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RESEARCH ARTICLE

On generalisations of the Aharoni–Pouzet base exchange theorem

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Abstract

The Greene–Magnanti theorem states that if M is a finite matroid, B_0 and B_1 are bases and $B_0 = \bigcup_{i=1}^n X_i$ is a partition, then there is a partition $B_1 = \bigcup_{i=1}^n Y_i$ such that $(B_0 \setminus X_i) \cup Y_i$ is a base for every i. The special case where each X_i is a singleton can be rephrased as the existence of a perfect matching in the base transition graph. Pouzet conjectured that this remains true in infinite-dimensional vector spaces. Later, he and Aharoni answered this conjecture affirmatively not just for vector spaces but also for infinite matroids. We prove two generalisations of their result. On the one hand, we show that 'being a singleton' can be relaxed to 'being finite' and this is sharp in the sense that the exclusion of infinite sets is really necessary. In addition, we prove that if B_0 and B_1 are bases, then there is a bijection F between their finite subsets such that $(B_0 \setminus I) \cup F(I)$ is a base for every I. In contrast to the approach of Aharoni and Pouzet, our proofs are completely elementary, they do not rely on infinite matching theory.

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1 | INTRODUCTION

In the usual axiomatisation of finite matroids in the terms of bases, one of the axioms demands that if B_0 and B_1 are bases, then for every $x \in B_0$, there is a $y \in B_1$ such that $B_0 - x + y$ is a base. An important research direction in matroid theory is looking for stronger base exchange properties. Let us mention a few fundamental results in this subfield. First of all, the y above can be chosen in such a way that the exchange is 'symmetric' in the sense that $B_1 - y + x$ is also a base. Greene's theorem [9] is a strengthening of this symmetric base exchange property, stating that symmetric exchange of subsets of bases is also possible. Namely, for every $X \subseteq B_0$ there is a $Y \subseteq B_1$ such that $(B_0 \setminus X) \cup Y$ and $(B_1 \setminus Y) \cup X$ are both bases. The Greene–Magnanti theorem [10] states that the following partition base exchange property also holds: If $B_0 = \bigcup_{i=1}^{n} X_i$ is a partition, then there is a partition $B_1 = \bigcup_{i=1}^{n} Y_i$ such that $(B_0 \setminus X_i) \cup Y_i$ is a base for every i. Note that Greene's theorem is equivalent with the special case $n = 2$ of this. A more recent result by Kotlar, Roda and Ziv [14, Theorem 1.1.] ensures that in the setting of the Greene–Magnanti theorem, one can choose the sets Y_i in such a way that not just $(B_0 \setminus X_i) \cup Y_i$ but also $(B_0 \setminus \bigcup_{i\leqslant j\leqslant n} X_j) \cup \bigcup_{i\leqslant j\leqslant n} Y_j$ is a base for every i.[†]

Pouzet initiated the investigation of base exchange properties of infinite-dimensional vector spaces. In particular, he was interested in the question whether the special case of the Greene–Magnanti theorem where all the sets X_i are singletons remains true in this more general setting. This has been settled affirmatively by Aharoni and Pouzet [1, Theorem 2.1] in an even more general context provided by Definition 1.1.

Vector spaces are the motivating examples of matroids but several important vector spaces are infinite (e.g. any non-trivial vector space over the reals) or even have infinite dimension. This led to the following matroid concept where the ground set and the bases are allowed to be infinite.

Definition 1.1. A finitary matroid is a pair $M = (E, \mathcal{I})$ with $\mathcal{I} \subseteq \mathcal{P}(E)$ such that

(I) $\emptyset \in \mathcal{I}$;
(II) $\mathcal{I}$ is closed under taking subsets;
(III) if $I, J \in \mathcal{I}$ with $|I| < |J|$, then there exists an $e \in J \setminus I$ such that $I + e \in \mathcal{I}$;
(IV) if all finite subsets of an infinite set X are in $\mathcal{I}$, then $X \in \mathcal{I}$.

Remark 1.2.

- If E is finite, then (I)–(III) is the usual axiomatisation of finite matroids in terms of independent sets, while (IV) is redundant.
- It is enough to demand axiom (III) for finite I and J.
- Some authors call the concept defined in Definition 1.1 simply 'matroid' (see, e.g. [2, 13, 16]), other authors refer to it as 'independence structure' [15] or 'independence space' [19] but the term 'finitary matroid' became dominant in the literature.
- The word 'finitary' reflects the fact that by axiom (IV) every circuit (i.e. minimal dependent set) is finite. In other words, the 'span' operator corresponding to the matroid is a finitary closure operator. This is in contrast to the more general concept of matroids (see the definition in Section 2) formulated by Higgs [12] and formulated again by Bruhn et al. [5] where 'infinitary matroids' (i.e. matroids with infinite circuits) also exist.
- Duals of finitary matroids are usually[‡] infinitary which was originally the main motivation of Rado to express the need [17] for a more general infinite matroid concept than finitary matroids.

[†] They formulated originally the dual of this and exchanged initial segments instead of terminal segments.

[‡] The exceptions are exactly the direct sums of finite matroids.

For more information about infinite matroids, we recommend the chapter 'Infinite Matroids' by Oxley in [18] and the habilitation thesis [3] of Bowler with the same title.

On the one hand, we give an example that Greene's theorem may fail even for finite-cycle matroids of infinite graphs.

Theorem 1.3. *There is a countably infinite graph $G = (V, E)$ with edge-disjoint spanning trees T_0 and T_1 such that there is a partition $E(T_0) = X_0 \cup X_1$ for which there are no edge-disjoint spanning trees S_0 and S_1 with $E(T_0) \cap E(S_i) = X_i$ for $i \in \{0, 1\}$.*

(Theorem 1.3 provides the promised counterexample because the symmetric exchange of the set X_0 corresponding to the bases $B_i := E(T_i)$ for $i \in \{0, 1\}$ is impossible.) On the other hand, we prove two generalisations of the Aharoni–Pouzet base exchange theorem.

Theorem 1.4. *Suppose that $M = (E, \mathcal{I})$ is a finitary matroid, B_0 and B_1 are bases of M and $B_0 = \bigcup_{i<\kappa} X_i$ is a partition where each X_i is finite. Then there is a partition $B_1 = \bigcup_{i<\kappa} Y_i$ such that $(B_0 \setminus X_i) \cup Y_i$ is a base for each $i < \kappa$.*

We will actually prove Theorem 1.4 in a slightly more general form (see Theorem 4.5).

Theorem 1.5. *Suppose that $M = (E, \mathcal{I})$ is a finitary matroid and B_0 and B_1 are bases of M. Then there is a bijection $F : [B_0]^{<\aleph_0} \to [B_1]^{<\aleph_0}$ such that $(B_0 \setminus I) \cup F(I)$ is a base for every $I \in [B_0]^{<\aleph_0}$.*

In the following section, we introduce some notation, and then in Section 3, we present our counterexample in Theorem 1.3. The positive results (Theorems 1.4 and 1.5) are proved in Section 4. Finally, the last section (Section 5) is devoted to some open problems.

2 | BASIC DEFINITIONS AND NOTATION

We use standard set-theoretic notation, in particular: the variable κ stands for cardinal numbers, α and β are ordinals, the set of natural numbers is denoted by ω, we write $[X]^{<\kappa}$ for the set of subsets of X of size less than κ and functions are represented as sets of ordered pairs.

A *matroid* is an ordered pair $M = (E, \mathcal{I})$ with $\mathcal{I} \subseteq \mathcal{P}(E)$ such that

(I) $\emptyset \in \mathcal{I}$;
(II) $\mathcal{I}$ is closed under taking subsets;
(III') for every $I, J \in \mathcal{I}$ where J is $\subseteq$-maximal in $\mathcal{I}$ and I is not, there exists an $e \in J \setminus I$ such that $I + e \in \mathcal{I}$;
(IV') for every $X \subseteq E$, any $I \in \mathcal{I} \cap \mathcal{P}(X)$ can be extended to a $\subseteq$-maximal element of $\mathcal{I} \cap \mathcal{P}(X)$.

We use the term 'edges' for the elements of the ground set of the matroid. While this might be a bit confusing, it is standard in the literature and consistent with a tradition of terminology, dating back to Edmonds' early papers. The sets in $\mathcal{I}$ are called *independent*, whereas the sets in $\mathcal{P}(E) \setminus \mathcal{I}$ are *dependent*. The maximal independent sets are called *bases*. The *rank* $\boldsymbol{r(M)}$ of a matroid M is the size of its bases.[†] The minimal dependent sets are the *circuits*. Every dependent set contains

[†] The rank is just consistently well defined, and for the details, see Subsection 4.1.

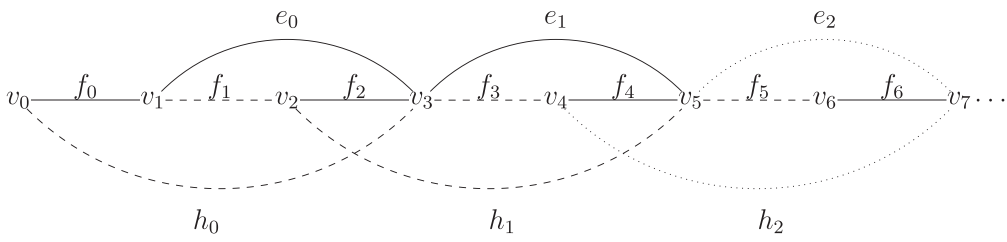

FIGURE 1 The graph G. Right after the step $n = 1$ of the induction, we already know that normal edges must belong to S_0 and dashed edges are in S_1. The affiliation of the dotted edges is unknown at this point.

a circuit (which is a non-trivial fact for infinite matroids). Clearly, a matroid is finitary (see Definition 1.1) if and only if all its circuits are finite. For an $X \subseteq E$, the pair $\boldsymbol{M} \upharpoonright \boldsymbol{X} := (X, \mathcal{I} \cap \mathcal{P}(X))$ is a matroid and it is called the *restriction* of M to X. We write $\boldsymbol{M} - \boldsymbol{X}$ for $M \upharpoonright (E \setminus X)$ and call it the minor obtained by the *deletion* of X. The *contraction* of X in M is a matroid on $E \setminus X$ in which $I \subseteq E \setminus X$ is independent if and only if $J \cup I$ is independent in M for a (equivalently: for every) maximal independent subset J of X. Contraction and deletion commute, that is, for disjoint $X, Y \subseteq E$, we have $(M/X) - Y = (M - Y)/X$. Matroids of this form are the *minors* of M. We say that $X \subseteq E$ *spans* $e \in E$ in matroid M if either $e \in X$ or $\{e\}$ is dependent in M/X. If I is independent in M but $I + e$ is dependent for some $e \in E \setminus I$, then there is a unique circuit $\boldsymbol{C_M(e, I)}$ of M through e contained in $I + e$ which is called the *fundamental circuit* of e on I.

3 | THE FAILURE OF GREENE'S BASE EXCHANGE THEOREM IN INFINITE MATROIDS

Assume that M is a finitary matroid, B_0 and B_1 are disjoint bases and $B_0 = X_0 \cup X_1$ is a partition. It was shown by McDiarmid in [15] that there are disjoint independent sets I_0 and I_1 with $I_0 \cup I_1 = B_0 \cup B_1$ and $B_0 \cap I_i = X_i$ for $i \in \{0, 1\}$. If M has a finite rank, then the sets I_i need to be bases because they have together $2 \cdot r(M)$ elements. This argument fails if $r(M)$ is infinite. Is it still true that they need to be bases? If not, is it always possible to choose them to be bases? We demonstrate a negative answer for these questions.

Theorem 1.3. *There is a countably infinite graph $G = (V, E)$ with edge-disjoint spanning trees T_0 and T_1 such that there is a partition $E(T_0) = X_0 \cup X_1$ for which there are no edge-disjoint spanning trees S_0 and S_1 with $E(T_0) \cap E(S_i) = X_i$ for $i \in \{0, 1\}$.*

Proof. To simplify the notation, let us identify the spanning trees with their edge sets. Let $G = (V, E)$ be the graph at Figure 1.

Let us denote the set of edges with exactly one end vertex in $U \subseteq V$ by $\delta(U)$. Consider the spanning trees $T_0 := \{f_n : n < \omega\}$ and $T_1 := E \setminus T_0$. We define $X_0 := \{f_{2n} : n < \omega\}$ and $X_1 := \{f_{2n+1} : n < \omega\}$. Suppose for contradiction that S_0 and S_1 are edge-disjoint spanning trees with $T_0 \cap S_i = X_i$ for $i \in \{0, 1\}$. We must have $h_0 \in S_1$ since otherwise v_0 would be an isolated vertex in S_1. But then all the edges in $\delta(\{v_0, v_1\})$ but e_0 are in S_1, thus we must have $e_0 \in S_0$. Suppose that we already know for some $n < \omega$ that $\{h_i : i \leqslant n\} \subseteq S_1$ and $\{e_i : i \leqslant n\} \subseteq S_0$. Consider

$$V_n := \left\{ v_{2n+2-4k} : k \leqslant \frac{n+1}{2} \right\} \cup \left\{ v_{2n+1-4k} : k \leqslant \frac{n}{2} \right\}.$$

(So, $V_0 = \{v_1, v_2\}$, $V_1 = \{v_0, v_3, v_4\}$, $V_2 = \{v_1, v_2, v_5, v_6\}$, $V_3 = \{v_0, v_3, v_4, v_7, v_8\}$, etc.) All the edges in $\delta(V_n)$ but h_{n+1} are in S_0, thus necessarily $h_{n+1} \in S_1$. But then all the edges in $\delta(\{v_m : m \leqslant 2n+3\})$ but e_{n+1} are in S_1 therefore $e_{n+1} \in S_0$. It follows by induction that

$$S_0 = X_0 \cup \{e_i : i < \omega\} \text{ and } S_1 = X_1 \cup \{h_i : i < \omega\}.$$

But then S_1 consists of two vertex-disjoint rays contradicting the assumption that S_1 is a spanning tree. □

4 | GENERALISATIONS OF THE AHARONI–POUZET BASE EXCHANGE THEOREM

In contrast to matroids of finite rank, in matroids of infinite rank, we cannot perform arbitrary arithmetic operations on the rank function. This makes the literal adaptation of some proofs to infinite matroids impossible. Indeed, the standard proof of the Greene–Magnanti theorem involves subtractions of certain values of the rank function but these subtractions are no longer well defined if the values in question are infinite. Furthermore, the contraction of a single edge in a base does not reduce the rank if it is infinite; thus, the direct adaptation of proofs based on induction on the rank after such a contraction is also impossible.

In contrast to the approach by Aharoni and Pouzet, our proofs do not rely on infinite matching theory but uses the elementary and powerful method by Kotlar, Roda and Ziv introduced in [14]. This method applies symmetric subset base exchange as a subroutine. We are going to show that symmetric subset base exchange works in any matroid as long as the set we intend to exchange is finite.

4.1 | Partition base exchange with finite sets

Recall that the definition of matroid we are using is what was given in Section 2. In contrast to finitary matroids, it is unprovable for general matroids that all the bases of a fixed matroid must have the same size (it is independent of the axiomatic set theory ZFC [4, 11]), but the following weakening is easy to prove.

Lemma 4.1 [5, Lemma 3.7]. *If B_0 and B_1 are bases of a matroid with $|B_0 \setminus B_1| < \aleph_0$, then $|B_0 \setminus B_1| = |B_1 \setminus B_0|$.*

Proposition 4.2. *Suppose that $M = (E, \mathcal{I})$ is a matroid, B_0 and B_1 are bases of M and X is a finite or cofinite subset of B_0. Then there is a $Y \subseteq B_1$ such that $(B_0 \setminus X) \cup Y$ and $(B_1 \setminus Y) \cup X$ are both bases.*

Proof. By the symmetry between X and $B_0 \setminus X$, we may assume that X is finite. We can also assume without loss of generality that $B_0 \cap B_1 = \emptyset$ since otherwise we consider the matroid $M/(B_0 \cap B_1)$, bases $B_0 \setminus B_1$ and $B_1 \setminus B_0$ and set $X' := X \setminus B_1$. If Y' is as desired for the new problem, then $Y := Y' \cup (B_0 \cap B_1 \cap X)$ is suitable for the original one.

We take disjoint sets $Y, Z \subseteq B_1$ (see Figure 2) such that $(B_0 \setminus X) \cup Y$ and $X \cup Z$ are both independent and for the set $U := B_1 \setminus (Y \cup Z)$ of uncovered edges, $|U|$ is as small as possible. The

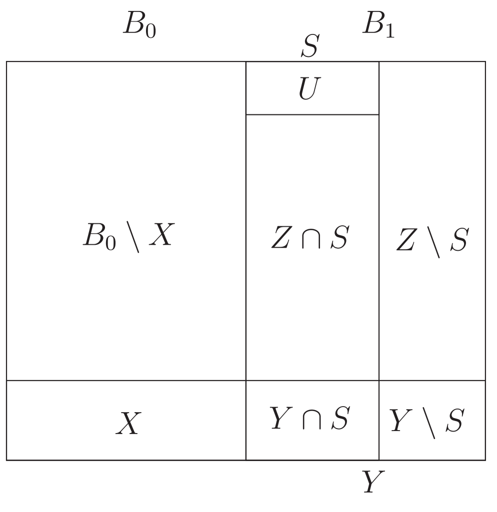

FIGURE 2 The symmetric exchange of X.

set Z misses at least $|X|$ edges from B_1 and it misses exactly $|X|$ edges if and only if $X \cup Z$ is a base (see Lemma 4.1). In particular, U must be finite. Similarly, Y can cover at most $|X|$ edges and $(B_0 \setminus X) \cup Y$ is a base if and on if $|X| = |Y|$. Therefore, it is enough to prove that $U = \emptyset$[†]. Suppose for contradiction that this is not the case. By removing edges from Z and adding them to Y, we can assume that $|X| = |Y|$.

Claim 4.3. There is a set S with $U \subseteq S \subseteq B_1$ such that S is spanned by both $(Y \cap S) \cup (B_0 \setminus X)$ and $(Z \cap S) \cup X$.

Proof. We need the following classical 'augmenting path lemma' developed by Edmonds and Fulkerson in [7]. A discussion of their method in the context of infinite matroids can be found, for example, in [8, Subsection 3.1].

Lemma 4.4 (Edmonds and Fulkerson). *Let $\{M_i : i < \kappa\}$ be a family of matroids defined on the common edge set E, let $\{I_i : i < \kappa\}$ be a family of pairwise disjoint sets such that I_i is independent in M_i and $U := E \setminus \bigcup_{i<\kappa} I_i \neq \emptyset$. Then there is either another family $\{J_i : i < \kappa\}$ of pairwise disjoint sets where J_i is independent in M_i and an $e \in U$ for which*

$$\bigcup_{i<\kappa} J_i = \{e\} \cup \bigcup_{i<\kappa} I_i$$

or there is a set S with $U \subseteq S \subseteq E$ such that $I_i \cap S$ spans S in M_i for every $i < \kappa$.

We apply Lemma 4.4 with the matroids $M/X \upharpoonright B_1$ and $M/(B_0 \setminus X) \upharpoonright B_1$ and sets Z and Y. By the choice of Y and Z, it is impossible to cover more edges; thus, the second case of Lemma 4.4 occurs which provides the desired S. □

[†] For finitary matroids, one can simplify the proof by applying the main result of [15] in order to choose Y and Z in such a way that $U = \emptyset$.

By the properties of S, the set $X \cup (Z \cap S)$ is a base of $M \restriction (X \cup S)$. Since S is independent in M, there is some $X' \subseteq X$ such that $X' \cup S$ is also a base of $M \restriction (X \cup S)$. Lemma 4.1 applied to $M \restriction (X \cup S)$ with these two bases and $|X| = |Y|$ ensure that

$$|X'| = |X| - |Y \cap S| - |U| = |Y \setminus S| - |U| < |Y \setminus S|.$$

Since $(Y \cap S) \cup (B_0 \setminus X)$ spans S and $X' \cup S$ spans $X \cup S$, the set $(Y \cap S) \cup (B_0 \setminus X) \cup X'$ spans X and therefore spans B_0 as well, and thus contains a base. But it is 'too small' to contain a base according to Lemma 4.1 because $B_0 \setminus X$ needs at least $|X|$ new edges to become a base and

$$|Y \cap S| + |X'| < |Y \cap S| + |Y \setminus S| = |Y| = |X|. \qquad \square$$

We prove a slightly stronger statement than Theorem 1.4 because we will need it in the proof of Theorem 1.5.

Theorem 4.5. *Suppose that $M = (E, \mathcal{I})$ is a finitary matroid, B_0 and B_1 are bases and $B_0 = \bigcup_{i<\kappa} X_i$ is a partition where all the sets X_i are finite. Then there is a partition $B_1 = \bigcup_{i<\kappa} Y_i$ and a bijection $\sigma : \kappa \to \kappa$ such that $(B_0 \setminus X_i) \cup Y_i$ and $(B_0 \setminus \bigcup_{i\leqslant j<\kappa} X_{\sigma(j)}) \cup \bigcup_{i\leqslant j<\kappa} Y_j$ are bases for every $i < \kappa$. For $\kappa \leqslant \omega$, the σ can be chosen to be the identity.*

Proof. For $\kappa < \omega$, the conditions imply that E is finite, and hence, the statement is (equivalent to) [14, Theorem 1.1.]. Suppose first that $\kappa = \omega$. Assume that the sets Y_i are already defined for $i < n$ for some $n < \omega$ such that for every $i < n$, the sets $(B_0 \setminus X_i) \cup Y_i$ and $(B_0 \setminus \bigcup_{i\leqslant j<\omega} X_j) \cup (B_1 \setminus \bigcup_{j<i} Y_j)$ are bases. We apply Proposition 4.2 with the matroid $M / \bigcup_{i<n} X_i$, bases $B_0 \setminus \bigcup_{i<n} X_i$ and $B_1 \setminus \bigcup_{i<n} Y_i$, and set X_n to obtain Y_n. The recursion is done. Since $Y_n \subseteq B_1 \setminus \bigcup_{i<n} Y_i$ for every $n < \omega$, the sets Y_i are pairwise disjoint. It follows by induction via Proposition 4.2 that $(B_0 \setminus X_i) \cup Y_i$ and $(B_0 \setminus \bigcup_{i\leqslant j<\omega} X_j) \cup (B_1 \setminus \bigcup_{j<i} Y_j)$ are bases for every $i < \omega$. In order to show that $B_1 = \bigcup_{i<\omega} Y_i$, let $e \in B_1$ be arbitrary. Since $C(e, B_0)$ is finite, there is an $i < \omega$ that $C(e, B_0) - e \subseteq \bigcup_{j<i} X_j$. But then we must have $e \in \bigcup_{j<i} Y_j$ because $(B_0 \setminus \bigcup_{i\leqslant j<\omega} X_j) \cup (B_1 \setminus \bigcup_{j<i} Y_j)$ is a base. Therefore $B_1 \setminus \bigcup_{j<i} Y_j = \bigcup_{i\leqslant j<\omega} Y_j$ which completes the proof of the case $\kappa = \omega$.

Suppose that $\kappa > \omega$. We reduce this to the countable case by the following technical lemma. Let us state the lemma in a slightly more general form than it is actually needed (it does not change the proof and having a reference in more general form could be helpful).

Lemma 4.6. *Assume that $M = (E, \mathcal{I})$ is a matroid without uncountable circuits, B_0 and B_1 are bases of M with $\aleph_0 < |B_0| =: \kappa$ and $B_0 = \bigcup_{i<\kappa} X_i$ is a partition where each X_i is countable. Then there is a family $\{B_j^\alpha : j \in \{0,1\}, \alpha < \kappa\}$ such that*

(1) $B_0^0 = B_1^0 = \emptyset$;
(2) $\bigcup_{\alpha<\kappa} B_j^\alpha = B_j$ *for* $j \in \{0,1\}$;
(3) $B_j^\alpha \subseteq B_j^{\alpha+1}$ *with* $|B_j^{\alpha+1} \setminus B_j^\alpha| = \aleph_0$ *for every* $\alpha < \kappa$ *and* $j \in \{0,1\}$;
(4) *for a limit ordinal* $\alpha < \kappa$ *and* $j \in \{0,1\}$ *we have* $B_j^\alpha = \bigcup_{\beta<\alpha} B_j^\beta$;
(5) B_0^α *and* B_1^α *span each other in* M *for every* α;
(6) *whenever* $B_0^\alpha \cap X_i \neq \emptyset$ *for some* $\alpha, i < \kappa$, *then* $X_i \subseteq B_0^\alpha$.

Proof. The proof is a straightforward transfinite recursion (or alternatively a basic application of a chain of elementary submodels). Suppose that B_0^α and B_1^α are already defined. We set $B_j^{\alpha,0} := B_j^\alpha$

for $j \in \{0,1\}$ and let $B_0^{\alpha,1} := B_0^\alpha \cup X_i$ for the smallest i with $B_0^\alpha \cap X_i = \emptyset$. If $B_0^{\alpha,n+1}$ and $B_1^{\alpha,n}$ are defined for some $n < \omega$, then

$$B_1^{\alpha,n+1} := B_1^{\alpha,n} \cup \bigcup\{C(e,B_1) - e\,:\; e \in B_0^{\alpha,n+1} \setminus B_0^{\alpha,n}\},$$

$$B_0^{\alpha,n+2} := B_0^{\alpha,n+1} \cup \bigcup\{X_i\,:\; (\exists e \in B_1^{\alpha,n+1} \setminus B_1^{\alpha,n})(C(e,B_0) \cap X_i \neq \emptyset)\}.$$

It is easy to check that $B_j^{\alpha+1} := \bigcup_{n<\omega} B_j^{\alpha,n}$ for $j \in \{0,1\}$ are suitable. Since limit steps obviously preserve all the conditions, we are done. □

Let B_j^α for $j \in \{0,1\}$ and $\alpha < \kappa$ as in Lemma 4.6. Properties (3) and (6) guarantee that for every $\alpha < \kappa$, the set $B_0^{\alpha+1} \setminus B_0^\alpha$ is the union of countably infinitely many X_i. Let σ_α be an ω-type enumeration of the sets X_i that are contained in $B_0^{\alpha+1} \setminus B_0^\alpha$. We choose σ to be the concatenation of the sequences σ_α. Properties (3) and (5) guarantee that $B_{0,\alpha} := B_0^{\alpha+1} \setminus B_0^\alpha$ and $B_{1,\alpha} := B_1^{\alpha+1} \setminus B_1^\alpha$ are bases of $M_\alpha := M \upharpoonright (B_0^{\alpha+1} \cup B_1^{\alpha+1})/B_0^\alpha$. For every $\alpha < \kappa$, we apply the already proved countable case with matroid M_α, bases B_0^α and B_1^α and partition $B_{0,\alpha} = \bigcup_{n<\omega} X_{\sigma_\alpha(n)}$. Let $B_{1,\alpha} = \bigcup_{n<\omega} Y_{\alpha,n}$ be the resulting partition. We shall prove that letting $Y_{\omega\alpha+n} := Y_{\alpha,n}$ results in a desired partition of B_1. The sets $B_{1,\alpha}$ for $\alpha < \kappa$ form a partition of B_1 by the properties (1)–(4). The sets $Y_{\alpha,n}$ for $n < \omega$ partition $B_{1,\alpha}$ by construction. Thus, the sets Y_i for $i < \kappa$ partition B_1. Let $\omega\alpha + n < \kappa$ be arbitrary. By construction,

$$(B_{0,\alpha} \setminus X_{\sigma(\omega\alpha+n)}) \cup Y_{\omega\alpha+n} \text{ and } \left[\left(B_{0,\alpha} \setminus \bigcup_{n\leqslant m<\omega} X_{\sigma(\omega\alpha+m)}\right)\right] \cup \bigcup_{n\leqslant m<\omega} Y_{\omega\alpha+m}$$

are bases of M_α. But then their respective union with B_0^α results in bases of $M \upharpoonright (B_0^{\alpha+1} \cup B_1^{\alpha+1})$. By property (5), such bases can be extended to bases of M by adding any of $B_0 \setminus B_0^{\alpha+1}$ and $B_1 \setminus B_1^{\alpha+1} = \bigcup_{\omega(\alpha+1)\leqslant\beta<\kappa} Y_\beta$. Thus, the desired exchange properties hold. □

From the proof above, it is clear that Proposition 4.2 has the following extension.

Corollary 4.7. *Suppose that $M = (E,\mathcal{I})$ is a matroid, B_0 and B_1 are bases, $n < \omega$ and $B_0 = \bigcup_{i\leqslant n} X_i$ is a partition where all but at most one X_i are finite. Then there is a partition $B_1 = \bigcup_{i\leqslant n} Y_i$ such that $(B_0 \setminus X_i) \cup Y_i$ and $(B_0 \setminus \bigcup_{i\leqslant j\leqslant n} X_j) \cup \bigcup_{i\leqslant j\leqslant n} Y_j$ are bases of M for each $i \leqslant n$.*

4.2 | Exchanging all finite subsets of a base

In this subsection, we prove another generalisation of [1, Theorem2.1], namely, the extension of a theorem due to Donald and Tobey (see [6, Theorem 1]) to finitary matroids. We repeat it here for convenience.

Theorem 1.5. *Suppose that $M = (E,\mathcal{I})$ is a finitary matroid and B_0 and B_1 are bases of M. Then there is a bijection $F : [B_0]^{<\aleph_0} \to [B_1]^{<\aleph_0}$ such that $(B_0 \setminus I) \cup F(I)$ is a base for every $I \in [B_0]^{<\aleph_0}$.*

Proof. We will make use of the following special case of Theorem 4.5 where the partition consists of singletons.

Corollary 4.8. *Assume that $M = (E, \mathcal{I})$ is a finitary matroid and B_0 and B_1 are bases. Then there are enumerations $B_0 = \{e_\alpha : \alpha < \kappa\}$ and $B_1 = \{f_\alpha : \alpha < \kappa\}$ such that $B_0 - e_\alpha + f_\alpha$ and $(B_0 \setminus \{e_\beta : \alpha \leqslant \beta < \kappa\}) \cup \{f_\beta : \alpha \leqslant \beta < \kappa\}$ are bases for every $\alpha < \kappa$.*

It is enough to show that for every $k < \omega$, there is a bijection $F_k : [B_0]^k \to [B_1]^k$ for which $(B_0 \setminus I) \cup F_k(I)$ is a base for every $I \in [B_0]^k$ because then $F := \bigcup_{k<\omega} F_k$ is suitable. We define $F_0 := \emptyset$. Suppose that we already know for some k and every M, B_0 and B_1 that such a bijection $F_k = F_{k,M,B_0,B_1}$ exists. Let M, B_0 and B_1 be fixed. We also fix enumerations as in Corollary 4.8 and let us well-order B_0 and B_1 according to these enumerations. In order to define a desired F_{k+1}, it is enough to give for every $\alpha < \kappa$ a bijection $F_{k+1,\alpha}$ between the k-subsets of B_0 with the smallest edge e_α and the k-subsets of B_1 with the smallest edge f_α. Indeed, if this is done, then $F_{k+1} := \bigcup_{\alpha<\kappa} F_{k+1,\alpha}$ is appropriate. Corollary 4.8 guarantees that $B_{0,\alpha} := \{e_\beta : \alpha < \beta < \kappa\}$ and $B_{1,\alpha} := \{f_\beta : \alpha < \beta < \kappa\}$ are bases in $M_\alpha := M/(\{e_\beta : \beta < \alpha\} \cup \{f_\alpha\})$. Let $F'_{k+1,\alpha}$ be what we get by applying the induction hypothesis for k with $M_\alpha, B_{0,\alpha}$ and $B_{1,\alpha}$. Then Corollary 4.8 and the induction hypothesis ensure that defining

$$F_{k+1,\alpha}(I) := F'_{k+1,\alpha}(I - e_\alpha) + f_\alpha$$

is suitable. □

5 | OPEN PROBLEMS

Our positive results (Theorems 1.4 and 1.5) are about finitary matroids. Their extension to arbitrary matroids cannot be provable because the equicardinality of bases is independent of ZFC and the statements fail when $|B_0| \neq |B_1|$. Even so, they might be true for other important matroid classes. A matroid is called *cofinitary* if its dual is finitary and it is *tame* if the intersection of any circuit with any cocircuit is finite.

Question 5.1. Do Theorems 1.4 and 1.5 remain true for matroids having only countable circuits? Do they hold for cofinitary or even for tame matroids? Are they consistently true for every matroid?

One can obtain equivalent forms of [14, Theorem 1.1] (see in the Introduction) by reversing the enumeration of the partition or phrasing it from the perspective of the dual matroid. Since the reverse of an infinite well-order is no longer a well-order and the dual of a finitary matroid may fail to be finitary, these lead to new problems in the infinite case.

Question 5.2. Suppose that $M = (E, \mathcal{I})$ is a finitary matroid, B_0 and B_1 are bases and $B_0 = \bigcup_{n<\omega} X_n$ is a partition where all the sets X_n are finite. Is there always a partition $B_1 = \bigcup_{n<\omega} Y_n$ such that $(B_0 \setminus X_n) \cup Y_n$ and $(B_0 \setminus \bigcup_{m\leqslant n} X_m) \cup \bigcup_{m\leqslant n} Y_m$ are bases for every $n < \omega$?

Question 5.3. Suppose that $M = (E, \mathcal{I})$ is a finitary matroid, B_0 and B_1 are bases and $B_1 = \bigcup_{n<\omega} Y_n$ is a partition where all the sets Y_n are finite. Is there always a partition $B_0 = \bigcup_{n<\omega} X_n$ such that $(B_0 \setminus X_n) \cup Y_n$ and $(B_0 \setminus \bigcup_{m\leqslant n} X_m) \cup \bigcup_{m\leqslant n} Y_m$ are bases for every $n < \omega$?

Question 5.4. Is it true for every finitary matroid M and bases B_0 and B_1 that there exists a bijection $F : \mathcal{P}(B_0) \to \mathcal{P}(B_1)$ such that $(B_0 \setminus I) \cup F(I)$ is a base for every $I \subseteq B_0$?

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REFERENCES

1. R. Aharoni and M. Pouzet, *Bases in infinite matroids*, J. Lond. Math. Soc. **2** (1991), no. 3, 385–392. https://doi.org/10.1112/jlms/s2-44.3.385.
2. R. Aharoni and R. Ziv, *The intersection of two infinite matroids*, J. Lond. Math. Soc. **58** (1998), no. 3, 513–525. https://doi.org/10.1112/s0024610798006723.
3. N. Bowler, *Infinite matroids*, Habilitation thesis, University of Hamburg, 2014. https://www.math.uni-hamburg.de/spag/dm/papers/Bowler_Habil.pdf.
4. N. Bowler and S. Geschke, *Self-dual uniform matroids on infinite sets*, Proc. Amer. Math. Soc. **144** (2016), no. 2, 459–471. https://doi.org/10.1090/proc/12667.
5. H. Bruhn, R. Diestel, M. Kriesell, R. Pendavingh, and P. Wollan, *Axioms for infinite matroids*, Adv. Math. **239** (2013), 18–46. https://doi.org/10.1016/j.aim.2013.01.011.
6. J. Donald and M. Tobey, *A generalised exchange theorem for matroid bases*, Bull. Aust. Math. Soc. **43** (1991), no. 2, 177–180. https://doi.org/10.1017/s0004972700028938.
7. J. Edmonds and D. R. Fulkerson, *Transversals and matroid partition*, J. Res. Nal. Bureau Standards Sect. B Math. Math. Phys. **69B** (1965), no. 3, 147. https://doi.org/10.6028/jres.069b.016.
8. J. Erde, J. P. Gollin, A. Joó, P. Knappe, and M. Pitz, *Base partition for mixed families of finitary and cofinitary matroids*, Combinatorica **41** (2021), no. 1, 31–52. https://doi.org/10.1007/s00493-020-4422-4.
9. C. Greene, *A multiple exchange property for bases*, Proc. Amer. Math. Soc. **39** (1973), no. 1, 45–50. https://doi.org/10.1090/s0002-9939-1973-0311494-8.
10. C. Greene and T. L. Magnanti, *Some abstract pivot algorithms*, SIAM J. Appl. Math. **29** (1975), no. 3, 530–539. https://doi.org/10.1137/0129045.
11. D. A. Higgs, *Equicardinality of bases in B-matroids*, Canad. Math. Bull **12** (1969), 861–862. https://doi.org/10.4153/cmb-1969-112-6.
12. D. A. Higgs, *Matroids and duality*, Colloq. Math. **2** (1969), 215–220. http://eudml.org/doc/267207.
13. P. Komjáth, E. C. Milner, and N. Polat, *A compactness theorem for perfect matchings in matroids*, J. Combin. Theory Ser. B **44** (1988), no. 3, 253–262. https://doi.org/10.1016/0095-8956(88)90035-4.
14. D. Kotlar, E. Roda, and R. Ziv, *On sequential basis exchange in matroids*, SIAM J. Discrete Math. **35** (2021), no. 4, 2517–2519. https://doi.org/10.1137/19M1291261.
15. C. J. H. McDiarmid, *An exchange theorem for independence structures*, Proc. Amer. Math. Soc. **47** (1975), no. 2, 513–514. https://doi.org/10.1090/s0002-9939-1975-0363960-9.
16. N. Polat, *A compactness result for perfect matchings*, Discrete Math. **64** (1987), no. 2–3, 229–237. https://doi.org/10.1016/0012-365x(87)90192-0.
17. R. Rado, *Abstract linear dependence*, Colloquium Mathematicum, vol. 14, Institute of Mathematics Polish Academy of Sciences, Warsaw, pp. 257–264, https://doi.org/10.4064/cm-14-1-257-264.
18. N. White, *Matroid applications*, vol. 40, Cambridge University Press, Cambridge, 1992. https://doi.org/10.1017/cbo9780511662041.
19. J. Wojciechowski, *Infinite matroidal version of Hall's matching theorem*, J. Lond. Math. Soc. **71** (2005), no. 3, 563–578. https://doi.org/10.1112/S0024610705006320.