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ORIGINAL RESEARCH

The κ-core and the κ-balancedness of TU games

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Abstract

We consider transferable utility cooperative games with infinitely many players. In particular, we generalize the notions of core and balancedness, and also the Bondareva–Shapley Theorem for infinite TU games with and without restricted cooperation, to the cases where the core consists of κ-additive set functions. Our generalized Bondareva–Shapley Theorem extends previous results by Bondareva (Problemy Kibernetiki 10:119–139, 1963), Shapley (Naval Res Logist Q 14:453–460, 1967), Schmeidler (On balanced games with infinitely many players, The Hebrew University, Jerusalem, 1967), Faigle (Zeitschrift für Oper Res 33(6):405–422, 1989), Kannai (J Math Anal Appl 27:227–240, 1969; The core and balancedness, handbook of game theory with economic applications, North-Holland, 1992), Pintér (Linear Algebra Appl 434(3):688–693, 2011) and Bartl and Pintér (Oper Res Lett 51(2):153–158, 2023).

Keywords TU games with infinitely many players · Bondareva–Shapley Theorem · κ-Core · κ-Balancedness · κ-Additive set function · Duality theorem for infinite LPs

1 Introduction

The core (Aumann, 1961; Gillies, 1959; Shapley, 1955) is definitely one of the most important solution concepts of cooperative game theory. In the transferable utility setting (henceforth games) the Bondareva–Shapley Theorem (Bondareva, 1963; Faigle, 1989; Shapley, 1967) provides a necessary and sufficient condition for the non-emptiness of the core; it states that the core of a game with or without restricted cooperation is not empty if and only if

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the game is balanced. The textbook proof of the Bondareva–Shapley Theorem goes by the strong duality theorem for linear programs (henceforth LPs), see e.g. Peleg and Sudhölter (2007). The primal problem corresponds to the concept of balancedness and so does the dual problem to the notion of core. However, this result is formalized for games with finitely many players. It is a question how one can generalize this result to the infinitely many player case.

The finitely many player case is special in (at least) two counts: (1) it can be handled by finite linear programs, (2) since the power set of the player set is also finite, it is natural to take the solution of a game from the set of additive set functions (additive games).

There are two main directions to reformulate the notion of additive set function. The first, when we weaken (generalize) the notion of additivity; this leads to the notion of k-additive core (Grabisch & Miranda, 2008), where k is a finite cardinal (natural number). The second, when we use a notion stronger than additivity (e.g. σ-additivity). This latter approach is considered here.

Schmeidler (1967), Kannai (1969, 1992), Pintér (2011), and Bartl and Pintér (2023) considered games with infinitely many players. All these papers studied the additive core; that is, the case when the core consists of bounded additive set functions. Schmeidler (1967) and Kannai (1969) showed that the additive core of a non-negative game without restricted cooperation with infinitely many players is not empty if and only if the game is Schmeidler balanced (Definition 17). Bartl and Pintér (2023) extended these results and showed that the additive core of a game bounded below with our without restricted cooperation with infinitely many players is not empty if and only if the game is (bounded-)Schmeidler balanced.

Kannai (1969, 1992) considered the following question: When does there exist a bounded σ-additive set function in the core? In this paper we generalize the above question as follows: When does there exist a bounded κ-additive set function in the core, where κ is an infinite cardinal number? Moreover, we consider this question in the case of games with restricted cooperation too.

Addressing this question, we introduce the notions of κ-core and κ-balancedness (Definitions 15 and 21). Then, we apply the strong duality theorem for infinite LPs by Anderson and Nash (1987) (Proposition 3) and prove that the κ-core of a game with or without restricted cooperation and with arbitrarily many players is not empty if and only if the game is κ-balanced (Theorem 25).

The set-up of the paper is as follows. In the next section we recall the main mathematical notions and results, which are related to infinite LPs, and used in this paper. In Sect. 3 we define the notion of κ-additive set functions and discuss some related concepts and results. In Sect. 4 we present game theory notions and define various cores (such as κ-core) and balancedness conditions (such as κ-balancedness) we consider in this paper. Section 5 presents our main result. We give an answer to the question we have raised: there is a bounded κ-additive set function in the core if and only if the game is κ-balanced (Theorem 25). The last section briefly concludes.

2 Duality theorem

In this section we discuss the duality theorem for infinite linear programs that we will use later.

Let X and Y be real vector spaces. The *algebraic dual* of X is the space of all linear functionals on X; that is, all linear mappings $\varphi \colon X \to \mathbb{R}$, which are also known as linear forms on X. We denote the algebraic dual of X by X'. Similarly Y' denotes the algebraic dual of Y. Moreover, $Y^* \subseteq Y'$ denotes a linear subspace of Y' such that (Y, Y^*) is a *dual pair of spaces*; that is, if $f \in Y$ is non-zero, then there exists a $y \in Y^*$ such that $y(f) \neq 0$. For any linear mapping $A\colon X \to Y$ its *adjoint mapping* is $A'\colon Y' \to X'$ with $(A'(y))(x) = y\big(A(x)\big)$ for all $x \in X$ and $y \in Y'$. Moreover, a subset $P \subseteq X$ of the vector space X is a *convex cone* if $\alpha x + \beta y \in P$ for all $x, y \in P$ and all non-negative $\alpha, \beta \in \mathbb{R}$. For any two functionals $f, g\colon X \to \mathbb{R}$ we write $f \geq_P g$ if $f(x) \geq g(x)$ for all $x \in P$.

Now, given a linear mapping $A\colon X \to Y$, a point $b \in Y$ and a linear functional $c\colon X \to \mathbb{R}$, let us consider the following infinite LP-pair (cf. Anderson & Nash, 1987, Section 3.3):

$$\begin{array}{lrcllrcl} (\mathrm{P_{LP}}) & c(x) & \to & \sup \qquad & (\mathrm{D_{LP}}) & y(b) & \to & \inf \\ & \text{s.t. } A(x) & = & b & & \text{s.t. } A'(y) & \geq_P & c \\ & x & \in & P & & y & \in & Y^* \end{array} \tag{1}$$

where $P \subseteq X$ is a convex cone and Y^* is a subspace of Y' such that (Y, Y^*) is a dual pair of spaces.

Definition 1 The program $(\mathrm{D_{LP}})$ is *consistent* if there exists a linear functional $y \in Y^*$ such that $\big(A'(y)\big)(x) \geq c(x)$ for all $x \in P$. The *value* of a consistent program $(\mathrm{D_{LP}})$ is $\inf\big\{\, y(b) : A'(y) \geq_P c,\ \ y \in Y^* \,\big\}$.

In the next definition we assume the weak topology on the space Y with respect to Y^*. To define that, we describe all the neighborhoods of a point. A set $U \subseteq Y$ is a *weak neighborhood* of a point $f_0 \in Y$ if there exist a natural number n and functionals $y_1, \ldots, y_n \in Y^*$ such that $\bigcap_{j=1}^n \big\{ f \in Y : \big|y_j(f) - y_j(f_0)\big| < 1 \big\} \subseteq U$.

Definition 2 Put $D = \big\{\big(A(x),\, c(x)\big) : x \in P\big\}$. The program $(\mathrm{P_{LP}})$ is *superconsistent* if there exists a $z \in \mathbb{R}$ such that $(b, z) \in \overline{D}$, where $\overline{D}$ is the closure of D. The *supervalue* of a superconsistent program $(\mathrm{P_{LP}})$ is $\sup\big\{z : (b, z) \in \overline{D}\big\}$.

We recall that a pair $(I, \leq)$ is *right-directed* if I is a preordered set and for any $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. A *net* (generalized sequence) of X is $(x_i)_{i \in I}$ where $(I, \leq)$ is a right-directed pair and $x_i \in X$ for all $i \in I$.

Notice that the program $(\mathrm{P_{LP}})$ is superconsistent if there exists a net $(x_i)_{i \in I}$ from P such that $A(x_i) \xrightarrow{w} b$, which means that $A(x_i)$ converges to b in the weak topology, and $(c(x_i))_{i \in I}$ is bounded. Furthermore, a number z^* is the supervalue of a superconsistent program $(\mathrm{P_{LP}})$ if it is the least upper bound of all numbers z such that there exists a net $(x_i)_{i \in I}$ from P such that $A(x_i) \xrightarrow{w} b$ and $c(x_i) \longrightarrow z$.

Proposition 3 *Consider the programs in* (1). *Program* $(\mathrm{P_{LP}})$ *is superconsistent and* z^* *is its finite supervalue if and only if program* $(\mathrm{D_{LP}})$ *is consistent and* z^* *is its finite value.*

Proposition 3 is a restatement of Theorem 3.3, p. 41, in Anderson and Nash (1987). Notice that we differ from Anderson and Nash (1987) in the point that Anderson and Nash use slightly different notions of superconsistency and supervalue. However, they also remark that their notions and the ones we use here are equivalent (p. 41 above Theorem 3.3). This is why we omit the proof of Proposition 3 here.

3 The κ-structures

Throughout this section κ is an infinite cardinal number. Let N be a non-empty set and let $\mathcal{A} \subseteq \mathcal{P}(N)$ be a field of sets; that is, if $S_1, \ldots, S_n \in \mathcal{A}$, then $\bigcup_{j=1}^{n} S_j \in \mathcal{A}$, and $N \in \mathcal{A}$ with $N \backslash S \in \mathcal{A}$ for any $S \in \mathcal{A}$. The pair $(N, \mathcal{A})$ is called *chargeable space*.

Given a chargeable space $(N, \mathcal{A})$, let $\text{ba}(\mathcal{A})$ and $\text{ca}(\mathcal{A})$ denote, respectively, the set of bounded additive set functions and the set of bounded σ-additive (i.e. countably additive) set functions $\mu \colon \mathcal{A} \to \mathbb{R}$.

Let $(S_i)_{i \in I}$ be a net of sets of $\mathcal{A}$; a net $(S_i)_{i \in I}$ is a *κ-net* if $\#I \leq \kappa$, where $\#I$ is the cardinality of the set I. In addition, the net $(S_i)_{i \in I}$ is *monotone decreasing* or *monotone increasing* if $i \leq j$ implies $S_i \supseteq S_j$ or $S_i \subseteq S_j$, respectively, for any $i, j \in I$.

Let $\mu \colon \mathcal{A} \to \mathbb{R}$ be a set function. We say that μ is *upper κ-continuous* or *lower κ-continuous* at $S \in \mathcal{A}$ if for any monotone decreasing or increasing κ-net $(S_i)_{i \in I}$ from $\mathcal{A}$ with $\bigcap_{i \in I} S_i = S$ or $\bigcup_{i \in I} S_i = S$, respectively, it holds that $\lim_{i \in I} \mu(S_i) = \mu(S)$. The set function μ is *κ-continuous* if it is both upper and lower κ-continuous at every set $S \in \mathcal{A}$.

Next we define the notion of κ-additivity. Our definition is similar to the one by Armstrong and Prikry (1980) or Schervish et al. (2017).

Definition 4 A set function $\mu \colon \mathcal{A} \to \mathbb{R}$ is *κ-additive* if it is additive and κ-continuous. Let $\text{ba}^{\kappa}(\mathcal{A})$ denote the set of κ-additive set functions over $\mathcal{A}$.

Note that $\text{ba}^{\kappa}(\mathcal{A})$ is a linear subspace of $\text{ba}(\mathcal{A})$. Furthermore, the following proposition is easy to see.

Proposition 5 *If the set function $\mu \colon \mathcal{A} \to \mathbb{R}$ is additive, then it is*

- *upper κ-continuous if and only if it is lower κ-continuous;*
- *κ-continuous if and only if it is lower κ-continuous at $\emptyset$;*
- *$\aleph_0$-continuous if and only if it is σ-additive.*

Example 6 The Lebesgue measure on $B([0, 1])$, the Borel σ-field of $[0, 1]$, is not κ-additive for any $\kappa \geq \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the real numbers; but it is κ-additive for $\kappa = \aleph_0$.

However, if μ is a measure such that it is a linear combination of Dirac measures, then it is κ-additive for every cardinal number κ.

If κ is not countable and the field of sets on which the κ-additive set function is defined is rich enough, then one may ask whether there are enough or just few κ-additive set functions. Without going into the details we remark that this problem is related to the notion of measurable cardinal (Ulam, 1930).

The next example shows that there are many κ-additive set functions in the space $\text{ba}^{\kappa}(\mathcal{A})$ even in the case when the field $\mathcal{A}$ is large; that is, the theory is not trivial nor vacuous.

Example 7 Let X be an arbitrary set such that $\#X = \kappa \geq \aleph_0$. Consider $\mathcal{P}(X)$, the power set of X. It is clear that the Dirac measures on $\mathcal{P}(X)$ are κ-additive. Let

$$\Delta = \left\{ \sum_{n=1}^{\infty} \alpha_n \delta_n : (\alpha_n)_{n=1}^{\infty} \in \ell^1, \delta_n \text{ being Dirac measures on } \mathcal{P}(X) \text{ for } n = 1, 2, 3, \ldots \right\}.$$

It is clear that each $\mu \in \Delta$ is a κ-additive set function on $\mathcal{P}(X)$. Notice that $\#\Delta \geq \#\text{ca}\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big)$; that is, even in the "worst" case, when there does not exist a non-trivial

{0, 1}-valued κ-additive set function on $\mathcal{P}(X)$, which means the cardinal κ is not measurable (Ulam, 1930), the collection Δ of the trivial κ-additive set functions on $\mathcal{P}(X)$ is at least as large as the collection of the σ-additive ones on $\mathcal{P}(\mathbb{N})$. In other words, even in the "worst" case, the problem of the non-emptiness of the κ-core is at least as complex as the non-emptiness of the σ-core with player set $\mathbb{N}$ and all coalitions feasible, the case considered by Kannai (1969, 1992).

Given a set system $\mathcal{A}$, the space $\mathbb{R}^{(\mathcal{A})}$ consists of all functions $\lambda\colon \mathcal{A} \to \mathbb{R}$ with a finite support; that is,

$$\mathbb{R}^{(\mathcal{A})} = \left\{\lambda \in \mathbb{R}^{\mathcal{A}} : \#\{\, S \in \mathcal{A} : \lambda_S \neq 0\,\} < \infty\right\}.$$

Denoting $\lambda(S)$ and the characteristic function of a set $S \in \mathcal{A}$ by λ_S and χ_S, respectively, let $\Lambda(\mathcal{A}) = \{\lambda_{S_1}\chi_{S_1} + \dots + \lambda_{S_n}\chi_{S_n} : n \in \mathbb{N},\ \lambda_{S_1}, \dots, \lambda_{S_n} \in \mathbb{R},\ S_1, \dots, S_n \in \mathcal{A}\}$ be the space of all simple functions on $(N, \mathcal{A})$; that is,

$$\Lambda(\mathcal{A}) = \left\{\sum_{S \in \mathcal{A}} \lambda_S \chi_S : \lambda \in \mathbb{R}^{(\mathcal{A})}\right\}.$$

We define a norm on $\Lambda(\mathcal{A})$ as follows. For a simple function $f = \lambda_{S_1}\chi_{S_1} + \dots + \lambda_{S_n}\chi_{S_n} \in \Lambda(\mathcal{A})$ let

$$\|f\| = \sup_{x \in N} |f(x)|.$$

The defined norm induces a topology on $\Lambda(\mathcal{A})$, and the topological dual $(\Lambda(\mathcal{A}))^\star$ of the vector space $\Lambda(\mathcal{A})$ consists of all linear functionals $\mu'\colon \Lambda(\mathcal{A}) \to \mathbb{R}$ continuous with respect to the norm. It is well-known that the dual $(\Lambda(\mathcal{A}))^\star$ is isometrically isomorphic to $\mathrm{ba}(\mathcal{A})$, the space of all bounded additive set functions on $\mathcal{A}$ (see e.g. Dunford and Schwartz (1958), Theorem IV.5.1, p. 258); that is, we can identify the space $(\Lambda(\mathcal{A}))^\star$ with $\mathrm{ba}(\mathcal{A})$ for simplicity. Indeed, a set function $\mu \in \mathrm{ba}(\mathcal{A})$ induces a continuous linear functional $\mu' \in (\Lambda(\mathcal{A}))^\star$ on $\Lambda(\mathcal{A})$ as follows:

$$\mu'(f) = \lambda_{S_1}\mu(S_1) + \dots + \lambda_{S_n}\mu(S_n) \tag{2}$$

for any $f = \lambda_{S_1}\chi_{S_1} + \dots + \lambda_{S_n}\chi_{S_n} \in \Lambda(\mathcal{A})$.

Lemma 8 *It holds that* $\left(\Lambda(\mathcal{A}), \mathrm{ba}^\kappa(\mathcal{A})\right)$ *is a dual pair of spaces.*

Proof Let $f \in \Lambda(\mathcal{A})$ be non-zero, whence there is an $x \in N$ such that $f(x) \neq 0$. Then δ_x, the Dirac measure concentrated at point x on $\mathcal{A}$, is a κ-additive set function, and $\delta'_x(f) = f(x) \neq 0$. □

4 The κ-core and the κ-balancedness of TU games

Let κ be an arbitrary infinite cardinal number as in the previous section. First, we recall the notion of TU games. Let N be a non-empty set of players and let $\mathcal{A}' \subseteq \mathcal{P}(N)$ be a collection of sets such that $\emptyset, N \in \mathcal{A}'$. Then a *TU game* (henceforth a *game*) on $\mathcal{A}'$ is a set function $v\colon \mathcal{A}' \to \mathbb{R}$ such that $v(\emptyset) = 0$. Every $A \in \mathcal{A}'$ is called a (feasible) *coalition*, the set $A = N$ is the *grand coalition*, and $v(A)$ is the *payoff* of A if the coalition A is formed. We denote the class of games on $\mathcal{A}'$ by $\mathcal{G}^{\mathcal{A}'}$. Let $\mathcal{A}$ denote the field hull of $\mathcal{A}'$; that is, the smallest field of sets that contains $\mathcal{A}'$. We say that $v \in \mathcal{G}^{\mathcal{A}'}$ is a game *without restricted cooperation* if $\mathcal{A}' = \mathcal{A}$; that is, $\mathcal{A}'$ is a field. Otherwise, if $\mathcal{A}'$ is not a field, we say $v \in \mathcal{G}^{\mathcal{A}'}$ is a game *with restricted cooperation*.

Example 9 Consider the following game: Let the player set N be the set of natural numbers; that is, let $N = \mathbb{N}$. Moreover, let $\mathcal{A}'$, the class of the feasible coalitions, be the class of the finite and co-finite subsets of $\mathbb{N}$; that is, $\mathcal{A}' = \{A \subseteq \mathbb{N}\colon \text{either } \#A < \infty \text{ or } \#(N \setminus A) < \infty\}$. Furthermore, let game v be defined as follows:

$$v(A) = \begin{cases} 0 & \text{if } \# A < \infty, \\ 1 & \text{otherwise.} \end{cases}$$

Since $\mathcal{A}'$ is a field, we have that v is a game without restricted cooperation.

Let us modify the game v above as follows: Let the set of the feasible coalitions be $\mathcal{A}'' = \mathcal{A}' \cup \{2k : k \in \mathbb{N}\}$; that is, the coalitions from $\mathcal{A}'$ and the set of the even numbers. Moreover, let the modified game v' be the following:

$$v'(A) = \begin{cases} v(A) & \text{if } A \in \mathcal{A}', \\ 10 & \text{otherwise.} \end{cases}$$

Since $\mathcal{A}'' = \mathcal{A}' \cup \{2k : k \in \mathbb{N}\}$ is not a field, we conclude that v' is a game with restricted cooperation.

In the following subsections we introduce the three notions of core and the three notions of balancedness that we consider in this paper.

4.1 The core of a TU game

First, we recall the notion of additive core of a game, which was considered by Schmeidler (1967), Kannai (1969, 1992), Pintér (2011), and Bartl and Pintér (2023).

Definition 10 For a game $v \in \mathcal{G}^{\mathcal{A}'}$ its *additive core* (henceforth ba-core) is defined as follows:

$$\text{ba-core}(v) = \left\{\mu \in \text{ba}(\mathcal{A}) : \mu(N) = v(N) \text{ and } \mu(S) \geq v(S) \text{ for all } S \in \mathcal{A}'\right\}.$$

Example 11 Consider the games v and v' from Example 9. Since v itself is an additive set function, we have that

$$\text{ba-core}(v) = \{v\}.$$

Since the domain $\mathcal{A}'' = \mathcal{A}' \cup \{2k : k \in \mathbb{N}\}$ of v' is not a field, it is meaningless to use the notion of additivity to characterize v'. Moreover, since the domain of v is a proper subset of the domain of v', and v' itself is an extension of v from $\mathcal{A}'$ onto $\mathcal{A}''$, we have the following:

$$\text{ba-core}(v') = \left\{\mu \in \text{ba}(\overline{\mathcal{A}''}) : \mu\big(\{2k : k \in \mathbb{N}\}\big) \geq 10,\ \mu(A) = v(A) \text{ for all } A \in \mathcal{A}'\right\},$$

where $\overline{\mathcal{A}''}$ is the field hull of $\mathcal{A}''$.

In other words, each element of ba-core(v') is such an extension of v, the only element of ba-core(v), that its value at the set of the even numbers is at least 10. It is clear that there exists such an extension of v; that is, ba-core$(v') \neq \emptyset$. Even more, it is easy to see that there are continuum many such extensions of v because the value of any extension at the set of even numbers can be any real number not smaller than 10, hence the cardinality of ba-core(v') is continuum.

We shall also need the notion of σ-additive core of a game.

Definition 12 For a game $v \in \mathcal{G}^{\mathcal{A}'}$ its σ*-additive core* (henceforth ca-core) is defined as follows:

$$\text{ca-core}(v) = \left\{\mu \in \text{ca}(\mathcal{A}) : \mu(N) = v(N) \text{ and } \mu(S) \geq v(S) \text{ for all } S \in \mathcal{A}'\right\}.$$

Example 13 Consider the games v and v' from Example 9 again. Since the ca-core is a subset of the ba-core for any game, and v is not σ-additive ($\sum_{n\in\mathbb{N}} v\big(\{n\}\big) = 0 \neq 1 = v(N)$), we conclude

$$\text{ca-core}(v) = \emptyset.$$

Since the restriction of any element of ca-core(v') onto $\mathcal{A}'$ is an element of ca-core(v), but ca-core$(v) = \emptyset$, we conclude that

$$\text{ca-core}(v') = \emptyset.$$

Example 14 Consider the following game: Let the player set N be the set of real numbers, that is, let $N = \mathbb{R}$. Moreover, let $\mathcal{A}'$, the class of the feasible coalitions, be the class of the finite and co-finite subsets of $\mathbb{R}$; that is, $\mathcal{A}' = \{A \subseteq \mathbb{R} : \text{either } \#A < \infty \text{ or } \#(N \setminus A) < \infty\}$. Furthermore, let game w be defined as follows:

$$w(A) = \begin{cases} 0 & \text{if } \# A < \infty, \\ 1 & \text{otherwise.} \end{cases}$$

Then it is easy to see that w itself is a measure (non-negative and σ-additive), hence

$$\text{ca-core}(w) = \{w\}.$$

In general, for an infinite cardinal number κ we introduce the notion of κ-core of a game.

Definition 15 For a game $v \in \mathcal{G}^{\mathcal{A}'}$ its κ*-core* is defined as follows:

$$\kappa\text{-core}(v) = \left\{\mu \in \text{ba}^{\kappa}(\mathcal{A}) : \mu(N) = v(N) \text{ and } \mu(S) \geq v(S) \text{ for all } S \in \mathcal{A}'\right\}.$$

Example 16 Consider the game w from Example 14. Then it is clear that w is not κ-additive for any $\kappa \geq \mathfrak{c}$, hence

$$\kappa\text{-core}(w) = \emptyset$$

for every $\kappa \geq \mathfrak{c}$.

In words, the ba-core, the ca-core, and the κ-core consists of bounded additive, bounded σ-additive, and bounded κ-additive, respectively, set functions defined on the field hull $\mathcal{A}$ of the feasible coalitions $\mathcal{A}'$ that meet the conditions of efficiency ($\mu(N) = v(N)$) and coalitional rationality ($\mu(S) \geq v(S)$ for all $S \in \mathcal{A}'$). Observe that the ca-core is a special case of the κ-core when $\kappa = \aleph_0$.

Notice that in the finite case all the three notions of ba-core, ca-core, and κ-core are equivalent with the notion of (ordinary) core.

4.2 Balancedness of a TU game

In the case of infinite games without restricted cooperation with additive core Schmeidler (1967) defined the notion of balancedness. Here, letting

$$\mathbb{R}_{+}^{(\mathcal{A}')} = \left\{\lambda \in \mathbb{R}^{(\mathcal{A}')} : \lambda_S \geq 0 \text{ for all } S \in \mathcal{A}'\right\},$$

we generalize his notion to the restricted cooperation case, and call it Schmeidler balancedness.

Definition 17 We say that a game $v \in \mathcal{G}^{\mathcal{A}'}$ is *Schmeidler balanced* if

$$\sup \left\{ \sum_{S \in \mathcal{A}'} \lambda_S v(S) : \sum_{S \in \mathcal{A}'} \lambda_S \chi_S = \chi_N,\ \lambda \in \mathbb{R}_+^{(\mathcal{A}')} \right\} \leq v(N). \tag{3}$$

Example 18 Consider the game v from Example 9. Take any balancing weights ($\lambda \in \mathbb{R}_+^{(\mathcal{A}')}$ such that $\sum_{S \in \mathcal{A}'} \lambda_S \chi_S = \chi_N$). Then for any ba-core element μ we have that

$$\sum_{S \in \mathcal{A}'} \lambda_S v(S) \leq \sum_{S \in \mathcal{A}'} \lambda_S \mu(S) = \mu(N) = v(N),$$

therefore the game v is Schmeidler balanced.

Notice that for finite games the notions of Schmeidler balancedness and (ordinary) balancedness (Bondareva, 1963; Faigle, 1989; Shapley, 1967) coincide, hence Schmeidler balancedness is an extension of (ordinary) balancedness.

Recall that $Y^* = \mathrm{ba}^\kappa(\mathcal{A})$ is a linear subspace of $Y^\star = \mathrm{ba}(\mathcal{A})$, which can be identified with the topological dual of the normed linear space $Y = \Lambda(\mathcal{A})$. In the next two definitions, where we introduce two new notions of balancedness, we consider the weak topology on $Y = \Lambda(\mathcal{A})$ with respect to $Y^* = \mathrm{ba}^\kappa(\mathcal{A})$ (see Lemma 8).

First, for a game $v \in \mathcal{G}^{\mathcal{A}'}$ consider the convex cone

$$K_v^+ = \left\{ \left(\sum_{S \in \mathcal{A}'} \lambda_S \chi_S, \sum_{S \in \mathcal{A}'} \lambda_S v(S) \right) : \lambda \in \mathbb{R}_+^{(\mathcal{A}')} \right\}. \tag{4}$$

Definition 19 We say that a game $v \in \mathcal{G}^{\mathcal{A}'}$ is *Schmeidler κ-balanced* if

$$z \leq v(N)$$

for all $z \in \mathbb{R}$ such that $(\chi_N, z) \in \overline{K_v^+}$, where $\overline{K_v^+}$ is the closure of K_v^+.

Example 20 Consider the game w from Example 14. Knowing that $\text{ca-core}(w) \neq \emptyset$, it is easy to see that the game w is Schmeidler balanced. We show that it is Schmeidler $\aleph_0$-balanced too. Take a net of weights $(\lambda^i)_{i \in I} \subseteq \mathbb{R}_+^{(\mathcal{A}')}$ such that $\sum_{S \in \mathcal{A}'} \lambda_S^i \chi_S \xrightarrow{w} \chi_N$; that is, for any $\mu \in \mathrm{ca}(\mathcal{A}')$ we have $\sum_{S \in \mathcal{A}'} \lambda_S^i \mu(S) \longrightarrow \mu(N)$; and such that $\sum_{S \in \mathcal{A}'} \lambda_S^i w(S) \longrightarrow z$. Then for any ca-core element μ and for any index $i \in I$ we have that

$$\sum_{S \in \mathcal{A}'} \lambda_S^i v(S) \leq \sum_{S \in \mathcal{A}'} \lambda_S^i \mu(S).$$

Moreover, since both sides converge, it follows that

$$z = \lim_{i \in I} \sum_{S \in \mathcal{A}'} \lambda_S^i v(S) \leq \lim_{i \in I} \sum_{S \in \mathcal{A}'} \lambda_S^i \mu(S) = \mu(N) = w(N),$$

therefore the game w is Schmeidler $\aleph_0$-balanced.

Observe that Schmeidler κ-balancedness implies Schmeidler balancedness, which implies (ordinary) balancedness.

Lastly, for a game $v \in \mathcal{G}^{\mathcal{A}'}$ let

$$\mathbb{R}_*^{(\mathcal{A}')} = \left\{ \lambda \in \mathbb{R}^{(\mathcal{A}')} : \lambda_S \geq 0 \text{ for all } S \in \mathcal{A}' \backslash \{N\} \right\}$$

and consider the convex cone

$$K_v = \left\{ \left(\sum_{S \in \mathcal{A}'} \lambda_S \chi_S, \sum_{S \in \mathcal{A}'} \lambda_S v(S) \right) : \lambda \in \mathbb{R}_*^{(\mathcal{A}')} \right\}. \quad (5)$$

Definition 21 A game $v \in \mathcal{G}^{\mathcal{A}'}$ is κ-*balanced* if

$$z \leq v(N)$$

for all $z \in \mathbb{R}$ such that $(\chi_N, z) \in \overline{K_v}$, where $\overline{K_v}$ is the closure of K_v.

Example 22 Consider the game w from Example 14. We know by Example 20 that the game is Schmeidler $\aleph_0$-balanced, hence Schmeidler balanced. We now show that the game w is not Schmeidler κ-balanced if $\kappa \geq \mathfrak{c}$. Take the index set $I = \left\{ i \subseteq N : \#(N \setminus i) < \infty \right\}$, and define its ordering by $i \leq j$ if $i \supseteq j$. Observe that $\# I = \mathfrak{c}$. Consider the net of weights $(\lambda^i)_{i \in I}$, with $\lambda^i \in \mathbb{R}_+^{(\mathcal{A})}$, defined as follows: for any $i \in I$ and for any $S \in \mathcal{A}$ let

$$\lambda_S^i = \begin{cases} 1 & \text{if } S = i \text{ or } S = N, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_{S \in \mathcal{A}'} \lambda_S^i \chi_S = \chi_i + \chi_N \xrightarrow{w} \chi_N$; that is, for any $\mu \in \text{ba}^\kappa(\mathcal{A}')$, with $\kappa \geq \mathfrak{c}$, we have $\sum_{S \in \mathcal{A}'} \lambda_S^i \mu(S) = \mu(i) + \mu(N) \longrightarrow \mu(N)$. However, for every $i \in I$ we have that $\sum_{S \in \mathcal{A}'} \lambda_S^i w(S) = w(i) + w(N) = 2$, therefore $\lim_{i \in I} \sum_{S \in \mathcal{A}'} \lambda_S^i w(S) = 2 > 1 = w(N)$. In other words, the game w is not Schmeidler κ-balanced, hence it is not κ-balanced, if $\kappa \geq \mathfrak{c}$.

Remark 23 The notion of κ-balancedness and Schmeidler κ-balancedness is very closely related to the notion of supervalue given in Definition 2. The cone K_v or K_v^+ is precisely the set D if $A(\lambda) = \sum_{S \in \mathcal{A}'} \lambda_S \chi_S$ and $c(\lambda) = \sum_{S \in \mathcal{A}'} \lambda_S v(S)$ with $P = \mathbb{R}_+^{(\mathcal{A}')}$ or $P = \mathbb{R}_*^{(\mathcal{A}')}$, respectively, in Definition 2. Then the game is κ-balanced or Schmeidler κ-balanced, respectively, if and only if the supervalue of the related primal problem (P_{LP}) is not greater than $v(N)$.

Notice that the notion of κ-balancedness is a "double" extension of Schmeidler balancedness. First, we do not take the balancing weight system alone, but we take nets of weight systems. Second, we let the weight of the grand coalition be sign unrestricted. It is worth noticing that the notion of κ-balancedness applies its full strength when in a net of weight systems the net of the weights of the grand coalition is not bounded below (see Lemma 24 below).

The insight why we need the "double" extension is the following: As we shall see, the proof of our generalized Bondareva–Shapley theorem is based on the strong duality theorem for infinite LPs (Proposition 3), which is based on separation of a closed convex set from a point (not in the set). Therefore, we need to take the weak closure of a convex set and to approach a point in the closure. This is why we need to use the nets of weight systems.

Regarding that the weight of the grand coalition is sign unrestricted, notice that the linear combinations of Dirac measures are κ-additive for any κ, moreover, it is easy to see that the linear space spanned by the Dirac measures is weak* dense in the set of bounded additive set functions. Hence, by the results of Schmeidler (1967), Kannai (1969, 1992), and Bartl and Pintér (2023), we have a necessary and sufficient condition for the non-emptiness of the "approximate" κ-core for any κ for free: Schmeidler balancedness. However, we analyze the non-emptiness of the (exact) κ-core for any κ. Therefore, we set the appropriate variable (the weight of the grand coalition) in the primal problem be sign unrestricted, by which we get equality in the related constraint in the dual problem (the total mass of an allocation must exactly be the value of the grand coalition), hence we will have a necessary and sufficient condition for the non-emptiness of the κ-core for any κ: κ-balancedness.

Between Schmeidler balancedness and κ-balancedness, there lies Schmeidler κ-balancedness, where only the first step is taken: we take nets of weight systems. Even though we shall see later that Schmeidler κ-balancedness does not lead to new characterization results, it provides deeper understanding of the problem.

Since Schmeidler κ-balancedness is the same as κ-balancedness except that K_v in Definition 19 is replaced by K_v^+ in Definition 21, by $K_v^+ \subseteq K_v$, it is clear that κ-balancedness implies Schmeidler κ-balancedness. Furthermore, Schmeidler κ-balancedness and κ-balancedness are related by the following lemma.

Lemma 24 *For a game $v \in \mathcal{G}^{\mathcal{A}'}$ it holds*

$$\sup_{\substack{(\lambda^i)_{i\in I}\subseteq \mathbb{R}_+^{(\mathcal{A}')} \\ A(\lambda^i)\xrightarrow{w}\chi_N \\ c(\lambda^i)\longrightarrow z}} z \leq v(N) \qquad \text{if and only if} \qquad \sup_{\substack{(\lambda^j)_{j\in J}\subseteq \mathbb{R}_*^{(\mathcal{A}')} \\ A(\lambda^j)\xrightarrow{w}\chi_N \\ c(\lambda^j)\longrightarrow z \\ \liminf \lambda_N^j > -\infty}} z \leq v(N).$$

where $A(\lambda) = \sum_{S\in\mathcal{A}'} \lambda_S \chi_S$ and $c(\lambda) = \sum_{S\in\mathcal{A}'} \lambda_S v(S)$ for any $\lambda \in \mathbb{R}_+^{\mathcal{A}'}$.

Proof The "if" part is obvious. Given a net $(\lambda^i)_{i\in I} \subseteq \mathbb{R}_+^{(\mathcal{A}')}$, consider the same net $(\lambda^j)_{j\in J} = (\lambda^i)_{i\in I} \subseteq \mathbb{R}_*^{(\mathcal{A}')}$. Notice that $\liminf \lambda_N^j \geq 0$.

We prove the "only if" part indirectly. Suppose the right-hand side does not hold. Then there exists a net $(\lambda^j)_{j\in J} \subseteq \mathbb{R}_*^{(\mathcal{A}')}$ such that $\liminf \lambda_N^j = L > -\infty$ and $A(\lambda^j) \xrightarrow{w} \chi_N$ with $c(\lambda^j) \longrightarrow z > v(N)$.

If $L > 0$, then there exists a $j_0 \in J$ such that $j \geq j_0$ implies $\lambda_N^j \geq 0$. Consider the index set $I = \{j \in J : j \geq j_0\}$ and the net $(\lambda^i)_{i\in I} \subseteq \mathbb{R}_+^{(\mathcal{A}')}$, which satisfies $A(\lambda^i) \xrightarrow{w} \chi_N$ and $c(\lambda^j) \longrightarrow z > v(N)$.

Assume $L \leq 0$. There exists a subnet $(\lambda^{j_i})_{i\in I}$ of $(\lambda^j)_{j\in J}$ such that $\lambda_N^{j_i} \longrightarrow L$. Define the net $(\bar{\lambda}^i)_{i\in I}$ as follows: for any $i \in I$ and for any $S \in \mathcal{A}'$ let

$$\bar{\lambda}_S^i = \begin{cases} 0 & \text{if } S = N, \\ \lambda_S^{j_i}/(1-L) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} A(\bar{\lambda}^i) &= \sum_{\substack{S\in\mathcal{A}' \\ S\neq N}} \frac{\lambda_S^{j_i}\chi_S}{1-L} = \frac{A(\lambda^{j_i}) - \lambda_N^{j_i}\chi_N}{1-L} \\ &\xrightarrow{w} \frac{\chi_N - L\chi_N}{1-L} = \chi_N, \end{aligned}$$

and

$$\begin{aligned} c(\bar{\lambda}^i) &= \sum_{\substack{S\in\mathcal{A}' \\ S\neq N}} \frac{\lambda_S^{j_i} v(S)}{1-L} = \frac{c(\lambda^{j_i}) - \lambda_N^{j_i} v(N)}{1-L} \\ &\longrightarrow \frac{z - Lv(N)}{1-L} > \frac{v(N) - Lv(N)}{1-L} = v(N). \end{aligned}$$

It follows that the left-hand side does not hold in either case, which concludes the proof. □

5 The main result

The next result is our generalized Bondareva–Shapley Theorem.

Theorem 25 *For any game $v \in \mathcal{G}^{\mathcal{A}'}$ it holds that κ-core$(v) \neq \emptyset$ if and only if the game is κ-balanced.*

Proof Put $X = \mathbb{R}^{(\mathcal{A}')}$, $P = \mathbb{R}_*^{(\mathcal{A}')}$, $Y = \Lambda(\mathcal{A})$, and $Y^* = \text{ba}^\kappa(\mathcal{A})$, moreover define the mapping $A\colon \mathbb{R}^{(\mathcal{A}')} \to \Lambda(\mathcal{A})$ by $A(\lambda) = \sum_{S\in\mathcal{A}'} \lambda_S \chi_S$, let $b = \chi_N$, and define the functional $c\colon \mathbb{R}^{(\mathcal{A}')} \to \mathbb{R}$ by $c(\lambda) = \sum_{S\in\mathcal{A}'} \lambda_S v(S)$. Now, consider the programs (P_{LP}) and (D_{LP}) of (1).

Notice that program (P_{LP}) is superconsistent and its supervalue is at least $v(N)$. (Consider that $\big(A(\lambda), c(\lambda)\big) \in K_v \subseteq \overline{K_v}$ for $\lambda \in \mathbb{R}^{(\mathcal{A}')}$ with $\lambda_N = 1$ and $\lambda_S = 0$ for $S \neq N$.) Then the game is κ-balanced (Definition 21) if and only if the supervalue of (P_{LP}) is finite and not greater than $v(N)$ (Remark 23).

Moreover, observe that a set function $\mu \in \text{ba}^\kappa(\mathcal{A})$ is feasible for (D_{LP}) if and only if $\mu(S) \geq v(S)$ for all $S \in \mathcal{A}'$ and $\mu(N) = v(N)$. Thus program (D_{LP}) is equivalent to finding an element of κ-core(v), and its value is $v(N)$ if it is consistent, and its value is $+\infty$ otherwise.

Therefore by Proposition 3 the game has a non-empty κ-core (program (D_{LP}) is consistent) if and only if it is κ-balanced (the supervalue of program (P_{LP}) is not greater than $v(N)$). □

If the player set N is finite, then so is $\mathcal{A}' \subseteq \mathcal{P}(N)$, whence the cone K_v is closed. Then by Lemma 24 κ-balancedness reduces to Schmeidler balancedness, which is (ordinary) balancedness (Bondareva, 1963; Faigle, 1989; Shapley, 1967), and the κ-core is the (ordinary) core in the finite case. We thus obtain the classical Bondareva–Shapley Theorem as a corollary of Theorem 25:

Corollary 26 (Bondareva–Shapley Theorem) *If N is finite, then the core of a game with or without restricted cooperation is non-empty if and only if the game is balanced.*

Regarding Theorem 25, it is worth mentioning that while Bondareva (1963) applied the strong duality theorem to prove the Bondareva–Shapley Theorem, Shapley (1967) used a different approach. We do not go into the details, but we remark that the common point in both approaches is the application of a separating hyperplane theorem. In other words, both Bondareva's and Shapley's approaches are based on the same separating hyperplane theorem, practically their result is a direct corollary of that. Here we use the strong duality theorem for infinite LPs (Proposition 3, Anderson & Nash, 1987), which is also a direct corollary of the same separating hyperplane theorem.

5.1 The σ-additive case

In this subsection let $\kappa = \aleph_0$. Then $\text{ba}^\kappa(\mathcal{A}) = \text{ca}(\mathcal{A})$, the space of all bounded countably additive set functions on $\mathcal{A}$. Given a game $v \in \mathcal{G}^{\mathcal{A}'}$, its κ-core is the σ-additive core ca-core(v) given in Definition 12.

In the next example we demonstrate that there exists a Schmeidler $\aleph_0$-balanced non-negative game without restricted cooperation having its ca-core empty.

Example 27 Let the player set $N = \mathbb{N}$, the system of coalitions $\mathcal{A} = \mathcal{P}(\mathbb{N})$, and the game v be defined as follows: for any $S \in \mathcal{A}$ let

$$v(S) = \begin{cases} 1 & \text{if } \#(N \setminus S) \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We show that $\text{ca-core}(v) = \emptyset$. If $\mu \in \text{ca-core}(v)$, then $\mu\big(N\backslash\{n\}\big) \geq v\big(N\backslash\{n\}\big) = 1$ and $v(N) = 1$, whence $\mu\big(\{n\}\big) \leq 0$. So $0 \geq \sum_{n=1}^{\infty} \mu\big(\{n\}\big) = \mu(N) = v(N) = 1$, a contradiction.

We now show that, if $(\chi_N, z) \in \overline{K_v^+}$, see (4), then $z \leq 1 = v(N)$. We have $(\chi_N, z) \in \overline{K_v^+}$ if and only if each neighborhood of the point (χ_N, z) intersects the cone K_v^+. In particular, if $(\chi_N, z) \in \overline{K_v^+}$, then for any natural number m and for any $\varepsilon > 0$ there exists a point $(f, t) \in K_v^+$ such that f belongs to the weak neighborhood

$$\big\{f \in \Lambda(\mathcal{A}) : \big|\delta_i'(f) - 1\big| < \varepsilon \text{ for } i = 1, \ldots, m\big\},$$

where δ_i' is the continuous linear functional induced by the Dirac measure δ_i concentrated at i, see (2), and t belongs to the neighborhood $\big\{t \in \mathbb{R} : |t - z| < \varepsilon\big\}$. Hence, we have a natural number n, some distinct sets $S_0, S_1, \ldots, S_n \in \mathcal{A}$, and some non-negative $\lambda_{S_0}, \lambda_{S_1}, \ldots, \lambda_{S_n}$ such that $f = \lambda_{S_0}\chi_{S_0} + \lambda_{S_1}\chi_{S_1} + \cdots + \lambda_{S_n}\chi_{S_n}$ and

$$\left|\sum_{\substack{j=0 \\ S_j \ni i}}^{n} \lambda_{S_j} - 1\right| < \varepsilon \quad \text{for} \quad i = 1, \ldots, m \tag{6}$$

with

$$\left|\sum_{j=0}^{n} \lambda_{S_j} v(S_j) - z\right| = \left|\sum_{\substack{j=0 \\ \#(N\backslash S_j)\leq 1}}^{n} \lambda_{S_j} - z\right| < \varepsilon. \tag{7}$$

We can assume w.l.o.g. that $S_0 = N$, as well as $\#(N\backslash S_j) = 1$ for $j = 1, \ldots, n_1$ and $\#(N\backslash S_j) > 1$ for $j = n_1 + 1, \ldots, n$, where $n_1 \leq n$.

Everything is clear if there exists an $i \in \{1, \ldots, m\}$ such that $i \in \bigcap_{j=1}^{n_1} S_j$. Then by (6)

$$\sum_{\substack{j=0 \\ \#(N\backslash S_j)\leq 1}}^{n} \lambda_{S_j} = \sum_{j=0}^{n_1} \lambda_{S_j} \leq \sum_{\substack{j=0 \\ S_j \ni i}}^{n} \lambda_{S_j} < 1 + \varepsilon,$$

whence $z < 1 + 2\varepsilon$ by (7).

In the other case we have $m \leq n_1$ and, because the sets $S_0, S_1, \ldots, S_n$ are pairwise distinct, for $i = 1, \ldots, m$ we can assume w.l.o.g. that $S_i = N\backslash\{i\}$. By (6)

$$\sum_{j=0}^{n_1} \lambda_{S_j} - \lambda_{S_i} = \sum_{\substack{j=0 \\ j \neq i}}^{n_1} \lambda_{S_j} \leq \sum_{\substack{j=0 \\ S_j \ni i}}^{n} \lambda_{S_j} < 1 + \varepsilon \quad \text{for} \quad i = 1, \ldots, m.$$

Summing up, we get $m \sum_{j=0}^{n_1} \lambda_{S_j} - \sum_{i=1}^{m} \lambda_{S_i} < m + m\varepsilon$, whence $m \sum_{j=0}^{n_1} \lambda_{S_j} - \sum_{j=0}^{n_1} \lambda_{S_i} < m + m\varepsilon$. It then follows

$$\sum_{\substack{j=0 \\ \#(N\backslash S_j)\leq 1}}^{n} \lambda_{S_j} = \sum_{j=0}^{n_1} \lambda_{S_j} < \frac{m}{m-1}(1 + \varepsilon).$$

Taking (7) into account, we obtain

$$z < \frac{m}{m-1}(1+\varepsilon)+\varepsilon. \tag{8}$$

Since $1+2\varepsilon < (1+\varepsilon)m/(m-1)+\varepsilon$, inequality (8) holds in both cases. By that $m \geq 2$ and $\varepsilon > 0$ can be arbitrary, we conclude that $z \leq 1$.

Remark 28 Consider the game v from Example 27. Since ca-core$(v) = \emptyset$, the game is not $\aleph_0$-balanced. To see this, consider the sequence $(\lambda^i)_{i=1}^{\infty}$, with $\lambda^i \in \mathbb{R}_*^{(\mathcal{A})}$, defined as follows: for any $i \in \mathbb{N}$ and for any $S \in \mathcal{A}$ let

$$\lambda_S^i = \begin{cases} -(i-2) & \text{if } S = N, \\ 1 & \text{if } S = N \setminus \{n\} \textit{ for } n = 1, \ldots, i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_{S \in \mathcal{A}} \lambda_S^i \chi_S = 2\chi_N - \chi_{\{1,\ldots,i\}} \xrightarrow{w} \chi_N$, where the weak convergence in the space $\Lambda(\mathcal{A})$ is with respect to ca$(\mathcal{A})$, and $\sum_{S \in \mathcal{A}} \lambda_S^i v(S) = 2 > 1 = v(N)$.

Notice again that the sequence $(\lambda_N^i)_{i=1}^{\infty} = (2-i)_{i=1}^{\infty}$ is unbounded below. If $(\lambda_N^i)_{i=1}^{\infty}$ were bounded below, then by Lemma 24 we would get a contradiction with Example 27.

Example 27 demonstrates that it is not sufficient to use $\mathbb{R}_+^{(\mathcal{A})}$ and K_v^+ in the definition of κ-balancedness; that is, Schmeidler κ-balancedness is unable to reveal that the ca-core is empty even for non-negative games without restricted cooperation.

Remark 29 Reconsidering Schmeidler balancedness for the additive case, it is somehow tempting to ask whether the following "σ-extension" of condition (3) could lead to a similar result in the σ-additive case too:

$$\sup \left\{ \sum_{S \in \mathcal{A}'} \lambda_S v(S) : \sum_{S \in \mathcal{A}'} \lambda_S \chi_S = \chi_N,\ \lambda \in \mathbb{R}_+^{[\mathcal{A}']} \right\} \leq v(N), \tag{9}$$

where $\mathbb{R}^{[\mathcal{A}']} = \left\{\lambda \in \mathbb{R}^{\mathcal{A}'} : \#\{S \in \mathcal{A}' : \lambda_S \neq 0\} \leq \aleph_0\right\}$ and $\mathbb{R}_+^{[\mathcal{A}']} = \left\{\lambda \in \mathbb{R}^{[\mathcal{A}']} : \lambda_S \geq 0 \text{ for all } S \in \mathcal{A}'\right\}$. Moreover, the convergence of the sum $\sum_{S \in \mathcal{A}'} \lambda_S \chi_S$ is understood pointwise. In this case it is equivalent to say that the convergence is weak in the space $\Lambda(\mathcal{A})$ with respect to ca$(\mathcal{A})$. If the sum $\sum_{S \in \mathcal{A}'} \lambda_S v(S)$ is convergent, but not absolutely convergent, then we put $\sum_{S \in \mathcal{A}'} \lambda_S v(S) := +\infty$.

Denoting $A(\lambda) = \sum_{S \in \mathcal{A}'} \lambda_S \chi_S$ and $c(\lambda) = \sum_{S \in \mathcal{A}'} \lambda_S v(S)$, we can also consider the following generalization of (9). Let $z \leq v(N)$ whenever there exists a net $(\lambda^i)_{i \in I} \subseteq \mathbb{R}_+^{[\mathcal{A}']}$ such that $A(\lambda^i) \xrightarrow{w} \chi_N$ and $c(\lambda^i) \longrightarrow z$ where z is finite. Then for each $i \in I$ there exists a sequence $(\lambda^{in})_{n=1}^{\infty} \subseteq \mathbb{R}_+^{(\mathcal{A}')}$ such that $A(\lambda^{in}) \xrightarrow{w} A(\lambda^i)$ and $c(\lambda^{in}) \longrightarrow c(\lambda^i)$. Consequently, there exists a net $(\lambda^j)_{j \in J} \subseteq \mathbb{R}_+^{(\mathcal{A}')}$ such that $A(\lambda^j) \xrightarrow{w} \chi_N$ and $c(\lambda^j) \longrightarrow z$. In other words, Schmeidler κ-balancedness covers such extensions of Schmeidler balancedness (Definition 17) like (9).

Moreover, in Example 27 we presented a non-negative Schmeidler κ-balanced game. Therefore, the presented game is balanced according to (9) too, but the ca-core of the game is empty.

6 Conclusion

We have generalized the Bondareva–Shapley Theorem to TU games with and without restricted cooperation, with infinitely many players, and with at least σ-additive cores: we have proved for an arbitrary infinite cardinal κ that the κ-core of a TU game with or without restricted cooperation is not empty if and only if the TU game is κ-balanced.

Perhaps the most interesting result of this paper is that in the proper notion of balancing weight system the weight of the grand coalition is sign unrestricted. More precisely, it is not surprising at all that in the definition of κ-balancedness we need nets (generalized sequences) since we have to take a weak closure of a set and to approach a point in the closure. However, the fact that in the elements of the net the weight of the grand coalition must be sign unrestricted, even more unbounded below, is interesting.

The reason why we need sign unrestricted weight of the grand coalition is simple. In the proof of Theorem 25 in the dual problem (which describes of the κ-core), when we set $\mu(N) = v(N)$ (equality), then it means the corresponding variable λ_N in the primal problem must be sign unrestricted. If we do not set $\mu(N) = v(N)$, but only $\mu(N) \geq v(N)$, in the dual problem, then we get Schmeidler κ-balancedness. However, Example 27 shows that Schmeidler κ-balancedness does not imply the non-emptiness of the κ-core.

Notice that Kannai (1969, 1992) gave another necessary and sufficient condition for that the σ-additive core of a non-negative game without restricted cooperation is not empty. Kannai's result is based on a very different approach and not related directly to our $\aleph_0$-balancedness condition.

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Declarations

Conflict of interest Both authors declare that they have no conflict of interest.

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