k are the 'lowest' and 'highest' quantities set by a firm at price p, respectively, when its opponent plays its symmetric equilibrium strategy μ. For simplicity we omit the subscript μ of $q^*(p)$.

We continue with a statement on the shape of the profit function in function of q.

Lemma 2. *Let Assumptions 1-3 hold. If $\overline{p} > \max\{p^*, c\}$, then in a symmetric MSE both firms' profit functions are constant in quantity on $(q^*(p), k]$ at any price $p \in [\overline{p}, \widehat{p}]$ given that its opponent plays its respective equilibrium strategy.*

Proof. Without loss of generality we consider π_1. Assume that $q^*(p) \leq q < q' \leq \min\{k, D(p)\}$ are two different quantities that can be set at p in a symmetric MSE (μ, μ). Then

$$\pi_1((p, q), \mu) = \pi_1\left((p, q'), \mu\right) = \overline{\pi}. \tag{6}$$

Note that

$$\begin{aligned}\pi_1((p, q), \mu) = {} & (1 - F(p))pq + (F(p) - \lim_{p^* \uparrow p} F(p^*)) \cdot \\ & p \int_{(D(p)-k)^+}^{D(p)/2} \min\left\{q, \frac{q}{q + \overline{q}} D(p)\right\} dG(\overline{q}|p) + \\ & p \int_{\underline{p}}^{p} \int_0^k \min\{q, D(p) - \overline{q}\} dG(\overline{q}|r) dF(r) - cq, \end{aligned} \tag{7}$$

where the first summand equals firm 1's revenue if it is the low-price firm, the second summand equals its revenue if both firms set price p and the third summand equals its revenue if it is the high-price firm. We have taken Lemma 1 in the limits of the first integral into account. Since the first integrand is concave (and also increasing) in q for any $\overline{q} \in [q^*(p), \min\{k, D(p)\}]$ its integral is also concave. The second summand equals zero if F does not have an atom at p and does not contribute to the revenue at p. Let $q(\lambda) = \lambda q + (1 - \lambda)q'$, where $\lambda \in [0, 1]$. By considering four cases it can be verified that for any given $\overline{q} \in [0, D(p)]$ and any $\lambda \in [0, 1]$

$$\begin{aligned}&\min\{q(\lambda), D(p) - \overline{q}\} \\ &\quad \geq \lambda \min\{q, D(p) - \overline{q}\} + (1 - \lambda) \min\{q', D(p) - \overline{q}\}.\end{aligned} \tag{8}$$

Therefore, for the third summand in (7) we have

$$\begin{aligned}&\int_{\underline{p}}^{p} \int_0^k \min\{q(\lambda), D(p) - \overline{q}\} dG(\overline{q}|r) dF(r) \geq \\ &\qquad \int_{\underline{p}}^{p} \int_0^k \lambda \min\{q, D(p) - \overline{q}\} dG(\overline{q}|r) dF(r) + \\ &\qquad \int_{\underline{p}}^{p} \int_0^k (1 - \lambda) \min\{q', D(p) - \overline{q}\} dG(\overline{q}|r) dF(r),\end{aligned}$$

while the first summand minus costs in (7) is linear, and therefore

$$\pi_1((p, q), \mu) = \pi_1((p, q(\lambda)), \mu) = \pi_1\left((p, q'), \mu\right) = \overline{\pi},$$

which completes the proof. □

To have that $s(p)$ is at most single-valued we would require strict concavity in addition to the result of Lemma 2. However, by striving for strict concavity we can exclude at least the following price-quantity pairs in a symmetric MSE.

Lemma 3. *Let Assumptions 1-3 and $\overline{p} > \max\{p^*, c\}$ hold. Then at any price $p \in [\overline{p}, \widehat{p}]$ in a symmetric MSE strategy firms never produce less than $D(p)/2$.*

Proof. Suppose that (6) holds true and let

$$\lambda^* = \frac{q' - (D(p) - \overline{q})}{q' - q}.$$

Inspecting inequality (8), we find that the inequality is strict if and only if $q < D(p) - \overline{q} < q'$.

The properties of the residual demand curve imply $(D(p) - k)^+ \leq q^*(p)$ for any $p \in [\overline{p}, \widehat{p}]$. The profit function (7) is larger than $\overline{\pi}$ in $[q^*(p), k]$ if for the second integrand in (7) we have $q^*(p) < D(p) - \overline{q} < k$ on a subset of quantities $\overline{q} \in [q^*(p), k]$ of positive measure within the limits of the integral since the remaining part of the profit function (7) is linear or convex in q. Hence, $\overline{q}$ has to satisfy $q^*(p) + \overline{q} < D(p)$ and $D(p) < k + \overline{q}$, where the former implies $q^*(p) < D(p)/2$ and the latter is fulfilled by $p > \overline{p}$. For any price $p \in (\overline{p}, \widehat{p}]$ we conclude that we can exclude the case of $q^*(p) < D(p)/2$, since otherwise profits would be higher for quantities within the interval $(q^*(p), k)$; a contradiction.

Finally, let us consider price $p = \overline{p}$. If there is no atom at $\overline{p}$, then the quantity at this price can be chosen arbitrarily since changing quantities for a set of prices of probability zero does still yield a symmetric MSE strategy. By Lemma 1 and the argument of the paragraph above the one and only case which has to be covered is when there is an atom at price $\overline{p}$ and firms only produce quantity $D(\overline{p}) - k$ at that price. Let $\alpha = \mu\left(\{(\overline{p}, D(\overline{p}) - k)\}\right)$. Intuitively, this leads to a more favorable residual demand than $D(p) - k$, and therefore firm A can benefit from a price increase. Formally, we get a lower bound on firm A's profit if firm B has at price $\overline{p}$ and quantity k another atom of mass $\beta = 1 - \alpha - (1 - \overline{p}/c)$. Any other mixed strategy of firm B distributing β differently on prices greater than or equal to $\overline{p}$ leads to even higher profits for firm A. Note that if firm A has an atom at $(\overline{p}, D(\overline{p}) - k)$ too but distributes the probability mass β over prices higher than $\overline{p}$, then firm B achieves still $\overline{\pi}$ profits. For simplicity we assume that firm A produces $D(p) - (D(\overline{p}) - k)$ and we show that firm A can gain from setting prices higher than $\overline{p}$. Then the 'modified' residual profit function equals

$$\begin{aligned}\pi^*(p) &= \alpha p\left(D(p) - (D(\overline{p}) - k)\right) + (1-\alpha)p(D(p) - k) - \\ &\quad c(D(p) - (D(\overline{p}) - k)) \\ &= \alpha(p - c)\left(D(p) - (D(\overline{p}) - k)\right) + \\ &\quad (1-\alpha)(p-c)(D(p) - k) - (1-\alpha)c(2k - D(\overline{p}))\end{aligned}$$

for any $p \geq \overline{p}$, and taking its derivative

$$\begin{aligned}\frac{d}{dp}\pi^*(p) &= \alpha\left(D(p) - (D(\overline{p}) - k)\right) + \alpha(p-c)D'(p) + \\ &\quad (1-\alpha)(D(p) - k + pD'(p)) - (1-\alpha)cD'(p) \\ &= (D(p) - k + (p-c)D'(p)) + \alpha(2k - D(\overline{p})),\end{aligned}$$

which is positive at $p = \overline{p}$ since the first summand equals zero (because it is the derivative of π^r with a unique maximum at $\overline{p}$) and the second summand is positive. Therefore, firm A would benefit from increasing its price, and thus we cannot have a symmetric MSE in which firm B has an atom at $(\overline{p}, D(\overline{p}) - k)$. □

From Lemmas 1 and 3 we arrive to the next corollary.

Corollary 1. *An atom is only possible at p if firms' supply equals $D(p)/2$ and at that p they share the market without superfluous production.*

We have not shown that the supply correspondence is deterministic. However, to the contrary, we will point out that the supply can be stochastic.

By Lemma 1 and Corollary 1 firms either set different prices or the tie-breaking does not matter. Therefore, considering profit function (7), or more specifically, focusing on the inner integral

$$\begin{aligned}\int_0^k \min\{q, D(p) - \overline{q}\} dG(\overline{q}|p) &= \int_0^k D(p) - \overline{q} dG(\overline{q}|p) \\ &= D(p) - E(\overline{q}|p),\end{aligned} \tag{9}$$

we can see that only the expected supply curve $E(\overline{q}|p)$ matters. The intuition behind this result is that if a firm is undercut by its opponent at price p, then only the quantity distribution of the opponent matters given that the latter is the low-price firm. Therefore, at which lower prices certain quantities are set is indifferent. This and the even stronger result (9) also implies that a deterministic supply curve exits, which equals the expected supply curve. Henceforth, with a slight abuse of notation we write $s(p)$ for both the expected supply curve and the deterministic supply curve. From equation (9) it follows that there are infinitely many nondeterministic supply correspondences resulting in a symmetric MSE. Furthermore, since as already mentioned only the quantity distribution of the low-price firm matters there is no coordination problem between the firms, or put it otherwise, we can pair equilibrium strategies arbitrarily to obtain a MSE. Then all underlying either symmetric or asymmetric equilibria are payoff equivalent.

We summarize our findings in the next proposition.

Proposition 2. *Under Assumptions 1-3 and $\overline{p} > \max\{p^*, c\}$, there exists a class of infinitely many MSE in which both firms' expected supply functions are the same and the expected supply functions specify a symmetric MSE with a deterministic supply function.*

4. Mixed-strategy equilibrium

In this section we calculate a symmetric MSE for the missing range of intermediate capacities on which $s(p)$ is only piecewise strictly decreasing. Nevertheless, the price distribution F specified in Proposition 1 remains still the equilibrium price distribution in the upper range of intermediate capacities. Furthermore, the expression on the right-hand side of (4) still specifies $s(p)$ on $[\overline{p}, P(k)]$ since in this case in the proof of Proposition 1 $D(p) - s(r)$ is non-negative for any $p \in [\overline{p}, P(k)]$ and any $r \in [\overline{p}, p]$. Since s will be defined piecewise on a finite set of disjoint and consecutive intervals, we shall denote by s_1 the expression on the right-hand side of (4). Since we determine s iteratively and at the same time the respective intervals with the boundary points (i.e. prices) too for notational convenience we let $p_0 = \overline{p}$, $p_1 = P(k)$ and $s_0(p) = k$ for any $p \in \left[\underline{p}, \overline{p}\right]$. From here on the subscripts of p stand for indexing the steps of the iterative process and not for the labeling of firms, which we highlight by using i and j as indexes instead of A and B.

When extending s to prices above $\overline{p}$ one needs to integrate $D(p) - s_1(r)$ only above prices r on which the integrand is nonnegative. To determine the lowest price from which the integration of $D(p) - s_1(r)$ should start for a given p we define $t_1(p) = s_1^{-1}(D(p)) = r$. The strategy for constructing the MSE is to determine the next piece of s denoted by s_2. Then we arrive either to a solution delivering an r^* satisfying $s_2(r^*) = D(r^*)/2$ and $r^* \leq p_2 = P(s_2(p_1))$ or we define $t_2(p) = s_2^{-1}(D(p)) = r$ and continue with determining the next piece of s denoted by s_3. We repeat the whole process until we obtain an r^* satisfying $s_n(r^*) = D(r^*)/2$ and $r^* \leq p_n = P(s_n(p_{n-1}))$, where n stands for the required number of steps. We shall denote by r_i the value of r^* obtained at the ith step, that is $s_i(r_i) = D(r_i)/2$.

The next proposition contains the results of the described procedure and the proof of their correctness.

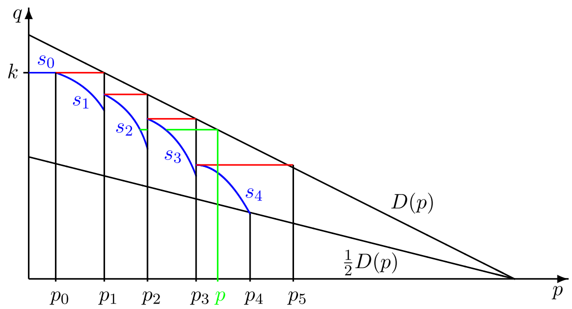

Fig. 2. Supply function. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Proposition 3. *Let Assumptions 1-3 hold. If $\overline{p} > \max\{p^*, c\}$ and $P(k) = p_1 < r_1$, then there exists an $n \in \{1, \ldots\}$ such that a symmetric MSE (μ^*, μ^*) of the production-in-advance game is given by the equilibrium price distribution (3) and by the 'supply' function $s(p)$ given by $s(p) = s_0(p) = k$ for all $p \in \left[\underline{p}, \overline{p}\right]$, given by (4) for all $p \in \left(\overline{p}, p_1\right]$, and given by*

$$s_{i+1}(p) = D'(p)\left(\frac{p^2}{t_i(p)} - p + \sum_{j=l(p)}^{i-1}\left(\frac{p^2}{t_j(p)} - \frac{p^2}{p_j}\right)\right) + D(p) + \frac{\overline{\pi}}{c} \tag{10}$$

for all $p \in (p_i, \min\{p_{i+1}, r^*\}]$ and all $i \in \{1, \ldots, n\}$ if

$$p_1 < r_1, \ldots, p_n < r_n, r_{n+1} \le p_{n+1}, \tag{11}$$

where r_i is the unique solution of $s_i(r_i) = D(r_i)/2$, $t_i(p) = s_i^{-1}(D(p))$, $p_i = P(s_i(p_{i-1}))$ for all $i \in \{1, \ldots, n+1\}$, and $l(p) \in \{1, \ldots, i\}$ is increasing in p. Then $\widehat{p} = r_{n+1}$. Furthermore, the numerical sequence $(s_i(p_{i-1}))_{i=1}^{n+1}$ is strictly decreasing, $s_i(p_i) \le s_{i+1}(p_i)$ and the functions $s_i(p)$ are strictly decreasing in p on $[p_{i-1}, \min\{p_i, r^*\}]$ for all $i \in \{1, 2, \ldots, n+1\}$.

Note that in (10), as usually, if $l(p) > i-1$, the sum equals zero.

Before giving a proof of Proposition 3 we illustrate the supply functions given by (10) and shed light on why the equilibrium supply function is discontinuous and has kinks. We depict a possible supply function in Fig. 2. We would like to emphasize that there is definitely a discontinuity at p_1, otherwise, we have either a discontinuity or kink at p_i, where the former case occurs if $l(p_i)$ is different for s_i and s_{i+1} in equation (10) and otherwise the latter case occurs. A concrete numerical example for the case of linear demand will be provided after the proof of Proposition 3. First, observe that if a firm sets price $p \in (p_0, p_1)$ and its opponent sets a price $r \in \left[\underline{p}, p\right)$ (i.e. it is the high-price firm), it can always sell a positive amount, and therefore $D(p) - s(r)$ is positive. However, if $p \in (p_1, p_2)$ and $r \in \left[\underline{p}, p_0\right)$, its residual demand equals zero, while $D(p) - s(r)$ becomes negative. Second, let us move a bit further to the right and pick a price $p \in (p_3, p_4)$ as indicated with a green p in Fig. 2. If the firm's opponent sets a price such that $s_3(r) < D(p)$ and $p \le p_3$, then the firm faces a positive residual demand, but this is also true if the firm's opponent sets a price such that $s_2(r) < D(p)$ and $p \le p_2$. Therefore, in this case we have $i(p) = 2$. A kink in s arises at a price satisfying $s_3(p_3) = D(p)$.

Now we turn to the proof of Proposition 3.

Proof. Proposition 1 can be considered as the initialization step of our recursive procedure, i.e. the statement of our proposition holds for $n = 1$. Then we assume that we have already obtained the sequence of prices $p_1, \ldots, p_i$, the sequence of supply functions $s_1, \ldots, s_i$, and the sequence of functions $t_1, \ldots, t_i$ recursively.

Since s and F are known for all $p \in \left[\underline{p}, p_i\right)$ in what follows we consider only prices such that $p \ge p_i$. We would like to emphasize that in line with the statement of Proposition 3 we are not showing the uniqueness of the symmetric MSE. Nevertheless, we have to deal with the successive construction of the supply function of the symmetric MSE. However, we do not derive the cumulative distribution function F given in Proposition 3, we just verify its correctness. When determining the next piece of s, we shall denote by $r_{i+1}^* \in [p_i, b]$ the price at which $s_{i+1}(r_{i+1}^*) = D(r_{i+1}^*)/2$ and assume that such a price exists uniquely. We will verify in the proof that the s_{i+1} given by (10) is continuous and strictly decreasing on $p \in [p_i, p_{i+1}]$, which implies that r_{i+1}^* is uniquely determined by the properties of D and s_{i+1}.

Step 1: *We simplify the functional form of the profit function. Intuitively, the simplification results from the observations that at any p each firm does not produce more than $D(p)$, less than its worst case residual demand $D(p) - k$ and the functional form of F is utilized.*

Given that we are looking for a symmetric MSE we denote the rival firm 2's strategy simply by μ. Then firm 1's profit equals

$$\pi_1((p,q),\mu) = pq(1 - F(p)) + p\int_{p_i}^{p} \min\left\{(D(p) - s_{i+1}(r))^+, q\right\} dF(r) + \sum_{j=l(p)}^{i} p \int_{t_j(p)}^{p_j} \min\left\{D(p) - s_j(r), q\right\} dF(r) - cq \tag{12}$$

for any $p \in (p_i, p_{i+1}]$ and any $q \in [0, D(p)]$, where we have already taken into account that $D(p) < s_{i+1}(p) = q$ does not make sense since then the firms produce a superfluous amount for sure and $l(p) \ge 1$ is the smallest index for which $D(p) > s_{l(p)}(p_{l(p)})$. Note that $l(p) \ge 1$ since for any $p \in (p_i, p_{i+1}]$ we have $D(p) < D(p_0) = k$. (12) simplifies to

$$\pi_1((p,q),\mu) = pq(1 - F(p)) + p\int_{p_i}^{p} \min\left\{D(p) - s_{i+1}(r), q\right\} dF(r) + \sum_{j=l(p)}^{i} p \int_{t_j(p)}^{p_j} \min\left\{D(p) - s_j(r), q\right\} dF(r) - cq, \tag{13}$$

where we could drop the non-negativity operation in the first integral of (12) because we will speak only about the next piece of a solution if $p \le p_{i+1}$ and $p \le r_{i+1}^*$. In addition, if $i = n$, then (11) will hold.

Since the equilibrium price distribution is given by $F(p) = 1 - c/p$ on $\left(\overline{p}, \widehat{p}\right)$ (13) takes the following form

$$\pi_1((p,q),\mu) = pq\frac{c}{p} + p\int_{p_i}^{p} \min\left\{D(p) - s_{i+1}(r), q\right\} dF(r) + \sum_{j=l(p)}^{i} p \int_{t_j(p)}^{p_j} \min\left\{D(p) - s_j(r), q\right\} dF(r) - cq$$

$$= p \int_{p_i}^{p} \min\{D(p) - s_{i+1}(r), q\}\, dF(r)$$

$$+ \sum_{j=l(p)}^{i} p \int_{t_j(p)}^{p_j} \min\left\{D(p) - s_j(r), q\right\} dF(r). \quad (14)$$

Step 2: *We derive the supply function s_{i+1} by solving the integral equation which equates the equilibrium profit and the integral of the residual demand with respect to prices. In this way we obtain the integral of the supply function, and therefore finally we have to take its derivative.*

We can see that (14) is strictly increasing in q on $[0, \max_{j=l(p),l(p)+1,\ldots,i+1} D(p) - s_j(p_j)]$ and constant on $\left[\max_{j=l(p),l(p)+1,\ldots,i+1} D(p) - s_j(p_j), D(p)\right]$ since $F(p) = 1 - c/p$, and therefore it follows that we can derive s_{i+1} on the respective interval by solving $\pi_1((p,q),\mu) =$

$$\overline{\pi} = p \int_{p_i}^{p} (D(p) - s_{i+1}(r)) \frac{c}{r^2} dr$$

$$+ \sum_{j=l(p)}^{i} p \int_{t_j(p)}^{p_j} \left(D(p) - s_j(r)\right) \frac{c}{r^2} dr$$

$$= pD(p) \left(\frac{c}{t_i(p)} - \frac{c}{p} + \sum_{j=l(p)}^{i-1} \left(\frac{c}{t_j(p)} - \frac{c}{p_j} \right) \right)$$

$$- p \sum_{j=l(p)}^{i} \left(S_j(p_j) - S_j(t_j(p))\right) - p \int_{p_i}^{p} s_{i+1}(r) \frac{c}{r^2} dr, \quad (15)$$

where

$$S_j(p) = \int_{p_{j-1}}^{p} s_j(r) \frac{c}{r^2} dr \quad (16)$$

for any $p \in \left[p_{j-1}, p_j\right)$. By simple rearrangements we get (17)

$$S_{i+1}(p) = D(p) \left(\frac{c}{t_i(p)} - \frac{c}{p} + \sum_{j=l(p)}^{i-1} \left(\frac{c}{t_j(p)} - \frac{c}{p_j} \right) \right)$$

$$- \sum_{j=l(p)}^{i} \left(S_j(p_j) - S_j(t_j(p))\right) - \frac{\overline{\pi}}{p} \quad (17)$$

from which by differentiation we obtain

$$S'_{i+1}(p) = D'(p) \left(\frac{c}{t_i(p)} - \frac{c}{p} + \sum_{j=l(p)}^{i-1} \left(\frac{c}{t_j(p)} - \frac{c}{p_j} \right) \right)$$

$$+ D(p) \left(-\frac{ct'_i(p)}{t_i^2(p)} + \frac{c}{p^2} + \sum_{j=l(p)}^{i-1} -\frac{ct'_j(p)}{t_j^2(p)} \right)$$

$$+ \sum_{j=l(p)}^{i} \left(s_j(t_j(p)) \frac{c}{t_j^2(p)} t'_j(p) \right) + \frac{\overline{\pi}}{p^2}$$

$$= D'(p) \left(\frac{c}{t_i(p)} - \frac{c}{p} + \sum_{j=l(p)}^{i-1} \left(\frac{c}{t_j(p)} - \frac{c}{p_j} \right) \right)$$

$$+ D(p) \left(-\frac{ct'_i(p)}{t_i^2(p)} + \frac{c}{p^2} + \sum_{j=l(p)}^{i-1} -\frac{ct'_j(p)}{t_j^2(p)} \right)$$

$$+ \sum_{j=l(p)}^{i} D(p) \frac{c}{t_j^2(p)} t'_j(p) + \frac{\overline{\pi}}{p^2}$$

$$= D'(p) \left(\frac{c}{t_i(p)} - \frac{c}{p} + \sum_{j=l(p)}^{i-1} \left(\frac{c}{t_j(p)} - \frac{c}{p_j} \right) \right)$$

$$+ D(p) \frac{c}{p^2} + \frac{\overline{\pi}}{p^2}, \quad (18)$$

where the fact that $l(p)$ is an increasing step function of p implies that S_{i+1} in not differentiable at at most i points. Since F does not have an atom at these points the value of s can be set arbitrarily there. Rearranging (18), we get

$$s_{i+1}(p) = D'(p) \left(\frac{p^2}{t_i(p)} - p + \sum_{j=l(p)}^{i-1} \left(\frac{p^2}{t_j(p)} - \frac{p^2}{p_j} \right) \right) + D(p) + \frac{\overline{\pi}}{c} \quad (19)$$

It can be verified that $s'_{i+1}(p) < D'(p)$ for prices higher than p_i.

Step 3: *We derive s recursively and we describe the terminal condition. In particular, if at price r^*_{i+1} the supply is less than or equal to half of the demand, the iterative process terminates.*

The process of constructing the next piece of s has to be repeated if $p_{i+1} < r^*_{i+1}$. After a finite number of steps, we have to arrive at an n such that $r^*_{n+1} \le p_{n+1}$ since equilibrium profits are positive. Clearly, both S_{n+1} and s_{n+1} can be extended through equations (17) and (19) for prices higher than r^*_{n+1}, respectively, where for $p \ge r^*_{n+1}$ equation (15) takes the following form

$$\overline{\pi} = p \int_{p_n}^{p} s_{n+1}(r) \frac{c}{r^2} dr$$

$$+ \sum_{j=l(p)}^{n} p \int_{t_j(p)}^{p_j} \left(D(p) - s_j(r)\right) \frac{c}{r^2} dr$$

$$= pD(p) \left(1 - \frac{c}{r^*} + \sum_{j=l(p)}^{n-1} \left(\frac{c}{t_j(p)} - \frac{c}{p_j} \right) \right)$$

$$- p \sum_{j=l(p)}^{n} \left(S_j(p_j) - S_j(t_j(p))\right) - p \int_{p_i}^{p} s_{n+1}(r) \frac{c}{r^2} dr, \quad (20)$$

since $s_{n+1}(p) < D(p) - s_{n+1}(p)$ for any $p > r^*_{n+1}$.

For any $p \ge r^*_{n+1}$ let

$$Q(p) = \int_{r^*_{n+1}}^{p} s_{n+1}(r) \frac{c}{r^2} dr. \quad (21)$$

Then we have

$$Q(r^*_{n+1}) = 0 \text{ and } Q'(p) = s_{n+1}(p) \frac{c}{p^2} \quad (22)$$

for any $p \in \left[r^*_{n+1}, r'\right)$, where r' is uniquely defined by the implicit equation $s(r') = D(r') - k$. Clearly, setting prices above r' does no make sense, since playing these pure strategies against mixed strategy $\mu_{s,F}$ will result in less profits than pure strategy $(\overline{p}, D(\overline{p})-k)$. From (20) we get

$$Q(p) = D(p)\left(1 - \frac{c}{r^*_{n+1}} + \sum_{j=l(p)}^{n-1}\left(\frac{c}{t_j(p)} - \frac{c}{p_j}\right)\right) - \sum_{j=l(p)}^{n}\left(S_j(p_j) - S_j(t_j(p))\right) - \frac{\overline{\pi}}{p} \quad (23)$$

for any $p \in \left[r^*_{n+1}, r',\right)$ from which by differentiation we obtain Q' and finally by simple rearrangements $s_{n+1}(p)$. With a slight abuse of notation we will still denote the obtained function by $s_{n+1}(p)$ on $p \in \left(r^*_{n+1}, r'\right)$ though, as it will turn out, the firms will not produce at prices above r^*_{n+1}. These extensions will be helpful for us in the price interval $[r^*_{n+1}, r']$.

Step 4: *We show that the highest price set by the firms (specified by the terminal condition) is set with positive probability.*

Now we will verify that having an atom at price r^*_{n+1} of mass $c/r^*_{n+1} = 1 - F(r^*_{n+1})$ completes a symmetric MSE. Assume that firm 2 plays the same mixed strategy. Then we already know that for any $p \in \left[\underline{p}, r^*_{n+1}\right)$ producing an amount of $q = s(p)$ results in $\overline{\pi}$ profit. Furthermore, for any $p \in \left[\underline{p}, r^*_{n+1}\right)$ and any quantity $[D(p)-s(p), k]$ profits equal $\overline{\pi}$, while they are strictly less for quantities less than $D(p)-s(p)$ by (14).

We claim that in the derived symmetric MSE firms produce at r^*_{n+1} an amount of $s(r^*_{n+1}) = D(r^*_{n+1})/2$. Suppose that they would produce more than $D(r^*_{n+1})/2$. Then there will be superfluous production at r^*_{n+1}, and therefore by the continuity of profits for prices below r^*_{n+1} profits at r^*_{n+1} would be less than at prices $r^*_{n+1}-\varepsilon$ if ε is sufficiently small. Suppose that they would produce an amount of q^* less than $D(r^*_{n+1})/2$. Then $\pi_1\left((p,q), \mu_{s,F}\right)$ is continuous at (r^*_{n+1}, q^*), and therefore $\pi_1\left((r^*_{n+1}, q^*), \mu_{s,F}\right) < \overline{\pi}$; a contradiction. Thus, we must have indeed $s(r^*_{n+1}) = D(r^*_{n+1})/2$. By the left continuity at r^*_{n+1} it follows that $\pi_1\left((r^*_{n+1}, D(r^*_{n+1})/2), \mu_{s,F}\right) = \overline{\pi}$.

To verify that $(\widehat{p}, s, F)$ specified in the previous paragraphs specifies a strategy of a symmetric MSE it remains to be shown that prices above r^*_{n+1} combined with any quantity $q \in [0,k]$ result in less profits than $\overline{\pi}$.

Step 4a: *To any price above r^*_{n+1} (the price where the atom lies), we determine the optimal quantity.*

The profit function of firm 1 in response to firm 2 playing the mixed strategy associated with $(\widehat{p}, s, F)$ for prices $p \geq r^*_{n+1}$ equals

$$\pi_1\left((p,q), \mu_{s,F}\right) = p\min\left\{D(p) - \frac{D(r^*_{n+1})}{2}, q\right\}\frac{c}{r^*_{n+1}} + p\int_{p_n}^{r^*_{n+1}}\left(D(p) - s_{n+1}(r)\right)\frac{c}{r^2}dr + \sum_{j=l(p)}^{n} p\int_{t_j(p)}^{p_j}\left(D(p) - s_j(r)\right)\frac{c}{r^2}dr - cq, \quad (24)$$

from which we get

$$\frac{\partial \pi_1}{\partial q}((p,q),\mu) = \begin{cases} -c & \text{if } \quad D(p) - \frac{D(r^*_{n+1})}{2} < q, \\ p\frac{c}{r^*_{n+1}} - c & \text{if } \quad D(p) - \frac{D(r^*_{n+1})}{2} > q \geq D(p) - s(p) \end{cases} \quad (25)$$

for any $p > \widehat{p} = r^*_{n+1}$. Since $pc/r^*_{n+1} - c > 0$ we get that quantity $q = D(p) - D(r^*_{n+1})/2$ results in the highest profit in (24) for any price $p > \widehat{p} = r^*_{n+1}$.

Step 4b: *Utilizing the result of Step 4a, we show that prices higher than r^*_{n+1} lead to a decrease in profits.*

We define the profit function of firm 1 at the best quantities for prices $p \geq r^*_{n+1}$ by

$$\pi^*(p) = p\left(D(p) - \frac{D(r^*_{n+1})}{2}\right)\frac{c}{r^*_{n+1}} + p\int_{p_n}^{r^*_{n+1}}\left(D(p) - s_{n+1}(r)\right)\frac{c}{r^2}dr + \sum_{j=l(p)}^{n} p\int_{t_j(p)}^{p_j}\left(D(p) - s_j(r)\right)\frac{c}{r^2}dr - c\left(D(p) - \frac{D(r^*_{n+1})}{2}\right) \quad (26)$$

It can be verified that $\pi^*(p)$ is strictly concave, and it would be straightforward to check that the derivative $\pi^*(p)$ is non-positive at r^*_{n+1}, which unfortunately does not result in a manageable inequality. Therefore, we consider the equality in (20) defining s and let us denote by

$$\pi^s(p) = p\int_{r^*_{n+1}}^{p} s(r)\frac{c}{r^2}dr + p\int_{p_n}^{r^*_{n+1}}\left(D(p) - s_{n+1}(r)\right)\frac{c}{r^2}dr + \sum_{j=l(p)}^{n} p\int_{t_j(p)}^{p_j}\left(D(p) - s_j(r)\right)\frac{c}{r^2}dr = \overline{\pi} \quad (27)$$

for prices $p \in \left[r^*_{n+1}, r'\right]$. Clearly, $d\pi^s(p)/dp = 0$ for any $p \in \left[r^*_{n+1}, r'\right]$ by the definition of s, which we will utilize by considering $\Delta(p) = \pi^*(p) - \pi^s(p) =$

$$= p\left(D(p) - \frac{D(r^*_{n+1})}{2}\right)\frac{c}{r^*_{n+1}} - c\left(D(p) - \frac{D(r^*_{n+1})}{2}\right) - p\int_{r^*_{n+1}}^{p} s(r)\frac{c}{r^2}dr$$

$$= \left(D(p) - \frac{D(r^*_{n+1})}{2}\right)\left(p\frac{c}{r^*_{n+1}} - c\right) - p\int_{r^*_{n+1}}^{p} s(r)\frac{c}{r^2}dr. \quad (28)$$

Then

$$\Delta'(p) = D'(p)\left(p\frac{c}{r^*_{n+1}} - c\right) + \left(D(p) - \frac{D(r^*_{n+1})}{2}\right)\frac{c}{r^*_{n+1}} - \int_{r^*_{n+1}}^{p} s(r)\frac{c}{r^2}dr - ps(p)\frac{c}{p^2}. \quad (29)$$

By substituting r^*_{n+1} for p in (29) and taking $s(r^*_{n+1}) = D(r^*_{n+1})/2$ into consideration we get $\Delta'(r^*_{n+1}) = 0$, which implies $d\pi^*(p)/dp = 0$, which completes the proof. □

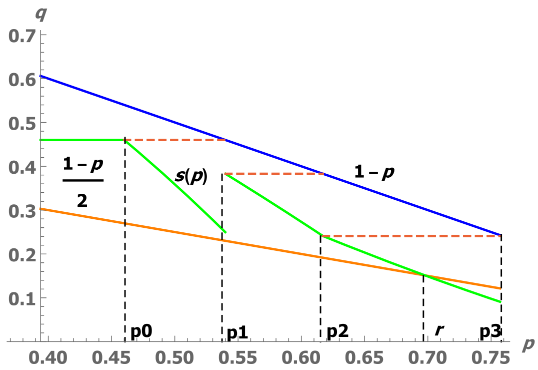

Fig. 3. The equilibrium supply function of Example 1.

Now we consider a numerical example and provide its numerical solution based on Proposition 3.

Example 1. Let $D(p) = 1 - p$, $c = 0.38$ and $k = 0.46$.

It can be verified that the cost and capacity pair given in Example 1 is very close to the Int1 region, but within the Int2 region shown in Fig. 1. Since the derivation of the cumulative distribution function is straightforward we only present the supply function $s(p)$ in Fig. 3, which is drawn in green and has four pieces s_0, s_1, s_2 and s_3. We can see that the termination condition is satisfied at $r^* = r_3$ since there s_3 crosses $D(p)/2$ left to p_3. r^* determines also the position of the atom of the cumulative distribution function F of the symmetric MSE prices. We can see that at p_2 the supply curve $s(p)$ just has a kink and no discontinuity.

It is worthwhile to note that in the case of linear demand there is no symmetric MSE just requiring two steps. Though s_1, s_2 and s_3 look linear in Fig. 3 they are highly nonlinear. To determine s_3, we need to find the appropriate root of a polynomial of degree 4. Thus, considering an example requiring an additional step seems to be intractable since the degree of the polynomial to be solved could be only approximated numerically and we would even need to determine t_3, which is an inverse function of s_3. For the same reasons we did not draw an extended version of Fig. 1 containing the area on which in case of linear demand we would have an equilibrium in three steps.

5. Concluding remarks

We found that though the cumulative distribution function of prices remains simple the construction of the supply function required a recursive procedure, which resulted in only a piecewise continuous supply function with kinks. From an economic point of view the discontinuities in s imply that certain unsold amounts are more likely than others. This may have implications on optimal store sizes or disposal units, but requires a richer model and further analysis.

It is straightforward to see that the closer we are coming to the large capacity region the number of discontinuities of $s(p)$ is increasing, the lowest price in the support of the equilibrium price distribution tends to c, the equilibrium profits tend to zero, r^* to b and the price distribution tends to the Montez and Schutz [8] equilibrium price distribution in distribution. Furthermore, it can be verified that $s(p)$ approximates $D(p)$, and therefore the solution approaches to the solution obtained in [8] for the case of large capacities.

Acknowledgements

I am grateful to an anonymous reviewer, László Á. Kóczy, Andreas Orland and Dóra G. Petróczy for helpful comments and suggestions. This research was supported by the Higher Education Institutional Excellence Program 2020 of the Ministry of Innovation and Technology of Hungary in the framework of the 'Financial and Public Services' research project (TKP2020-IKA-02) at Corvinus University of Budapest.

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