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Strong core and Pareto-optimality in the multiple partners matching problem under lexicographic preference domains

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ABSTRACT

We study strong core and Pareto-optimal solutions for multiple partners matching problem under lexicographic preference domains from a computational point of view. The restriction to the two-sided case is called stable many-to-many matching problem and the general one-sided case is called stable fixtures problem. We provide an example to show that the strong core can be empty even for many-to-many problems, and that deciding the non-emptiness of the strong core is NP-hard. On the positive side, we give efficient algorithms for finding a near feasible strong core solution and for finding a fractional matching in the strong core of fractional matchings. In contrast with the NP-hardness result for the stable fixtures problem, we show that finding a maximum size matching that is Pareto-optimal can be done efficiently for many-to-many problems. Finally, we show that for reverse-lexicographic preferences the strong core is always non-empty in the many-to-many case.

1. Introduction

Roth (1984) proposed the study of many-to-many matching markets in the context of job markets, where each worker can have multiple jobs, and each firm can employ multiple workers, but at most one contract can be signed between any worker and firm. The agents of such a market have choice functions over the possible contracts involving them, that specifies a subset for any given set of contracts. The most well studied solution concept is *stability*. A solution is setwise stable if there are no alternative contracts outside of the solution set that would be selected by all parties in a blocking coalition (possibly rejecting some existing contracts). Pairwise stability means the lack of a single blocking contract. Roth showed that setwise and pairwise stable solutions coincide and exist for specific substitutable choice functions, and a number of extensions and structural results have been obtained in the follow-up literature (Roth, 1985; Blair, 1988; Fleiner, 2003; Klaus and Walzl, 2009; Klijn and Yazıcı, 2014).

In this paper we are focusing on the concept of strong core and Pareto-optimality under lexicographic and reverse-lexicographic preferences. The strong core is a classical solution concept in cooperative game theory, meaning that there is no weakly blocking coalition with an alternative matching for this coalition (without using outside contracts) that is a weak improvement for all of them, and strict improvement for at least one member. A solution is Pareto-optimal if the coalition containing all agents (called grand coalition) is not weakly blocking. It was already observed by Blair (1988) that the (strong) core and the set of pairwise stable solutions can be independent for many-to-many matching problems under substitutable preferences. Further examples of this kind

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were provided in Sotomayor (1999) and Konishi and Ünver (2006) for more restricted responsive preferences. In this paper we provide new examples for even more restricted lexicographic preferences.

What is the relevance of strong core solutions in practice? The bilateral contracts between agents can create strong bounds, even if two agents are not transacting directly, but if they have a connection through third parties then they may care about the well-being of each other. In particular, one would not seek a new transaction with another agent, if this new transaction would result in a worse or terminated deal for an agent in their connected network.

Let us consider a simple example to make this point clear. Suppose that we have four agents, a, b, c, and d and currently they are transacting with each other through bilateral contracts ab, bc and cd. Now, a and d have a new potential collaboration, that would be beneficial for both a and d, however if this happens then a would cancel her partnership with b, as a may have capacity for one contract only, making b worse off. Note that b and d are connected through c by two bilateral contracts, and suppose that these contracts are important for each corresponding pair, so neither of these agents want to break them in any case. Therefore d cannot engage in the blocking deal with a, since as a result a would drop her contract with b, and b would be worse off. The fact that b is connected to d through c with important contracts means that they can only form a blocking coalition together with outsiders. So, in this situation there is no blocking coalition where some agent would strictly improve and the others weakly improve.

Are such situations realistic in real world markets? Let us just substitute a with Russia, b with Ukraine, c with USA, and d with Germany. Russia is trading gas with Ukraine, but they would prefer trading with Germany instead directly through a new channel (Nord Stream 2) and then terminate their deals with Ukraine. USA has a strong partnership with both Germany and Ukraine that neither want to break. Germany and Ukraine do not trade directly, but they are both strongly linked to USA, that make them all belong to an informal coalition. Since the new trade between Germany and Russia would be harmful for Ukraine, it won't be approved by this informal coalition if the links through the USA are strong enough.[1]

Why do we study lexicographic preferences? From a theoretical point of view this is the simplest case of preferences over bundles. When the agents are providing their strict rankings over their potential partners then lexicographic preferences over the bundles are generated in a unique, straightforward way. The responsive and the even more general substitutable preferences have a large spectrum, and a central coordinator of such a market cannot expect the agents to express their preferences over the bundles, since these can be very complex and also exponential in size. Studying the concept of (pairwise) stability can be still tractable based on the preferences over the individual partners, but for regarding the strong core or Pareto-optimality one would need to make certain assumptions to deal with the ambiguity of possible preference extensions for bundles.[2] We shall also note that our counterexamples and hardness results for lexicographic preferences are naturally valid for all the above mentioned domains, namely for additive and responsive preferences as well.

1.1. Related literature

Many-to-many matching markets have been studied first by Roth in (1984) and (1985). He considered a model with multiple possible contract terms between any worker-firm pair, from which they may select at most one. The agents at both sides select the best contracts from a possible set according to their choice functions. Roth showed that if these choice functions are substitutable then a (pairwise) stable matching always exists, and can be obtained by a generalised deferred-acceptance algorithm. The lattice property of (pairwise) stable solutions was proved in Blair (1988), and even more general results for the existence and lattice structure were obtained by Fleiner for substitutable choice functions by using Tarski's fixpoint theorem (Fleiner, 2003). Klaus and Walzl studied special versions of setwise stability under different domain restrictions on substitutable preferences (Klaus and Walzl, 2009). Klijn and Yazici proved that the rural hospitals theorem holds for substitutable and weakly separable preferences in many-to-many markets (Klijn and Yazıcı, 2014).

The efficient computation of pairwise stable solutions for the stable many-to-many matching problem was demonstrated in Baıou and Balinski (2000), and the problem of computing an optimal solution with respect to the overall rank of the matching was given in Bansal et al. (2003). For the nonbipartite stable fixtures problem, Irving and Scott (2007) provided a linear time algorithm for finding a pairwise stable solution, if one exists. Finally, Cechlárová and Fleiner (2005) extended these tractability results for the case of multiple contracts for the stable fixtures problem.

Regarding the concept of strong core, for many-to-one stable matching problems under responsive preferences the strong core coincides with the set of pairwise stable solutions, as shown e.g. in Roth and Sotomayor (1990). However, for the stable many-to-many matching problem Sotomayor provided examples to show that the strong core and the set of pairwise stable solutions can be disjoint (Sotomayor, 1999). Konishi and Ünver (2006) gave an example for a many-to-many matching problem under responsible preferences where the strong core is empty. (However, we shall remark that their example allowed preferences, where one agent finds another agent unacceptable alone, but acceptable when bundled with another agent.) In this paper we strengthen these results by giving an example for the emptiness of the strong core under the restricted domain of lexicographic preferences (where, by definition, an unacceptable agent can never be part of an acceptable bundle).

[1] We provided this example in an earlier version of the paper, well before Russia invaded Ukraine. We decided to keep it, although we know that the current situation is much more serious and complex. We note that a game theoretical analysis about the case of Nord Stream 2 can be found in Balázs et al. (2020).

[2] As an example, we can mention the concept of possible and necessary Pareto-optimality for responsive preferences, that was studied for allocation problems in Aziz et al. (2019). For given linear orders by the agents over individual partners, a solution is *possibly Pareto-optimal* if it is Pareto-optimal for at least one possible responsive extension of the individual preferences, and it is *necessarily Pareto-optimal* if it is Pareto-optimal for all possible responsive extension of the individual preferences.

Table 1
A summary of our results for the lexicographic preference domain.? denotes that the corresponding problem is left open in this paper.

Lexicographic preferences	Many-to-many/Fixtures	
	Verification	Search Complexity
Strong core	coNP-c/coNP-c [T 3.4]	NP-h/NP-h [T 3.2]
Weak core	coNP-c/coNP-c [T 5.3]	?/NP-h [T 5.1]
Pareto optimal solution	coNP-c/coNP-c [T 3.3]	P/P

Table 2
A summary of our results for the Reverse-lexicographic preference domain.

RL-preferences	Many-to-many/Fixtures	
	Verification	Search Complexity
Strong core	coNP-c/coNP-c [T 6.4]	P [T 6.2] /NP-h [T 6.3]
Pareto optimal solution	coNP-c/coNP-c [T 6.5]	P/P

Table 3
A summary of our main positive results about relaxations of the strong core and maximum size Pareto-Optimal solutions.

Relaxed/max size solutions	Many-to-many/Fixtures	
	Lexicographic	RL
Fractional Strong core	P/P [T 4.2]	P/P [T 6.8]
Near-feasible strong core	P/P [T 4.1]	P/P [T 6.10]
Maximum size Pareto-optimal	P [T 3.5]/ NP-h [T 3.6]	P/P [T 6.6]

Lexicographic preferences for many-to-many matching problems with one-sided preferences have been studied in Aziz et al. (2019); Cechlárová et al. (2014), and Hosseini and Larson (2019). However, we are not aware of any paper on lexicographic preferences for multiple partners matching problems.

Furthermore, we consider reverse-lexicographic preferences, where the agents care mostly about their number of (acceptable) partners they receive in the matching, and this being equal they prefer the matching where their worst matched partner is as good as possible, and so on. Such preferences were studied in Kwanashie et al. (2014) under the name of *generous maximum matching* in the context of Student / Project allocation, where only one side (students') has preferences. Finally, we also study the weak core of stable fixtures problem under lexicographic preferences.

1.2. Our contribution

First we provide an example showing that the strong core of stable many-to-many matching problems can be empty even for lexicographic preferences in Section 3. Later in that section we prove several hardness results. We show that deciding whether a stable many-to-many matching problem has non-empty strong core is NP-hard. We also prove that it is co-NP-complete to decide whether a given matching for a stable many-to-many matching problem is Pareto-optimal or whether it is in the strong core. We conclude Section 3 by showing that finding a maximum size Pareto-optimal matching for the stable fixtures problem is NP-hard, but on the positive side it can be done efficiently in the many-to-many variant. More on the positive side, in Section 4 we give efficient algorithms for finding a strong core solution for slightly adjusted capacities (which we call near feasible strong core solution), and also for finding a fractional matching that is in the strong core of fractional matchings, even for the stable fixtures problem. In Section 5 we consider the weak core and show that for the stable fixtures problem, both the verification and existence problems are computationally hard in general, while we leave open the existence problem and its complexity in the many-to-many case. Finally, in Section 6 we consider the corresponding questions for a similar preference domain, which we call reverse-lexicographic (RL) preferences, and we show that every stable matching belongs to the strong core for such preferences, and thus we can always find a strong core solution efficiently for many-to-many markets. However, we show that deciding whether a given matching belongs to the strong core is coNP-complete and that deciding the existence of a strong core solution is NP-complete for the fixtures variant. Then, we show that similarly to the lexicographic case, we can find fractional and near feasible strong core solutions efficiently.

Our results are summarized in Table 1, 2 and 3.

2. Preliminaries

For a positive integer n, we denote $\{1, 2, \ldots, n\}$ by $[n]$. When defining the multiple partners matching problem, we distinguish between the two-sided *stable many-to-many matching problem*, and the one-sided *stable fixtures problem*, that we define as follows. Let $G = (N, E)$ denote the underlying graph, where the node set N represents the agents and we have an undirected edge $ab \in E(G)$ if the two corresponding agents find each other mutually acceptable. For each agent a, we denote by $A(a)$ the edges between a and the acceptable partners of a. Let $k(a)$ denote the (integer) capacity of agent a. We assume that every agent a has a strict preference order $>_a$ over the agents acceptable for her (or alternatively, over the adjacent edges to a), where $b >_a c$ means that a prefers b to c. We also assume, that each $>_a$ preference list has $\emptyset$ as its last element, meaning that a prefers each acceptable partner to being alone. Let $M \subset E$. If $M(a)$ denotes the set of edges incident to node a in M (that is the set of pairs in which agent a is involved in the solution), then the feasibility of the matching can be described with condition $|M(a)| \le k(a)$ for every agent $a \in N$. If, for a matching M the above condition is satisfied with equality then we say that the agent is *saturated*, otherwise she is *unsaturated*. The solution of our problem is a k-*matching, that is a set of edges $M \subset E$ that is feasible, so no capacity is violated. By abuse of notation, we call k-matchings simply* matchings *in this paper.* We call a matching M *complete*, if it saturates every agent. When G is non-bipartite then we get the stable fixtures problem (Irving and Scott, 2007), and when G is bipartite then we get the stable many-to-many matching problem, see e.g. Baıou and Balinski (2000). To follow the usual conventions, we call the two sides in this case the men and women respectively.

The classical solution concept for these problems is (pairwise) stability. A matching M is *stable* if there is no blocking pair. A pair $ab \notin M$ is *blocking*, if a is either unsaturated, or there is $ac \in M$ such that $b >_a c$, and likewise, b is either unsaturated or there is $bd \in M$ such that $a >_b d$.

When all the capacities are unit then for two-sided problems we get the *stable marriage problem*, and the one-sided case is called *stable roommates problem*, as defined by Gale and Shapley (1962). Gale and Shapley gave an efficient algorithm for finding a stable matching for the marriage case, and demonstrated with an example that stable matching may not exist for the roommates case. Irving (1985) gave a linear time algorithm that can find a stable solution for the roommates problem, if one exists. The results are similar for the capacitated case, a stable solution always exists for two-sided problems and can be computed in linear time by a generalised Gale-Shapley type algorithm, see e.g. Baıou and Balinski (2000). For the stable fixtures Irving and Scott (2007) provided a linear time algorithm to find a stable solution, if one exists.

In this paper we focus on the strong core and Pareto-optimality of the solutions, so we need to extend the preferences of the agents to the sets of partners. Let $\succ_a$ denote the preferences of agent a over the possible sets of partners (or alternatively over the possible sets of adjacent edges). We will assume that the preferences of the agents are lexicographic in the sense that they mostly care about their best partner, and then about their second best partner, and so on. Formally, we define the preference relation $\succ_a$ for agent a over each acceptable partner sets $S, T \subset A(a)$ in the following way. Consider the symmetric difference $S \triangle T = (S \setminus T) \cup (T \setminus S)$. We say that a lexicographically prefers S to T, denoted by $S \succ_a T$, if the best element of $S \triangle T$ according to $>_a$ is in S. Note that lexicographic preferences are strict over $A(a)$ by definition. Furthermore, we say that a matching M is lexicographically better than M' for a, if $(M \cap A(a)) \succ_a (M' \cap A(a))$.

We also extend these definitions to fractional matchings. We define *fractional matchings* as $f^M : E \to [0,1]$ functions, such that $\sum_{uv \in A(a)} f^M(uv) \le k(a)$ for every $a \in N$. A fractional matching is *half-integral*, if $f^M(e) \in \{0, \frac{1}{2}, 1\}$ for each edge $e \in E$. For a fractional matching, the *fractional partner set* $f^M(a)$ in f^M of an agent a is just f^M restricted to $\{ab \in A(a)\}$. We define a strict preference order over each such fractional partner set. Let $f^S, f^T : \{ab \in A(a)\} \to [0,1]$ be two fractional partner set. We define $f^S \triangle f^T$ to be those edges e, for which $f^S(e) \neq f^T(e)$ holds. We say that *a lexicographically prefers* f^S to f^T, if for the best element ab in $f^S \triangle f^T$ according to $>_a$ it holds that $f^S(ab) > f^T(ab)$. Note that this induces a strict ranking over all fractional acceptable partner sets. Also, if restricted to (integral) matchings, it just gives back the original definition.

In Section 6 we also consider *reverse-lexicographic* preferences, which are defined as follows. Take two sets $S, T \subset A(a)$, for some agent a. Then, S is *reverse-lexicographically preferred* to T by a, if and only if either $|S| > |T|$ holds, or $|S| = |T|$ and the worst element in $S \triangle T$ in the order $>_a$ is in T. Informally, we can describe reverse-lexicographic ordering as follows. Take two sets $S, T \subset A(a)$. Take the $k(a)$ places indexed by $i = 1, \ldots k(a)$ that a has, then fill these up by assigning the best partner of a in S (resp. T) to place 1, the second best to place 2, etc. Of course, some of the last places may remain empty, if a is unsaturated in S (resp. T). Now, let l be the index of the last place, where these two vectors are different. Then, a reverse-lexicographically prefers S to T, if this l-th coordinate is better for S than for T, according to the order $>_a$ (recall that for each acceptable partner b for a, it holds that $b >_a \emptyset$). We also call reverse-lexicographic preferences *RL-preferences* for short.

We can also extend the notion of RL-preferences for the fractional case, as follows. Let f^S, f^T be two fractional partner sets and let a be an agent. Then, a RL-prefers f^S to f^T, if and only if either $|f^S(a)| > |f^T(a)|$ (where $|f^S(a)| = \sum_{e \in A(a)} f^S(e)$) or $|f^S(a)| = |f^T(a)|$ and $f^S(ab) < f^T(ab)$ for the worst element ab in $f^S \triangle f^T$ according to $>_a$.

A matching M is in the *weak core*, if there is no *blocking coalition* $S \subset N$ with an alternative matching M' on S that is strictly preferred by all the members of S, that is $M'(a) \succ_a M(a)$ for every $a \in S$. A matching M is in the *strong core*, if there is no *weakly blocking coalition* S with alternative matching M' on S that is weakly preferred by all the members and strictly preferred by at least one member in S. To further distinguish the cases of weakly blocking and blocking coalitions, we sometimes call the latter strictly blocking coalitions. A matching is *Pareto-optimal* if it is not weakly blocked by N, which we call the *grand coalition*. A matching is *weakly Pareto-optimal*, if it is not (strictly) blocked by N. We say that a matching M' *(Pareto)-dominates* a matching M, if it holds that $M'(a) \succ_a M(a)$ or $M'(a) = M(a)$ for each agent a and $M'(a) \succ_a M(a)$ for at least one of them. A matching M' *strictly (Pareto)-dominates* a matching M, if $M'(a) \succ_a M(a)$ for each agent $a \in N$.

Examples for stability versus core property

Here we provide two examples to demonstrate the differences between stable matchings, strong core and Pareto-optimal matchings.

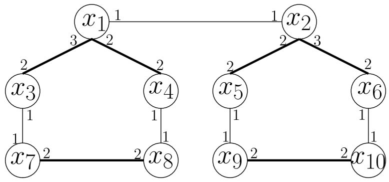

Fig. 1. Example 3. The numbers on the edges indicate the preferences, i.e. 1 means best, 2 means second best, etc. The only complete matching M is shown by the bold edges.

Example 1

We have four agents on both sides, $A = \{a, b, c, d\}$ and $B = \{x, y, z, w\}$ having capacity two each, and with the following linear preferences on their potential partners:

a :	$x > z > w > y$	x :	$b > c > d > a$
b :	$y > z > w > x$	y :	$a > c > d > b$
c :	$x > y$	z :	$a > b$
d :	$x > y$	w :	$a > b$

Here, the unique pairwise stable solution is $M = \{az, aw, bz, bw, cx, cy, dx, dy\}$, and the unique strong core solution is $M' = \{ax, ay, bx, by\}$ when we assume that agents have lexicographic preferences. Note that both of these solutions are Pareto-optimal.

The next, extended example shows that the unique stable solution may not even be Pareto-optimal.

Example 2

We have five agents on both sides, $A = \{a, b, c, d, p\}$ and $B = \{x, y, z, w, q\}$ having capacity two each, and with the following linear preferences on their potential partners:

a :	$x > y > z > q > w$	x:	$d > c > b > p > a$
b :	$y > x > w > q > z$	y :	$c > d > a > p > b$
c :	$z > w > x > q > y$	z :	$b > a > d > p > c$
d :	$w > z > y > q > x$	w :	$a > b > c > p > d$
p :	$x > y > z > w > q$	q :	$a > b > c > d > p$

Here, the unique pairwise stable solution is $M = \{ay, az, bx, bw, cw, cx, dz, dy, pq\}$, and the unique strong core solution is $M' = \{ax, aw, by, bz, cz, cy, dw, dx, pq\}$. Note that M' also Pareto-dominates M, so no stable matching is Pareto-optimal for this example.

Example 3

We have a stable fixtures problem with ten agents and the following preferences:

x_1 :	$x_2 > x_4 > x_3$
x_2 :	$x_1 > x_5 > x_6$
x_3 :	$x_7 > x_1$
x_4 :	$x_8 > x_1$
x_5 :	$x_9 > x_2$
x_6 :	$x_{10} > x_2$
x_7 :	$x_3 > x_8$
x_8 :	$x_4 > x_7$
x_9 :	$x_5 > x_{10}$
x_{10} :	$x_6 > x_9$

The capacities of agents x_1 and x_2 are 2, the capacities of the others are 1. Here the only complete matching is $M = \{x_1x_3, x_1x_4, x_2x_5, x_2x_6, x_7x_8, x_9x_{10}\}$, but the matching $M' = \{x_1x_2, x_3x_7, x_4x_8, x_5x_9, x_6x_{10}\}$ strictly Pareto-dominates it, so there is no complete Pareto-optimal matching in this instance. The example is illustrated in Fig. 1. We will see in Section 3 that for the stable many-to-many matching problem a maximum size Pareto-optimal matching always exists and one can be found in polynomial time, but the same problem is NP-hard for the stable fixtures problem.

3. Strong core and Pareto-optimal solutions

In this section we study the basic questions related to finding or verifying strong core or Pareto-optimal solutions. For the many-to-one case, it is known that a stable matching is always in the strong core, hence the strong core is always non-empty, even if the

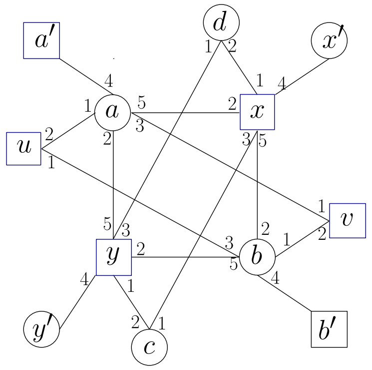

Fig. 2. An illustrative image of the counterexample used in Theorem 3.1. The numbers on the edges indicate the preferences (1 is best).

preferences are only responsive (Roth and Sotomayor, 1990). Hence, it would be tempting to think that strong core solutions should always exist in the many-to-many case, if the preferences are well-structured, i.e. lexicographic. However, we present an instance which does not admit a strong core solution, and then we also show that deciding if a stable many-to-many matching problem admits a strong core solution is NP-hard. We remark that in this section and the next section all preferences are lexicographic.

Theorem 3.1. *The strong core of a stable many-to-many matching problem may be empty under lexicographic preferences.*

Proof. We construct an instance where the strong core is empty, illustrated in Fig. 2. The construction is the following: There are 12 agents, on one side we have agents a and b with capacity 2, c, d, x', y' with capacity one, and on the other side we have agents x, y with capacity 2 and u, v, a', b' with capacity one. The preferences of the agents are shown as follows:

$a:$	$u > y > v > a' > x$	$x:$	$d > a > c > x' > b$
$b:$	$v > x > u > b' > y$	$y:$	$c > b > d > y' > a$
$c:$	$x > y$	$u:$	$b > a$
$d:$	$y > x$	$v:$	$a > b$
$x':$	x	$a':$	a
$y':$	y	$b':$	b

Let us suppose that there is a matching M in the strong core. Clearly, if any one of $\{c, d, u, v\}$ is unmatched, then they form a blocking coalition with their second choice, since they are their second choice's best option. This also means that the middle four cycle $C = \{ax, xb, by, ya\}$ cannot be included either, since they are the only possible partners of $\{c, d, u, v\}$. Hence any possible matching M in the strong core has to form an acyclic subgraph in the acceptability graph.

Observe that a, b, x, y must be saturated, as otherwise they would block either with their dummy partner a', b', x' or y' and the rest of the component in M containing them (if it includes some other agents) or with their third best partner (who considers them best), in the case when they only obtain their dummy partner in M.

Suppose that there is an edge of the four cycle C that is included in M, suppose by symmetry it is ax. If $aa' \notin M$, then aa' would block M with the rest of a's component in $M \setminus \{ax\}$, because x cannot be a part of that component, as M did not contain cycles. If $aa' \in M$, then the coalition $\{a, a', v\}$ with the matching $M' = \{aa', av\}$ block. So no edges of C can be in M.

If any of $\{au, bv, xd, yc\}$ is included in the matching, then u, v, c or d would block with b, a, x or y and their best partner in M respectively, as they must have a partner that they consider worse and can drop (they are saturated and no edge of C is in M). This means that the only possible choice left for M is $\{av, aa', bu, bb', yd, yy', xc, xx'\}$, but then the coalition $\{a, b, x, y\}$ with the four cycle C in the middle would block, a contradiction. So the strong core of the instance is indeed empty. □

3.1. Finding strong core solutions

In this section we show that it is computationally intractable to find strong core solutions for the stable many-to-many matching problem.

Theorem 3.2. *Deciding whether the strong core of a stable many-to-many matching problem is non-empty is NP-hard under lexicographic preferences, even if each capacity is at most two.*

Proof. We reduce from a special version of the NP-complete COM-SMTI problem, which was shown to be NP-hard by Manlove et al. (2002). Here, we are given an instance $I = (U, W, E, \succ)$ of the stable marriage problem with ties and incomplete lists, such that there are no ties in the preferences of the men $U = \{u_1, .., u_n\}$ and the set of woman can be partitioned into two parts $W = W^s \cup W^t = \{w_1, .., w_k\} \cup \{w_{k+1}, ..., w_n\}$ such that the preference lists of the women in W^s have no ties and the preference lists of the women in

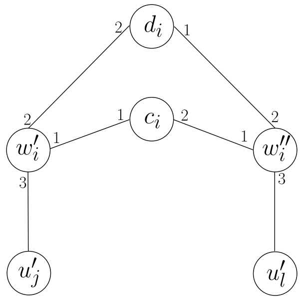

Fig. 3. The gadget G_i for Theorem 3.2 if the tie in w_i's list was $u_j \sim_{w_i} u_l$.

W^t consist of only a single tie of length exactly two. The task is to find a complete weakly stable matching, that is a matching M that covers every man and woman and there is no pair $(m, w) \notin M$ such that they both strictly prefer each other to their partner in M.

Suppose that we have an instance I of COM-SMTI. We construct an instance I' of the stable many-to-many matching problem such that the strong core is nonempty if and only if there is a complete weakly stable matching in I.

First of all, for every man $u_i \in U$ we create a vertex u_i' with capacity one, and for every woman $w_i \in W^s$ we create a vertex w_i' with capacity one. Denote these two sets by U' and $W^{s'}$. Then, for every woman $w_i \in W^t$ we create a gadget G_i illustrated in Fig. 3. We create four vertices: w_i', w_i'' and c_i having capacity two and d_i having capacity one. w_i' is connected to one man in w_i's preference list (the one with the smaller index) and w_i'' is connected to the other. The preferences are the following:

$$\begin{array}{rl} c_i: & w_i' > w_i'' \\ d_i: & w_i'' > w_i' \\ w_i': & c_i > d_i > u_j' \\ w_i'': & c_i > d_i > u_l' \end{array}$$

Finally, we add a gadget G, which is just a copy of the counterexample from Theorem 3.1 illustrated in Fig. 2 and a special agent g with capacity one.

The preferences of the agents in U' are the same just over the agents w_i' instead of w_i, except if there was a woman $w_i \in W^t$ in their preference list, then we substitute w_i with the appropriate copy from $\{w_i', w_i''\}$ (the one that he is connected to - so w_i' if he is the neighbour of w_i with smaller index and w_i'' otherwise). Finally, we add the special agent g to the end of all of their preference lists.

Similarly, for each $w_i' \in W^{s'}$, the preference lists are the same with u_i'-s instead of u_i-s. The agents in the gadget G have the same preferences, except that we add g to the beginning of $a \in G$'s list. Finally, the preference list of g has the agents in U' in an arbitrary order followed by $a \in G$ in the end.

Now let us suppose that we have a complete weakly stable matching M in I. We create a matching M' in I' by adding an edge $u_i'w_j'$ or $u_i'w_j''$ (the one which exists) for each $u_i w_j \in M$. Also for each gadget G_i we add the edges $w_i'c_i$ and $w_i''c_i$ to M'. If the partner of $w_i \in W^s$ was u_j, then we add $w_i''d_i$, if it was u_l, then we add $w_i'd_i$. Finally, we add the edges ag and $\{av, bu, yd, yy', xc, xx', bb'\}$ to M'.

We show that M' is in the strong core. Let us suppose that there is a blocking coalition $\mathcal{P}$ for M'. If there is a vertex $v_i \in \{w_i', w_i'', c_i, d_i\}$ from a gadget G_i in $\mathcal{P}$, then all of them are in $\mathcal{P}$, since if $v_i = d_i$, then w_i' or $w_i'' \in \mathcal{P}$, so their favourite partner c_i is also in $\mathcal{P}$ and so is the other copy of w_i. Similarly if w_i' or w_i'' or $c_i \in \mathcal{P}$, then all of them are in $\mathcal{P}$ and the two copies of w_i get the same partner, so none of them can achieve a strictly better situation. So no agents from G_i can improve their situation.

If a man u_i' is strictly better off in $\mathcal{P}$, then he has to have a better partner and also he has to be at least as good a partner for her if her capacity is one. But, he cannot be strictly better in that case, since M was weakly stable. So the partner w_i has to be from W^t. But then, the corresponding copy of w_i was matched to c_i and d_i in M', both of which it prefers to u_i', so they cannot be paired in a blocking coalition, a contradiction.

If an agent $w_i' \in W^{s'}$ gets a strictly better partner in $\mathcal{P}$, then she and her partner would form a blocking pair for M, a contradiction.

Special agent g cannot get a better partner in $\mathcal{P}$, because she is the worst choice for every other possible partner other than a, and every one of them is at full capacity, since M was a complete matching.

Finally, it is straightforward to check, that there are no blocking coalitions in the gadget G to M' either, so M' is in the strong core.

For the other direction suppose that M' is in the strong core of I'. This implies that $ga \in M'$, since there is no strong core solution among the agents in G. Therefore every agent in U' must be matched to someone in W', because otherwise they would block with g.

Now, we create M the following way: for each $u_i' \in U'$ we assign u_i the woman corresponding to the partner of u_i' in M'. To see that no two men get the same partner, suppose that u_j and u_l do. Then, $u_j'w_i'$ and $u_l'w_i''$ are both in M' for a woman in $W^{t'}$. However

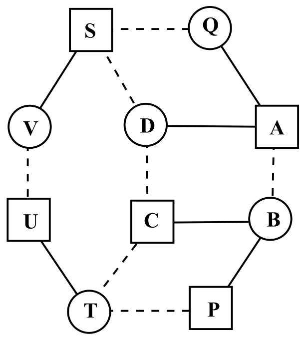

Fig. 4. The mutually acceptable pairs between the main sets of agents, where the solid edges denote the projection of M.

this would imply that $\{c_i, d_i, w'_i, w''_i, u'_j\}$ or $\{c_i, d_i, w'_i, w''_i, u'_l\}$ would form a blocking coalition, as w'_i or w''_i could switch to c_i or d_i and strictly improve without any other member of the coalition getting a worse partner set.

Since every man u_i is saturated and matched to different partners, it follows that M is a complete matching.

Now suppose that there is a strictly blocking pair (u_i, w_i). Then $w_i \in W^s$ and $\{u'_i, w'_i\}$ would form a blocking coalition for M', a contradiction.

So M is a complete and weakly stable matching. □

3.2. Verification problems related to strong core and Pareto-optimality

In the previous section we concluded that finding strong core solutions for the stable many-to-many matching problem under lexicographic preferences is NP-hard. Now we deal with the other natural question, which is verification. In this section we show that verifying strong core and Pareto-optimal solutions are both coNP-complete. As the reduction for the strong core verification problem builds on the reduction for the Pareto-optimality verification, we start by the latter one.

Theorem 3.3. *Deciding whether a given matching M is Pareto-optimal for the stable many-to-many matching problem under lexicographic preferences is co-NP-complete, even for complete matchings. It is also co-NP-hard to decide whether M is a maximum size Pareto-optimal matching.*

Proof. The problem of verifying whether a matching is Pareto-optimal is in co-NP, since checking that an alternative matching M' Pareto-dominates M can be done efficiently. We reduce from a special version of Exact-3-Cover, where we are given a set of $3n$ items $X = \{x_1, x_2, \dots, x_{3n}\}$ and a set of $m = 3n$ 3-sets, $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_m\}$, where each Y_j contains 3 items from X. The decision question is whether there exists a subset $\mathcal{Y}' \subset \mathcal{Y}$ of size n that contains all the elements of X exactly once (Hickey et al., 2008). Given an instance I of Exact-3-Cover, as described above, we create an instance I' of stable many-to-many matching problem as follows. We will have five gadgets, each with two sets of agents, $A \cup B$, $C \cup D$, $P \cup Q$, $S \cup T$ and $U \cup V$. More specifically, the sets of agents are as follows.

$$\begin{array}{ll} A = \{a_1, a_2, \dots, a_{3n}\} & B = \{b_1, b_2, \dots, b_{3n}\} \\ C = \{c_1, c_2, \dots, c_n\} & D = \{d_1, d_2, \dots, d_n\} \\ P = \{p_1, p_2, \dots, p_{3n}\} & Q = \{q_1, q_2, \dots, q_{3n}\} \\ S = \{s_1, s_2, \dots, s_m\} & T = \{t_1, t_2, \dots, t_m\} \\ U = \{u_1, u_2, \dots, u_{4m}\} & V = \{v_1, v_2, \dots, v_{4m}\} \end{array}$$

Let the capacity of every agent be 2 in $A \cup B$, 3 in $C \cup D$, 1 in $P \cup Q$, 4 in $S \cup T$ and 1 in $U \cup V$. Finally, we describe the linear orders of the agents on their acceptable partners. Here, for any set E, $[E]$ denotes the elements of E in the order of the elements' indices. Furthermore, we define $Q_j = \cup\{q_i : x_i \in Y_j\}$ and $S_i = \cup\{s_j : x_i \in Y_j\}$, and similarly $P_j = \cup\{p_i : x_i \in Y_j\}$ and $T_i = \cup\{t_j : x_i \in Y_j\}$.

a_1 :	$b_1 > q_1 > d_1 > b_{3n}$	b_1 :	$a_2 > p_1 > c_1 > a_1$
	$\dots$	$\dots$	
a_i :	$b_i > q_i > d_{\lfloor (i+2)/3 \rfloor} > b_{i-1}$	b_i :	$a_{i+1} > p_i > c_{\lfloor (i+2)/3 \rfloor} > a_i$
	$\dots$	$\dots$	
a_{3n} :	$b_{3n} > q_{3n} > d_n > b_{3n-1}$	b_{3n} :	$a_1 > p_{3n} > c_n > a_{3n}$
c_1 :	$d_1 > b_1 > b_2 > b_3 > d_n > [T]$	d_1 :	$c_2 > a_1 > a_2 > a_3 > c_1 > [S]$
	$\dots$	$\dots$	
c_i :	$d_i > b_{3i-2} > b_{3i-1} > b_{3i} > d_{i-1} > [T]$	d_i :	$c_{i+1} > a_{3i-2} > a_{3i-1} > a_{3i} > c_i > [S]$
	$\dots$	$\dots$	
c_n :	$d_n > b_{3n-2} > b_{3n-1} > b_{3n} > d_{n-1} > [T]$	d_n :	$c_1 > a_{3n-2} > a_{3(n-1)+2} > a_{3n} > c_n > [S]$
p_1 :	$[T_1] > b_1$	q_1 :	$[S_1] > a_1$
	$\dots$	$\dots$	
p_i :	$[T_i] > b_i$	q_i :	$[S_i] > a_i$
	$\dots$	$\dots$	
p_{3n} :	$[T_n] > b_n$	q_{3n} :	$[S_n] > a_n$
s_1 :	$[D] > v_1 > v_2 > v_3 > v_4 > [Q_1]$	t_1 :	$[C] > u_1 > u_2 > u_3 > u_4 > [P_1]$
	$\dots$	$\dots$	
s_j :	$[D] > v_{4j-3} > v_{4j-2} > v_{4j-1} > v_{4j} > [Q_j]$	t_j :	$[C] > u_{4j-3} > u_{4j-2} > u_{4j-1} > u_{4j} > [P_j]$
	$\dots$	$\dots$	
s_m :	$[D] > v_{4m-3} > v_{4m-2} > v_{4m-1} > v_{4m} > [Q_n]$	t_m :	$[C] > u_{4m-3} > u_{4m-2} > u_{4m-1} > u_{4m} > [P_m]$
u_1 :	$v_1 > t_1$	v_1 :	$u_1 > s_1$
	$\dots$	$\dots$	
u_j :	$v_j > t_{\lfloor (j+3)/4 \rfloor}$	v_j :	$u_j > s_{\lfloor (j+3)/4 \rfloor}$
	$\dots$	$\dots$	
u_{4m} :	$v_{4m} > t_m$	v_{4m} :	$u_{4m} > s_m$

We create a matching M in I' as follows. Let each agent in A and B be matched with their acceptable partners in $D \cup Q$ and in $C \cup P$, respectively. Furthermore, each agent in S is matched with all of her four acceptable agents in V, similarly, each agent in T is matched with all of her four acceptable agents in U. We depict the accessibility graph of I' in Fig. 4 with regard to the main sets of agents, where the solid edges mark that all of the mutually acceptable pairs between the two corresponding sets belong to M and the dashed edges denote when no edge between the corresponding sets belongs to M.

Now we shall prove that I has an exact 3-cover if an only if matching M is not Pareto-optimal in I'. As M is a complete matching (i.e. each agent is saturated), deciding whether M is Pareto optimal is equivalent to deciding whether M is a maximum size Pareto optimal matching. Hence, by proving the above claim we show hardness for both problems stated in the theorem.

First, let us suppose that we have an exact 3-cover $\mathcal{Y}'$ in I. We create a matching M' in I' that Pareto dominates M in the following way. In M' we match each agent in A to her two acceptable partners in B, which implies that we also match each agent in B to her two acceptable partners in A. Likewise, we match each agent in C to her two acceptable partners in D, which implies that we also match each agent in D to her two acceptable partners in C. For the rest of the agents we create M' according to the 3-cover $\mathcal{Y}'$ as follows. If $Y_j \in \mathcal{Y}'$ then we match s_j to the three acceptable agents in Q_j and also to an arbitrary agent in D such that we match exactly one agent from S to each agent in D, and similarly, we match t_j to the three agents in P_j and also to an arbitrary agent in C such that each agent in C is matched with exactly one agent from T, and finally we also match $u_{4(j-1)+k}$ to $v_{4(j-1)+k}$ for every $k \in \{1, 2, 3, 4\}$. For those agents in $s_j \in S$ and $t_j \in T$, where $Y_j \notin Y'$, we keep the edges of M. It is easy to see that all the agents that changed partners in I' improved according to their lexicographic preferences, since all of them become matched to their best potential partner in M'.

In the other direction, let us suppose that M is not-Pareto optimal, so there is an alternative matching M^* that Pareto-dominates it. We shall prove that I has an exact 3-cover. First we note that if any agent in $A \cup B \cup C \cup D$ has a different partner in M^* than in M (thus necessarily improves) then the matching between sets A and B and also between C and D must be complete, as we had in M'. To see this, observe that any of these agents can only improve by obtaining her best partner. Take a_i. For her, this implies that she gets b_i. But then, b_i gets her worst partner, so she also has to improve and get her best partner a_{i+1}. Now, a_{i+1} has to get b_{i+1} and by iterating this, we get that any edge between A and B must be added. For an agent in $C \cup D$, say c_i, if she improves and obtains d_i, then d_i gets a worse partner than the ones she was filled with, so she also has to get her best partner c_{i+1}. Again, iterating this, we get that all edges between $C \cup D$ must be inside the new matching, if someone from $C \cup D$ improves. Finally, if someone from $A \cup B$ improves, then by the previous argument, she has to drop a partner from $C \cup D$, so an agent in $C \cup D$ also has to improve. Similarly is someone from $C \cup D$ improves, then she has to drop a partner from $A \cup B$, so someone from $A \cup B$ also must improve. Hence, the improvement of any agent in $A \cup B \cup C \cup D$ implies that all such agents strictly improve and the matchings between A and B as well as C and D are complete. This also implies that all the agents in both P and Q must get new partners in M^* from the sets T and S, respectively. However, this is only possible if at least n agents from both S and T get also new partners from the sets D and C, respectively. But the agents in C and D have remaining capacity one each, so they should become matched with exactly n agents from each of T and S, respectively, so these are the only $2n$ agents from these sets that can change partners and help improve to those in P and Q. To summarise, if these agents all improve then we must be able to choose an exact 3-cover by adding Y_j to $\mathcal{Y}'$

if s_j has improved in M^*. What remained is to show that the improvement of any other agent outside the set $A \cup B \cup C \cup D$ would also lead to the same effect. Indeed, if any agent in $U \cup V$ improves in M^* then her/his partner in M from $S \cup T$ must also improve and this is only possible if the latter agent gets matched with someone from $A \cup B \cup C \cup D$. The same applies if any agent in $P \cup Q$ would improve. Thus we can conclude that the improvement of any agent in I' implies that all agents in $A \cup B \cup C \cup D$ must improve and thus we are able to find an exact 3-cover in I. □

Building on the hardness of the verification of Pareto-optimality, we also show hardness for strong core verification.

Theorem 3.4. *Deciding whether a given matching is in the strong core of a stable many-to-many matching problem under lexicographic preferences is co-NP-complete, even if each capacity is at most 5.*

Proof. The problem is in co-NP, since checking that M can be blocked by a coalition S with an alternative matching M_S can be done efficiently. We reduce from the problem of checking Pareto-optimality for stable many-to-many matching problem under lexicographic preferences, that we showed to be co-NP-complete in Theorem 3.3, even if capacities are at most 4 and the two sides have the same number of agents. Suppose that we have such an instance I of the stable many-to-many matching problem (with strict preferences of the agents), and a matching M that is to be checked to be Pareto-optimal. Let the agents in I be denoted by $u_1, \dots, u_n$ on side U and $v_1, \dots, v_n$ on side W. We create instance I' by adding $2n$ new agents $a_1^*, \dots, a_n^*$ to side W and $b_1^*, \dots, b_n^*$ to side U of the market, such that they have capacity 3. For each $i \in [n], j \in [n]$, we make a_i^* the best partner of u_i and b_j^* the best partner of v_j. Furthermore, a_i^* only considers u_i, b_i^* and b_{i+1}^* acceptable and ranks them by the order $u_i \succ b_i^* \succ b_{i+1}^*$, while b_i^* only considers v_i, a_i^* and a_{i-1}^* acceptable and ranks them by $v_i \succ a_i^* \succ a_{i-1}^*$ (where we let $a_0^* := a_n^*, a_{n+1}^* := a_1^*$, $b_0^* := b_1^*$ and $b_{n+1}^* := b_1^*$).

Let us also increase the capacity of all the agents in $U \cup W$ in I' by one and then we create M' as an extension of M in the following way. We keep each edge $u_i v_j$ of M. Then, we add edges $\{a_1^* b_1^*, a_1^* b_2^*, a_2^* b_2^*, \dots, a_n^* b_1^*\}$ and $\{u_i a_i^*, v_i b_i^* \mid i \in [n]\}$. Now, we show that M is Pareto-optimal in I if and only if M' is in the strong core in I'. On one hand a blocking coalition in I' with matching N' must involve every agent in I', because it must contain either a u_i or a v_j agent, hence also her best partner a_i^* or b_j^*. As each a_i^*, b_j^* obtains all her partners in M', all of $a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*$ and all of their partners must be in the blocking coalition, and that includes every agent. Hence, if there is a blocking coalition to M', then the grand coalition blocks M in I with a matching N (which is obtained by keeping the edges if N' that are also edges in I). On the other hand if the grand coalition blocks M in I with a matching N, then extending N to N' in I' the same way as we extended M, we get that the grand coalition is a blocking coalition for M' with N'. □

3.3. Maximum size Pareto-optimal matchings

While we proved that strong core solutions may not exist, it is clear that Pareto-optimal solutions always do. Hence, a natural question is to ask whether we can find such matchings that are optimal in some sense. Here we consider probably the most important optimality criteria, that is the size of a matching. We show that there is an interesting dichotomy here: finding a maximum size Pareto optimal matching, that is, a matching that is one of the largest amongst all Pareto optimal matchings, is easy in the two sided many-to-many setting, but it becomes NP-hard in the one-sided fixtures case.

We start with our positive result and describe an algorithm that finds a maximum size Pareto-optimal solutions. In fact, the algorithm returns a Pareto-optimal matching that is also a maximum cardinality matching. The techniques we use here are very similar to the ones in Cechlárová et al. (2014), where they investigated Pareto-optimal matchings in the case when only one side of the agents have preferences.

Now we state our algorithm. Denote the set of men as $U = \{u_1, ..., u_n\}$ and the set of women $W = \{w_1, ..., w_m\}$. Let $deg(v)$ be the degree of a vertex $v \in U \cup W$.

Algorithm 1 Maximum size Pareto-optimal matching.

```
Let M := ∅
Compute the size of the maximum cardinality matching η
for i = 1, .., n do
    l = 1
    while |M(u_i)| < k(u_i) and l ≤ deg(u_i) do
        Let w_j be the l-th choice of u_i
        if There is a feasible matching of size η containing M ∪ {u_i w_j} then
            M := M ∪ {u_i w_j}
        end if
        l = l + 1
    end while
end for
Output M
```

Theorem 3.5. *For the stable many-to-many matching problem Algorithm 1 finds a maximum size matching that is Pareto-optimal.*

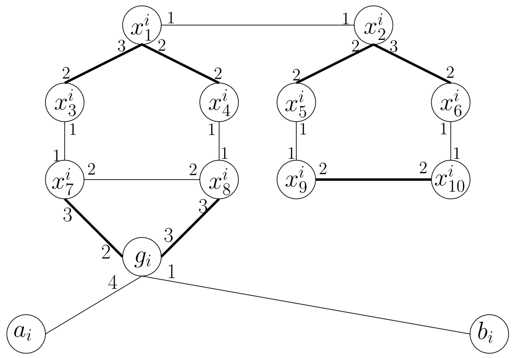

Fig. 5. An illustration for the edge gadgets in Theorem 3.6. The bold edges are the corresponding edges of the only possible Pareto-optimal complete matching.

Proof. During the algorithm, each edge is checked at most once. Also, deciding if there exists a maximum size matching containing a given set of edges can be done in polynomial time. Indeed, to decide if there is a maximum size matching containing a feasible edge set N, we just have to delete these edges from the graph, decrease each vertex capacity by the number of adjacent edges it had in N, and check whether the new graph has a matching of size $\eta - |N|$, where η is the size of the maximum size matching in the original graph. So the running time of the algorithm is polynomial.

We show that there is not even a matching M', that is weakly better for every man than the output matching M, and strictly better for at least one man u_j. We say that such a matching Pareto-dominates M for the men. This is enough to show that M is Pareto optimal, because if a matching M' Pareto-dominates it, then at least one agent from both sides must get a different partner set, hence at least one man must strictly improve. Suppose for the contrary that there is a matching M', where each man obtains an at least as good partner set as in the output matching M and there is a man u_j, who obtains a strictly better set. It is clear that no maximum size matching M' can be weakly better for all men and strictly better for at least one, because among the maximum size matchings, the algorithm chose one that is best possible for u_1, and with respect to that, best possible for u_2, etc. Hence, if all of $u_1, \dots, u_n$ weakly improves in a maximum size matching M', then $M(u_1) = M'(u_1)$ and so $M(u_2) = M'(u_2)$, and so $M(u_i) = M'(u_i)$ for $i \in [n]$, meaning that $M = M'$.

Let the best partner of u_j in $M' \setminus M$ be $w(u_j)$. If $w(u_j)$ is not saturated in M, then u_j has to be saturated by the maximality of M. But then, there is an edge $u_j w_l \in M \setminus M'$, so letting $M'' = M \cup u_j w(u_j) \setminus u_j w_l$, we obtain a maximum size matching, where each man has an at least as good situation and u_j is strictly better off, contradicting our above observation.

So for each man u_j, who is better off in M', their best partner in $M' \setminus M$ $w(u_j)$ is saturated in M. Hence, from each such woman, there is an edge that is in $M \setminus M'$. Therefore, starting from u_j and then, from each man going to their best partner in $M' \setminus M$ and from each woman, going to an arbitrary partner in $M \setminus M'$, we can find a cycle C that alternates between M and M'. Each man strictly prefers $M' \cap C$ to $M \cap C$ in this cycle C, because as M' Pareto-dominates M on the men's side, the best partner a man u_i gets in $M' \setminus M$ (which he gets in $M' \cap C$ too) is strictly better than the best partner in $M \setminus M'$ that they lose in M', by the lexicographicality of the preferences. Thus letting $M'' = M \cup (M' \cap C) \setminus (M \cap C)$, we obtain a maximum size matching, that is weakly better than M for each man, and strictly better for at least one, contradiction. □

On the negative side, we show that the same problem becomes NP-hard in the fixtures case.

Theorem 3.6. *Deciding whether there exists a complete Pareto-optimal matching for the stable fixtures problem under lexicographic preferences is NP-hard, even if each capacity is at most 4.*

Proof. Again we reduce from the problem EXACT-3-COVER. We use almost the same construction as in Theorem 3.3 with the only difference that here we substitute each $a_i b_i$ and $a_i b_{i-1}$ edge with an edge-gadget G_i and H_i respectively, illustrated in Fig. 5. Every gadget G_i and H_i are essentially just a copy of Example 3 in Section 2, illustrated in Fig. 1, only we add a special agent g_i or h_i respectively. We will denote the agents corresponding to $x_1, \dots, x_{10}$ by $x_1^i, \dots, x_{10}^i$ in G_i and by $y_1^i, \dots, y_{10}^i$ in H_i. An agent g_i has capacity 2 and preference $b_i > x_7^i > x_8^i > a_i$ and is added to the end of the preference lists of both x_7^i and x_8^i. An agent h_i has capacity 2 and preference $a_i > y_7^i > y_8^i > b_{i-1}$ and is added to the end of the preference lists of both y_7^i and y_8^i. Finally, we substitute b_i and b_{i-1} in the preference list of a_i by g_i and h_i, respectively, and similarly substitute a_i and a_{i+1} in b_i's preference list by g_i and h_{i+1}, respectively, for each $i = 1, \dots, 3n$. So the preference list of a_i is $g_i > q_i > d_{\lfloor (i+2)/3 \rfloor} > h_i$ and the preference list of b_i is $h_{i+1} > p_i > c_{\lfloor (i+2)/3 \rfloor} > g_i$.

Suppose that there is a complete Pareto-optimal matching M in this instance and suppose that there is an index i such that $a_i g_i$ and $g_i b_i$ are in M. Then the matching M restricted to the set of vertices $\{x_1^i, \dots, x_{10}^i\}$ has to be $\{x_1^i x_3^i, x_1^i x_4^i, x_2^i x_5^i, x_2^i x_6^i, x_7^i x_8^i, x_9^i x_{10}^i\}$ by the completeness of M, but then if we give each agent apart from $x_1^i, .., x_{10}^i$ the same partners and match $x_1^i, ... x_{10}^i$ such that each gets only their favourite, then we obtain a matching M' that Pareto-dominates M, a contradiction.

Similarly, if $a_i h_i$ and $h_i b_{i-1}$ are in M for some i, then M cannot be Pareto-optimal either, a contradiction.

Since a gadget G_i or H_i can be saturated only if g_i or h_i is matched to 0 or 2 agents in them, we obtain that there can be no edge in M that connects an agent a_i or b_i to an agent g_j or h_j. But M is a complete matching, hence each agent in $A \cup B$ is saturated, so all edges between A and $Q \cup D$ have to be included and also all edges between B and $C \cup P$. Then, since every agent in $T \cup S$ is saturated too, all edges between U and T and all edges between S and V are also included M, so M is basically the same matching that we constructed in Theorem 3.3, with some additional edges inside the edge gadgets.

Now if there would be an exact 3-cover in I, then we could construct a matching M' that Pareto-dominates M, implying that there can be no complete Pareto-optimal matching in the same way as before, with the addition that the agents in $A \cup B$ obtain their partners in $\bigcup_i \{g_i\} \cup \bigcup_i \{h_i\}$ instead of each other. In this way each agent of forms g_j and h_j obtains their best partner, so they are strictly better off, too. Finally, we also match each agent in the remaining parts of $\bigcup_i \{G_i\} \cup \bigcup_i \{H_i\}$ to their best choices. So the existence of an exact 3-cover implies that no complete Pareto-optimal matching exists.

In the other direction if there is no complete Pareto-optimal matching, then the matching M constructed above is not Pareto-optimal, so it is dominated by a matching M'. Again, the same proof works to show that there has to be a 3-cover of the original instance. The only additional thing we have to check in this case is that, if any agent from a gadget G_i or H_i improves their position, then so does every agent in $A \cup B \cup C \cup D$. But this is only possible if she gets her first choice in M', which implies that every agent in the gadget obtains its best partner, so g_i or h_i improves their position too, which leads to every edge between $A \cup B$ and $\bigcup_i \{g_i\} \cup \bigcup_i \{h_i\}$ being included in M', so the proof is complete. □

4. Relaxed strong core solutions

The NP-hardness of finding matchings in the strong core motivates us to relax the problem in some ways to make the problem tractable, and also to guarantee the existence of a desired solution.

We give two algorithms, that are heavily inspired by the Top Trading Cycle (TTC) algorithm of Gale (Shapley and Scarf, 1974). The first algorithm computes a matching M, such that M violates the original capacity constraints by at most one, but is guaranteed to be a strong core solution for this slightly modified instance. We call such a solution a *near-feasible strong core solution*. The second algorithm computes a fractional matching M, that is guaranteed to be in the strong core of fractional solutions of the original instance.

The algorithms described here not only work for the stable many-to-many matching case, but also for the non-bipartite stable fixtures problem. Moreover, both algorithms run in quadratic time in the number of edges.

The main idea of the algorithms is very simple: in each step, we create a directed graph $D_i = (V_i, A_i)$, such that the vertices of D_i are the agents who have remaining capacities at the i-th iteration, and there is a directed edge from a to b if b is a's best choice from the vertices of D_i who are not yet matched to a. Then we search for a directed cycle C_i in D_i and add the edges of C_i to the matching.

4.1. Near feasible solutions

Now we describe the algorithm for finding a near feasible matching that is in the strong core of the modified instance formally. Let $p_U^M(v)$ denote the best agent in v's preference list among the agents in U who are not matched to v in M. Let $k(v)$ denote the capacity of v. We will denote the remaining capacity of v by $k^r(v)$. Also we use $E(C_i)$ as the edges corresponding to the directed edges of $A(C_i)$ in the original graph G.

Algorithm 2 Near-feasible strong core matchings.

Set $M = \emptyset$, $i = 0$, $k^r(v) = k(v)$
$V_0 = N$, $A_0 = \{v p_V^M(v) :\ v \in V_0\}$,
while $A_i \neq \emptyset$ **do**
  Find a directed cycle C_i in $D_i = (V_i, A_i)$.
  For each $e \in E(C_i)$: $M := M \cup e$.
  if $|C_i| = 2$ **then**
    For each $v \in V(C_i)$: $k^r(v) = k^r(v) - 1$
  else
    For each $v \in V(C_i)$: $k^r(v) = k^r(v) - 2$
  end if
  $V_{i+1} = \{v \in N :\ k^r(v) \geq 1\}$
  $A_{i+1} = \{v p_{V_{i+1}}^M(v) :\ v \in V_{i+1}\}$
  $i = i + 1$
end while

Theorem 4.1. *Algorithm 2 produces a matching M in $\mathcal{O}(|E|^2)$ time for the stable fixtures problem that is in the strong core of the instance with modified capacities $k'(v)$, where $k'(v) = \max\{k(v), |M(v)|\} \leq k(v) + 1$.*

Proof. In each iteration we add at least one edge to M, so the algorithm terminates in at most $|E|$ iterations, each can be done in $\mathcal{O}(|E|)$ steps.

Also, we only add at most two edges containing a given vertex v in one step and only to vertices with $k^r(v) \geq 1$, so $|M(v)| \leq k(v) + 1$.

Finally, we show that M is in the strong core of this new instance. First of all it is easy to see, that if we run the algorithm with these new capacities we get the same output M, so we can suppose that the algorithm never violates the capacity constraints during its execution.

Suppose for a contradiction that there is a blocking coalition $\mathcal{P}$ for M and let $M'_{\mathcal{P}}$ be the matching for the vertices in $\mathcal{P}$ that blocks M. Let C_i be the first cycle that contains an edge that is not in $M'_{\mathcal{P}}$, but contains a vertex of $\mathcal{P}$ and let that vertex be u. Since $M'_{\mathcal{P}} \not\subset M$, such a cycle exists. Then, by the fact that u must have an at least as good partner set it $M'_{\mathcal{P}}$ we get that the edge corresponding to the arc uw starting from u in C_i is in $M'_{\mathcal{P}}$. (u cannot get a better partner than $w = p_{V_i}^{M_i}(u)$ that she did not already have in M, because then there would be a cycle C_j before C_i that contains a vertex in $\mathcal{P}$ but not every edge of $E(C_j)$ is in $M'_{\mathcal{P}}$, a contradiction). This also means that $w \in \mathcal{P}$, and by similar reasoning, the edge corresponding to the arc starting from w is in $M'_{\mathcal{P}}$, too, and continuing this argument we get that $E(C_i) \subset M'_{\mathcal{P}}$, a contradiction. □

4.2. Fractional strong core matchings

We describe the algorithm that finds a fractional matching that is in the strong core of fractional matchings. The notation is the same, and the algorithm itself is also very similar to Algorithm 2. The main idea of the algorithm is that in each round, everybody with some remaining capacity points to his favourite partner who also has remaining capacity such that the edge between them has value less than 1 in f^M at the moment. Then, we find a cycle in this graph, and add each edge with the same weight with as large weight as possible (to remain feasible).

For the sake of generality, here we assume that each vertex capacity $k(v)$ can be an arbitrary nonnegative real number instead of being integer. We also suppose that each edge has a nonnegative real capacity $k(e)$ (previously this was assumed to be 1). A fractional matching f^M is called feasible in this case, if $f^M(e) \le k(e)$ also holds for each $e \in E$. Let $p_U^M(v)$ denote the best agent in v's preference list among the agents in U who are not matched with v with full weight (so with weight $k(e)$) in f^M.

Algorithm 3 Fractional strong core.

Set $f_0^M \equiv 0$, $i = 0$.
Set $k^r(v) = k(v)$ ($\forall v \in N$)
Set $k^r(e) = k(e)$ ($\forall e \in E$).
Let $V_0 = \{v \in N : k(v) > 0\}$, $A_0 = \{vp_V^M(v) : v \in V_0\}$,
while $A_i \neq \emptyset$ **do**
  Find a directed cycle C_i in $D_i = (V_i, A_i)$.
  Let $f^{C_i}(e) = 1$, if $e \in E(C_i)$, $f^{C_i}(e) = 0$, if $e \notin E(C_i)$.
  Let $\varepsilon_i := \max\{\delta : f_i^M(e) + \delta f^{C_i}(e)$ is feasible$\}$
  $f_{i+1}^M = f_i^M + \varepsilon_i f^{C_i}$.
  if $|C_i| = 2$: **then**
    For each $v \in V(C_i)$: $k^r(v) = k^r(v) - \varepsilon_i$
    For each $e \in E(C_i)$: $k^r(e) = k^r(e) - \varepsilon_i$.
  else
    For each $v \in V(C_i)$: $k^r(v) = k^r(v) - 2\varepsilon_i$
    For each $e \in E(C_i)$: $k^r(e) = k^r(e) - \varepsilon_i$.
  end if
  $V_{i+1} = \{v \in N : k^r(v) > 0\}$
  $A_{i+1} = \{vp_{V_{i+1}}^M(v) : v \in V_{i+1}\}$
  $i = i + 1$
end while
$f^M = f_i^M$

Theorem 4.2. *For the stable fixtures problem, Algorithm 3 produces a fractional matching f^M in $\mathcal{O}(|E|^2)$ time that is in the strong core of fractional matchings.*

Proof. The running time is $\mathcal{O}(|E|^2)$, because in each iteration of the while loop, a vertex or an edge becomes saturated, as we choose the maximum possible value for ε_i (in the case of an edge this means $f^M(e) = k(e)$); and each iteration can be done in $\mathcal{O}(|E|)$ time.

The capacity constraints are obviously satisfied during the algorithm, and so is $f^M(e) \le k(e)$ ($\forall e \in E$) by the choice of ε_i.

We show that the algorithm returns a fractional matching in the strong core even in the case when the agents and the edges can have arbitrary fractional capacities too. We prove this statement by induction on the number t of iterations of the algorithm. Suppose the number of iterations is 0. Then, any edge induced by V_0 has capacity 0, so the only feasible matching is $f^M \equiv 0$ and hence the output is in the strong core.

Let I be an instance and suppose now that the number of iterations is $t \ge 1$ and suppose that the statement holds for $t-1$. Let the first cycle that the algorithm finds be C_1. If we decrease the vertex and edge capacities the same way as the algorithm in the first iteration, then in the obtained instance I', the algorithm runs exactly the same way as it does in I after the first iteration. Also, the number of iterations when run on I' is strictly smaller, so the fractional matching g^M that the algorithm outputs for I' is in the strong core of I' by induction. Also, it holds that $f^M = g^M + \varepsilon_1 f^{C_1}$.

Suppose now that there is a blocking coalition $\mathcal{P}$ for f^M with a fractional matching $f^M_{\mathcal{P}}$. We can suppose that $f^M_{\mathcal{P}}(uv) > 0$ only if $u, v \in \mathcal{P}$. If there is no agent $u \in V(C_1) \cap \mathcal{P}$, then each agent in $\mathcal{P}$ and each edge with positive value in $f^M_{\mathcal{P}}$ has the same capacity in I and I', so $f^M_{\mathcal{P}}$ is also feasible in I'. But, as $f^M \geq g^M$, we obtain that $\mathcal{P}$ blocks g^M with $f^M_{\mathcal{P}}$, contradicting that g^M is in the strong core of I'.

Hence there is an agent $u_1 \in V(C_1) \cap \mathcal{P}$. As u_1 points to his favourite partner u_2 in C_1, by the lexicographical preferences we obtain that $f^M_{\mathcal{P}}(u_1u_2) \geq \varepsilon_1$. In particular, $u_2 \in \mathcal{P}$ and by similar reasoning, $f^M_{\mathcal{P}}(u_2u_3) \geq \varepsilon_1$, where u_3 is u_2's favourite partner. Iterating this argument, we get that $V(C_1) \subset \mathcal{P}$ and $f^M_{\mathcal{P}}(e) \geq \varepsilon_1$ for each $e \in E(C_1)$. Therefore, $g^M_{\mathcal{P}} = f^M_{\mathcal{P}} - \varepsilon_1 f^{C_1}$ is a feasible fractional matching in I'. As $g^M = f^M - \varepsilon_1 f^{C_1}$, we get that $\mathcal{P}$ blocks g^M with the fractional matching $g^M_{\mathcal{P}}$ in I', contradiction. □

5. The weak core

In this section we consider another relaxation of the strong core, the weak core.[3] We settle most of the corresponding complexity questions, we only leave open the complexity of finding a weak core matching in the stable many-to-many matching problem.

We note the concept of weak core might be too broad to be considered as the sole criterion, because it allows rather sub-optimal solutions. As a very simple example take three agents a, b and c, where a and c have capacity one and b has capacity two, and the possible pairs are ab and bc, where b prefers a to c. In this example the unique strong core, Pareto-optimal and stable solution is to take both edges. However, edge ab alone is also in the weak core, since bc is not blocking and the grand coalition does not block either, since a would not strictly improve. Therefore a decision maker may want to add some additional requirements when selecting a solution from the weak core.

5.1. Finding weak core solutions

In the fixtures case, it is easy to see that the weak core can also be empty, just consider three agents with cyclic preferences and unit capacities. By using such a no-instance, we are able to prove the NP-hardness of finding a strong core solution in a stable fixtures problem in a similar fashion as the strong core.

Theorem 5.1. *It is NP-hard to decide whether an instance of the stable fixtures problem under lexicographic preferences admits a weak core solution.*

Proof. We use the same construction as in Theorem 3.2, reducing from COM-SMTI. We only change the gadget G_i for the women $w_i \in W^t$ and the gadget G.

Let the gadget G_i now only consist of two agents w'_i with capacity 2 and a dummy man u^*_i with capacity one, who is first choice for w'_i. w'_i ranks the two nodes corresponding to her original neighbours in an arbitrary strict way.

Let the gadget G consist of a triangle with agents $\{a, b, c\}$ such that $a >_b c$, $b >_c a$, and $c >_a b$. (Clearly, G has no weak core solution). Again, the special agent g is added to the top of a's preference list, while a is added to the end of g's one.

Suppose that there is a complete stable matching M in I. We simply extend M to M' by adding the edges $w'_iu^*_i$ for each $w_i \in W^t$ and the edges ag, bc. We claim that M' is in the weak core. Let $\mathcal{P}$ be a strictly blocking coalition. If no w'_i agent is in $\mathcal{P}$ with capacity 2, then we can suppose that $\mathcal{P}$ consists of two agents with capacity one, who mutually prefer each other to their partner. But there is no blocking edge between the agents with capacity one in M', as M was stable and complete. Hence, there is an agent w'_i in $\mathcal{P}$. Then, w'_i must keep her best partner u^*_i, and u^*_i cannot strictly improve, contradiction.

For the other direction, suppose that there is a weak core solution M' in I'. Then, g must be matched with a, otherwise $\{a, b\}, \{a, c\}$ or $\{b, c\}$ form a strictly blocking coalition. Hence, no gu'_j edge can be a blocking edge, so each man u'_j receives a partner w'_i in M'. We claim that these edges form a complete and stable matching M in I.

Suppose that two men u'_j, u'_k both receive the same partner w'_i. Then, w'_i has capacity 2 and is not matched to u^*_i. Hence, $\{w'_i, u^*_i\}$ form a strictly blocking coalition, contradiction. We conclude that M is indeed a complete matching. Suppose the edge u_jw_i blocks M. Then, $\{u_j, w_i\}$ is a strictly blocking coalition for M', a contradiction. □

5.2. Verifying weak core solutions

Finally, we also consider the verification problem related to the weak core. In this case, the problem is hard even in the many-to-many version. For the basis of our hardness result, we first show that verifying if a matching is weakly Pareto-optimal is also coNP-hard in the stable many-to-many matching problem.

Theorem 5.2. *It is coNP-complete to decide whether a matching M is weakly Pareto optimal in the stable many-to-many matching problem. Also, it is NP-hard to find a maximum size weakly Pareto optimal matching for the stable fixtures problem.*

[3] We thank one of our anonymous Referees to suggest this problem variant.

Proof. For the verification problem, containment in coNP is trivial, as it is easy to verify if a matching M' dominates M.

We start by showing hardness of the verification problem. To show hardness, we reduce from EXACT-3-COVER. Let I be an instance of EXACT-3-COVER with sets $\mathcal{Y} = \{Y_1, \dots, Y_m\}$ and elements $X = \{x_1, \dots, x_{3n}\}$. As previously discussed, we may assume that $m = 3n$ and that each element appears in exactly 3 sets.

For each set $Y_j \in \mathcal{Y}$ in I, we create agents s_j and t_j with capacity 4 along with agents $u_j^1, \dots, u_j^4, v_j^1 \dots, v_j^4$ with capacity 2 and agents $w_j^1, \dots, w_j^4, z_j^1 \dots, z_j^4$ with capacity 1.

For each element $x_i \in X$, we add agents p_i and q_i each with capacity 1.

Finally, we create $2(m-n)$ agents a_l, b_l for $l \in [m-n]$ with capacity 1 and $2n$ agents c_l and d_l for $l \in [n]$ with capacity 1.

The preferences are as follows.

$$
\begin{array}{rl|rl}
p_i: & [T_i] & q_i: & [S_i] \\
s_j: & [D] > v_j^1 > v_j^2 > v_j^3 > [B] > v_j^4 > [Q_j] & t_j: & [C] > u_j^1 > u_j^2 > u_j^3 > [A] > u_j^4 > [P_j] \\
c_l: & [T] & d_l: & [S] \\
u_j^l: & z_j^l > v_j^l > t_j & v_j^l: & w_j^l > u_j^l > s_j \\
w_j^l: & v_j^l & z_j^l: & u_j^l \\
a_l: & [T] & b_l: & [S]
\end{array}
$$

Here, $T = \{t_j \mid j \in [m]\}$, $S = \{s_j \mid j \in [m]\}$, T_i (S_i) denotes the t_j (s_j) agents such that $x_i \in Y_j$, $C = \{c_l \mid l \in [n]\}$, $D = \{d_l \mid l \in [n]\}$ and Q_j (P_j) denotes the set of q_i (p_i) agents such that $x_i \in Y_j$. Furthermore, for an indexed set A, $[A]$ means that the elements of A are ranked in the order of their indices.

So far the constructed instance is bipartite. Let the matching M consist of the edges $\{s_j v_j^l, t_j u_j^l, u_j^l v_j^l \mid j \in [m], l \in [4]\}$. We claim that there is matching M', where every agent strictly improves, if and only if I admits an exact 3-cover.

Suppose first that there is an exact 3-cover $\mathcal{Y}' \subset \mathcal{Y}$. Let $\mathcal{Y}' = \{Y_{j_1}, \dots, Y_{j_n}\}$. We create a matching M' that strictly dominates M. For each $Y_{j_k} \in \mathcal{Y}'$, we add the three edges between s_{j_k} and Q_{j_k} along with an edge between s_{j_k} and d_k, as well as the three edges between t_{j_k} and P_{j_k} along an edge between t_{j_k} and c_k. Then, all agents in P, Q, C and D get matched. For the rest of the sets $\{Y_{l_1}, \dots, Y_{l_{m-n}}\}$, we match s_{l_i} to $v_{l_i}^1, v_{l_i}^2, v_{l_i}^3$ and b_i and match t_{l_i} to $u_{l_i}^1, u_{l_i}^2, u_{l_i}^3$ and a_i. Finally, we add the edges $\{u_j^l z_j^l, v_j^l w_j^l \mid j \in [m], l \in [4]\}$. Then, all agents from $A = \{a_l \mid l \in [m-n]\}, B = \{b_l \mid l \in [m-n]\}, W = \{w_l^i \mid l \in [m], i \in [4]\}, Z = \{z_l^i \mid l \in [m], i \in [4]\}$ get matched too. Then, it is straightforward to verify that each agent strictly improves and that M' is feasible.

For the other direction, suppose that there is a matching M', where each agent strictly improves. Then, all agents in Q must get matched in M'. If an agent q_i gets matched to an agent s_j, then s_j must get a partner from D to improve. Otherwise, s_j must keep v_j^1, v_j^2, v_j^3, but she also has q_i now, so no other agents are matched to her, so she does not improve, contradiction. As $|D| = n$, we get that all q_i agents get matched to at most n s_j agents, which is only possible if the sets corresponding to those s_j agents form an exact 3-cover.

Now, we extend the constructed instance I' to an instance I'' to show NP-hardness of deciding whether there is a complete weakly Pareto optimal matching. For this, we add special agents g, h with capacity $3n + 5m$ along with agents g', h' with capacity 1 and agents r_1, r_2 with capacity 2 and agents r_3, r_4 with capacity 1. Their preferences are the following.

$$
\begin{array}{rl}
g: & g' > [Z] > [B] > [D] > [Q] \\
h: & h' > [W] > [A] > [C] > [P] \\
g': & g > r_1 \\
h': & h > r_1 \\
r_1: & r_2 > h' > g' \\
r_2: & r_1 > r_3 > r_4 \\
r_3: & r_4 > r_2 \\
r_4: & r_3 > r_2
\end{array}
$$

Furthermore, we add g as the worst acceptable partner to each agent in $Z \cup B \cup D \cup Q$ and h as the worst partner for each agent in $W \cup A \cup C \cup P$.

We illustrate the construction in Fig. 6.

With the help of this additional gadget, in the unique maximum size matching M, where every agent is saturated, r_2 is matched to r_3 and r_4, hence r_1 is matched to g' and h', hence g is matched to all agents in $Z \cup B \cup D \cup Q$ and h is matched to all agents in $W \cup A \cup C \cup P$, who have capacity one, so they have no other partner. Hence, the edges $\{s_j v_j^l, t_j u_j^l, u_j^l v_j^l \mid j \in [m], l \in [4]\}$ are also in M as this is the only way for the other agents to be saturated. So if we restrict the matching to the vertices of I', we exactly get back the same matching as before.

Now, if there is an exact 3-cover, then we can make a matching M' just as before, with the addition that we add the edges $gg', hh', r_1 r_2, r_3 r_4$ to M' to ensure that all newly added agents also strictly improve.

Also, if there is a matching M' strictly dominating M, then again, each agent q_i must get matched to agents s_j, who then must get an agent form D. So there can be at most n such agents and their corresponding sets must form an exact 3-cover. $\square$

Now we are ready to extend the hardness result to weak core verification.

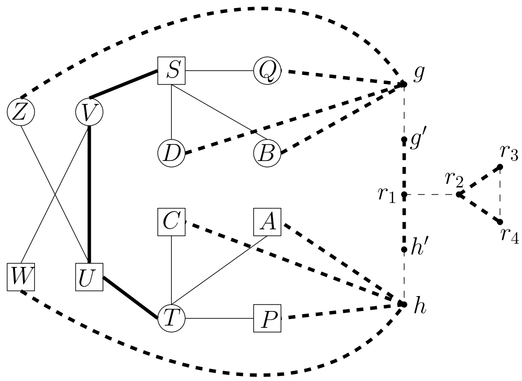

Fig. 6. An illustration for Theorem 5.2. The small nodes are the ones we add for the second construction along with the dashed edges. Bold edges are the corresponding edges of M (a bold edge between two sets means that each edge between the two sets is in M).

Theorem 5.3. *It is coNP-complete to decide if a matching M is in the weak core under lexicographic preferences.*

Proof. In Theorem 5.2 we showed that verifying whether a matching is weakly Pareto-optimal is coNP-complete even for the many-to-many case. Let I be an instance of that problem with bipartite graph $G = (U, W, E)$ and a matching M.

We create an instance I' of weak core verification as follows. Keep each agent and edge from I. For each agent in $U \cup W$ we increase the capacity by one. Furthermore, we add two super agents a to W and b to U. So let $U' = U \cup \{a\}$, $W' = W \cup \{b\}$. Add a as the best partner for each agent in U and b as the best partner for each agent in W. The preferences of a and b are such that they consider each other the worst, but otherwise their preferences are arbitrary (for example they rank the agents according to their indices).

Let the capacity of a be $|W'|$ and the capacity of b be $|U'|$. Create a matching M' by keeping all edges of M and add $\{au \mid u \in U\}$ and $\{bw \mid w \in W\}$. As each agent in $U \cup W$ increased capacity, M' is feasible.

We claim that M' is in the weak core if and only if M is weakly Pareto-optimal.

Suppose that M is not weakly Pareto optimal in I. Then, there is a matching N, where each agent strictly improves. Extend N to N' by adding the edges $\{au \mid u \in U\}$, $\{bw \mid w \in W\}$ and ab. Then, each agent in $U \cup W$ still improves from M' to N' and a and b also improve, as they received a new partner (each other). Hence M' is not in the weak core, the grand coalition strictly blocks.

Suppose that M' is not in the weak core. Let $\mathcal{P}$ be a blocking coalition. We claim that $\mathcal{P} = U' \cup W'$. Indeed, if any $u \in U$ is in $\mathcal{P}$, then they must keep their best partner b. If $a \in \mathcal{P}$, then she also must get b to strictly improve, as she received all other partners in M' and b is the worst. Hence $b \in \mathcal{P}$ necessarily. Agent b can only strictly improve by getting each $u \in U$ and her worst partner a too (she obtained all other partners in M'). So $U' \subset \mathcal{P}$. Similarly for a we get that $W' \subset \mathcal{P}$.

This means that there is a matching N', where each agent in $U \cup W$ strictly improves. The edges $\{au \mid u \in U\}$, $\{bw \mid w \in W\}$ must be in N', so N' restricted to I is a matching where each agent strictly improves from M, hence M is not weakly Pareto-optimal. $\square$

6. Reverse-lexicographic preferences

In this section we study another simple preference structure that we call reverse-lexicographical preferences (RL-preferences for short). The motivation here is that in some situations, the agents may want to have as many partners as possible (for example a company aims to have all its positions filled) but they also do not want to receive very bad partners, e.g. they want their worst partner to be as good as possible.

Another motivation is that strong core solutions fail to always exists even in the many-to-many settings, but we will show that RL-preferences behave nicer in several aspects, and here a strong core solution is always guaranteed and can be found for the many-to-many variant. It also behaves nicer for Pareto-optimal solutions: we show that, as opposed to lexicographic preferences, under RL-preferences we can find maximum size Pareto-optimal solutions even in the stable fixtures problem.

6.1. Finding strong core solutions

In this section we investigate the search problems related to the strong core. We start by proving that the strong core of a stable many-to-many matching problem is never empty if we assume RL preferences. For this, we first show that stable matchings always form a subset of the strong core.

Theorem 6.1. *For reverse-lexicographic preferences, if a matching M is stable then it is also in the strong core.*

Proof. Let M be a stable matching. Suppose for a contradiction that there is a weakly blocking coalition $\mathcal{P}$ for M. Let $M'_{\mathcal{P}}$ denote the matching of the agents contained in $\mathcal{P}$, that each of them weakly prefers, and at least one strictly prefers to M.

Since it must hold that $M'_{\mathcal{P}}$ is not contained in M, we get that there is an edge $uw \in M'_{\mathcal{P}} \setminus M$. We will show that uv is a blocking pair for M. If u was saturated in M, then so is she in $M'_{\mathcal{P}}$, and by the definition of the RL-preferences, it must hold that w is better for u, than the worst agent in $M'_{\mathcal{P}}$, according to the order $>_u$. Similar argument for w shows that either w was unsaturated by M, or u is preferred by w to the worst agent in $M(w)$. So both u and w are either unsaturated, or they have a partner in M that is worse, so uw is a blocking edge, a contradiction. □

By using the Gale-Shapley algorithm to find a stable matching in linear time, we get the following result immediately.

Theorem 6.2. *For the stable many-to-many matching problems with RL-preferences, the strong core is always nonempty, and strong core solutions can be found in* $\mathcal{O}(|E|)$ *time.*

However, as the following example illustrates, the strong core can be strictly larger than the set of stable solutions.

Example 4

We have three agents on both sides of a many-to-many market with capacity 2 each and the following preferences.

$$\begin{array}{lll|lll} a: & & z > y & x: & & c > b \\ b: & & x > y > z & y: & & a > b > c \\ c: & & y > x & z: & & b > a \end{array}$$

Here matching $M = \{ay, az, bx, bz, cx, cy\}$ is a strong core solution under RL-preferences, but not stable, since by is a blocking pair.

Although the stable matchings will always be a part of the strong core, the set of strong core solutions can form a strictly larger set. Moreover, the existence problem becomes NP-hard for the fixtures variant, as stated in the next theorem.

Theorem 6.3. *Deciding whether a given stable fixtures problem with RL-preferences admits a strong core solution is NP-hard, even if each capacity is at most 2.*

Proof. We reduce from the NP-complete COM-SMTI, as in the proof of Theorem 3.2. Let I be an instance of COM-SMTI, let the set of men be $U = \{u_1, \dots, u_n\}$, and the set of women be $W = \{w_1, \dots, w_m\}$, where $W^t = \{w_1, \dots, w_l\}$ is the set of women, whose preference list is a single tie of length exactly 2, and $W^s = \{w_{l+1}, \dots, w_n\}$ is the set of women with strict preferences.

We create an instance I' with $G' = (V', E')$ of our problem as follows. For each woman $w_i \in W^t$, we create a gadget G_i, with the same preferences and capacities as in the proof of Theorem 3.2. For each woman $w_i \in W^s$, we create a single agent w'_i with capacity 1. For each man $u_i \in U$, we create an agent u'_i with capacity 1. We also add a special agent x with capacity 1, and 3 more agents s_1, s_2, s_3 again with capacity 1.

The agents in the G_i gadgets have the same preferences as in the proof of Theorem 3.2. The w'_i agents for $w_i \in W^s$ have the same preferences as w_i, just on the u'_j agents instead of the u_j-s. The u'_j agents also have the same preferences as u_j, with the modification that agent x is added to the end of their preference list. Agent x has preference $u_1 >_x u_2 >_x \cdots >_x u_n >_x s_1$, agent s_1 has preference $x >_{s_1} s_2 >_{s_1} s_3$, s_2 has preference $s_3 >_{s_2} > s_1$ and s_3 has preference $s_1 >_{s_3} s_2$. The edges correspond exactly to the mutual acceptability relations according to the preference lists.

We show that the strong core is nonempty in I' if and only if there is a complete weakly stable matching in I.

First suppose that there is a complete weakly stable matching M in I. We create a matching M' in I' as follows. For each edge $u_j w_i \in M$, add the edge $u'_j w'_i$ or $u'_j w''_i$ to M'. For each $i = 1, \dots, l$ add the edges $c_i w'_i$ and $c_i wi''$ to M'. Then, for each pair $\{w'_i, w''_i\}$, $i = 1, \dots, l$ if w_i is matched to u_j, then add the edge $d_i w''_i$, otherwise add the edge $d_i w'_i$. Finally, add the edges xs_1 and $s_2 s_3$ to M'.

Suppose for a contradiction that there is a blocking coalition $\mathcal{P}$ with matching $M''_{\mathcal{P}}$ for M'. If there is an agent $v_i \in \{c_i, d_i, w'_i, w''_i\}$ for some $i = 1, .., l$ in U, then a straightforward argument shows that all of $\{c_i, d_i, w'_i, w''_i\}$ must be in $\mathcal{P}$ as every agent must be saturated in $M''_{\mathcal{P}}$ because they were saturated in M'. This also implies that they must get the same set of partners in $M''_{\mathcal{P}}$ as in M', since a strict improvement of one of them would cause another to be strictly worse off. Therefore, we can suppose that $\mathcal{P}$ does not contain any such agent, since if it does, then by removing all four of them along with their partners we would still get a blocking coalition.

Observe that all other agents have capacity one, therefore $\mathcal{P}$ only contains agents with capacity 1, so $M''_{\mathcal{P}}$ is a union of vertex disjoint edges. Therefore, there must be a single edge such that its two agents still form a blocking coalition. This edge cannot be of the form $u'_j w'_i$, since such an edge would contradict the weak stability of M, because for capacity 1 agents, the RL-ranking is the same as their linear ranking over their neighbours.

The only other possibilities are $u'_j x$ for some j, $s_1 s_2$ or $s_3 s_1$. It is straightforward to verify that none of them form a blocking coalition, as each u'_j is matched in M' to someone better than x and s_1 is with her best possible partner.

For the other direction, suppose that there is a strong core solution M' in I'. We show that M' must induce a complete weakly stable matching by taking the $u_j w_i$ edges such that $u'_j w'_i$ or $u'_j w''_i$ is contained in M. First suppose that there is a man u_j that does not get a partner this way. Then, since $\{u_j, x\}$ did not form a blocking coalition, x must not be matched to s_1 in M'. Hence, the agents s_1, s_2, s_3 must be matched among themselves, but then each of the 3 possible matchings would be strictly blocked by two agents of them. Therefore, each u_j gets a partner in M'. Suppose that u_j and u_l get the same partner. Then, $u'_j w'_i$ and $u'_l w''_i$ are contained in M'. However this would imply that $\{c_i, d_i, w'_i, w''_i, u'_j\}$ or $\{c_i, d_i, w'_i, w''_i, u'_l\}$ would form a blocking coalition, as w'_i or w''_i could switch to c_i or d_i and strictly improve without any other member of the coalition getting a worse partner set.

This implies, that M is a complete matching. Hence, it only remains to show that it is weakly stable. Suppose there is a blocking edge $u_j w_i$. This means that $w_i \in W^s$. But then, $\{u'_j, w'_i\}$ would form a blocking coalition for M', a contradiction. □

6.2. *Verifying strong core and Pareto-optimal solutions*

Next we turn our attention to the verification problem. We show that verifying if a matching M is in the strong core is NP-complete even for the many-to-many case. This also implies, that the set of strong core solutions can strictly contain the set of stable matchings even in this framework.

Theorem 6.4. *Verifying whether a matching M in a stable many-to-many matching problem with RL-preferences is in the strong core is coNP-complete even if each capacity is at most 5.*

Proof. The problem is in coNP, as given a blocking coalition and a matching M', we can check whether each agent weakly improves and at least one strictly improves, or not.

We reduce from EXACT-3-COVER. Take an instance $I = (\mathcal{Y} = \{Y_1, \dots, Y_m\}, X = (x_1, \dots, x_{3n}))$ of EXACT-3-COVER. We can also suppose that each element is contained in exactly three sets, which will be important, so $m = 3n$. The NP-completeness of this restricted version of EXACT-3-COVER was first shown by Hein et al. (1996), but only stated explicitly in Hickey et al. (2008). We create an instance I' of stable many-to-many matching problem, as follows.

For each set Y_j we add a set-agent s_j with capacity 3. Also, we add a local gadget for s_j consisting of four agents v_j^1, v_j^2, v_j^3 with capacity 2, and u_j with capacity 3. This will ensure that a set-agent must get all 3 element-agents to improve.

For each element x_i, we add an element-agent q_i with capacity 5 and four connector agents: p_i^1, p_i^2, that have capacity 4 and r_i^1, r_i^2 that have capacity 2. These will guarantee that all element-agents must be inside in any blocking coalition.

Finally, we add agents $z_1, z_2, \dots z_{3n}$ with capacity 3 and agents $w_1, \dots, w_{3n}$ with capacity 2. They will ensure that each q_i must get a set-agent in a blocking coalition.

Now we describe the underlying strict preferences.

$$
\begin{array}{rcl|rcl}
s_j & : & [Q_j] > v_j^1 > v_j^2 > v_j^3 & q_i & : & [S_i] > p_i^1 > p_i^2 > p_{i-1}^1 > p_{i-1}^2 > z_i \\
u_j & : & v_j^1 > v_j^2 > v_j^3 & v_j^l & : & s_j > u_j \\
p_i^1 & : & q_i > q_{i+1} > r_i^1 > r_{i-1}^2 & r_i^1 & : & p_i^2 > p_i^1 \\
p_i^2 & : & q_i > q_{i+1} > r_i^1 > r_i^2 & r_i^2 & : & p_i^2 > p_{i+1}^1 \\
w_i & : & z_{i+1} > z_i & z_i & : & q_i > w_i > w_{i-1}
\end{array}
$$

Where $i, j \in [3n]$, $l \in [3]$. Here, $[Q_j]$ denotes the q_i agents corresponding to the elements in Y_j in the order of their indices and $[S_i]$ denotes the s_j agents corresponding to the sets that include x_i in the order of their indices. The edges are exactly those uw pairs that consider each other acceptable (so they appear on each other's preference lists).

Now we construct the initial matching M by adding every edge except the ones between S and Q. Observe that every agent is at full capacity in M. The construction is illustrated in Fig. 7.

We prove that there is a weakly blocking coalition to M if and only if there is an exact 3-cover in I.

Suppose first that there is an exact 3-cover $\mathcal{Y}'$. Take the coalition $\mathcal{P}$, consisting of $\{s_j \mid Y_j \in \mathcal{Y}'\} \cup \{q_i, p_i^l, r_i^l \mid i \in [3n], l \in [2]\}$. Take the matching $M_{\mathcal{P}}$ we get by keeping all edges of M, where both endpoints are contained in $\mathcal{P}$, and then for all $s_j \in \mathcal{P}$ we add the three edges between s_j and Q_j. As $\mathcal{Y}'$ was an exact 3-cover, each agent remains at full capacity. Also, the q_i and s_j agents, who get different partner sets all strictly improve, as they only exchange their worst partner(s) for better one(s). This concludes that $\mathcal{P}$ is a weakly blocking coalition.

For the other direction, suppose that there is a weakly blocking coalition $\mathcal{P}$. As only the edges between the S and Q are not included in M, only these edges can contribute to a strict improvement for some agent. Therefore, there is an element-agent q_i in $\mathcal{P}$, who gets a set-agent s_j, thus both strictly improve in $M_{\mathcal{P}}$.

If a q_i agent strictly improves, then she must still be matched with her p_i^l, p_{i-1}^l, $l \in [2]$ partners. This holds because if q_i would drop a p_i^l agent, then that p_i^l agent cannot be at full capacity anymore in $M_{\mathcal{P}}$ so she is not in $\mathcal{P}$, which implies that her r_i^l neighbours also cannot be saturated in $M_{\mathcal{P}}$, hence they are also not in $\mathcal{P}$. By iterating this, none of p_i^l, r_i^l, $i \in [3n], l \in [2]$ are in $\mathcal{P}$. However, this leaves only $3 + 1 = 4$ possible partners for q_i (as each element is in at most 3 sets), whereas she had 5 partners in M. This reasoning also shows that p_i^1, p_i^2 are in $\mathcal{P}$ and so are all of p_i^l, r_i^l, $i \in [3n], l \in [2]$ and all of them get all of their acceptable partners, hence $q_1, .., q_{3n}$ must be in $\mathcal{P}$ and they keep their p_i^l partners, too. As there is an q_i who strictly improves, she can only do this by exchanging z_i to an acceptable set-agent s_j. By an analogous reasoning for the z_i, w_i agents, we get that none of them are in $\mathcal{P}$ (z_i cannot be saturated, hence w_i cannot be saturated, etc), so all q_i agents lose a partner in $M_{\mathcal{P}}$ and therefore all of them must get a set-agent as a partner in $M_{\mathcal{P}}$.

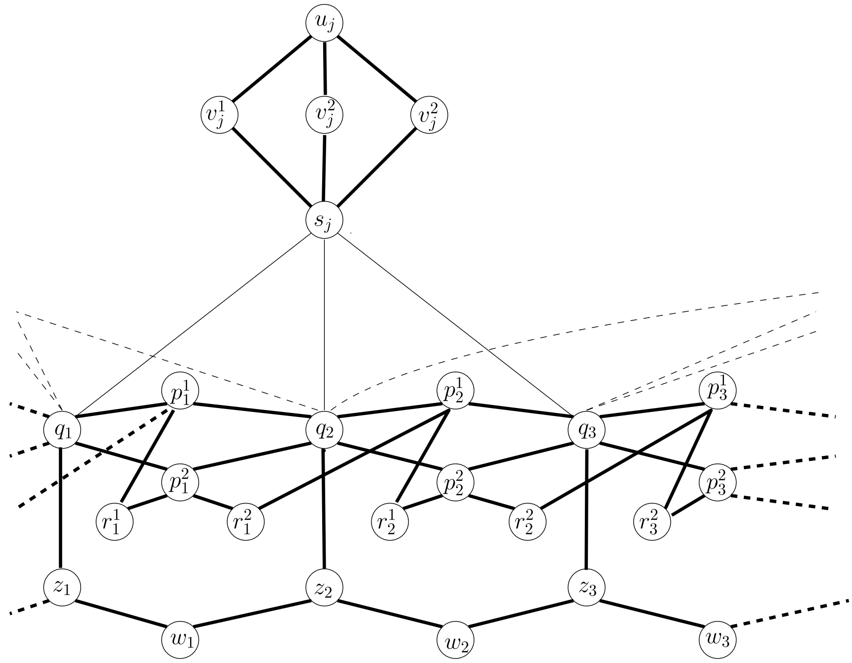

Fig. 7. An illustration for Theorem 6.4. The bold edges represent the edges of M. The s_j vertex corresponds to a set $Y_j = \{x_1, x_2, x_3\}$.

Finally, if a set-agent s_j receives a q_i element-agent as a partner, then she must drop one of her original partners, a v_j^l agent. Then, none of her v_j^l partners can remain in $\mathcal{P}$, as u_j cannot remain saturated but she would be needed for v_j^1, v_j^2, v_j^3 to remain saturated. Hence to remain saturated, s_j must get all three of her acceptable element-agents in $M_{\mathcal{P}}$.

This implies, that the set-agents in $\mathcal{P}$ all get 0 or 3 element-agents and all element-agents are matched to exactly one set-agent, which implies that the sets corresponding to the strictly improving set-agents in $\mathcal{P}$ must form an exact 3-cover. □

Verification for Pareto-optimality is also coNP-complete.

Theorem 6.5. *It is coNP-complete to verify whether a matching M is Pareto-optimal in the stable many-to-many matching problem under RL-preferences.*

Proof. Containment in coNP is trivial.

To show hardness, we reduce from EXACT-3-COVER. Let I be an instance of EXACT-3-COVER. We can suppose that the number of sets is $m = 3n$ and that each element is covered exactly three times in $\mathcal{Y}$.

For each set Y_j, we create an agent s_j with capacity 3 along with agents v_j^1, v_j^2, z_j with capacity 1.

For each element x_i, we create four agents p_i, q_i, p_i^1, p_i^2 with capacity 2.

Finally, we create an agent a with capacity $4n$ and an agent b with capacity $2n$.

The preferences are as follows:

p_i :	$q_{i+1} > p_i^1 > p_i^2 > q_i$	q_i :	$p_i > [S_i] > p_{i-1}$
s_j :	$v_j^1 > v_j^2 > [Q_j] > z_j$	p_i^1, p_i^2 :	p_i
a :	$[V]$	v_j^1, v_j^2 :	$a > s_j$
b :	$[Z]$	z_j :	$b > s_j$

where $V = \{v_j^1, v_j^2 \mid j \in [m]\}$ and $Z = \{z_j \mid j \in [m]\}$. Here, $[Q_j]$ denotes the q_i agents corresponding to the elements in Y_j in the order of their indices and $[S_i]$ denotes the s_j agents corresponding to the sets that include x_i in the order of their indices. Let M be the following matching. $M = \{q_i p_i, q_{i+1} p_i \mid i \in [3n]\} \cup \{s_j v_j^1, s_j v_j^2, s_j z_j \mid j \in [m]\}$. We claim that M is not Pareto-optimal if and only if I admits an exact 3-cover.

Suppose that I admits an exact 3-cover $\mathcal{Y}' \subset \mathcal{Y}$. Then, it must hold for $\mathcal{Y} \setminus \mathcal{Y}'$ that it covers each element exactly twice, as $m = 3n$ and each element is covered exactly three times in $\mathcal{Y}$. Create a matching M' as follows. Match each p_i to p_i^1 and p_i^2. For each $Y_j \notin \mathcal{Y}'$, match s_j to her three partners in Q_j and match z_j to b and v_j^1, v_j^2 to a. Then, each agent in Q is matched to two partners in M' (as is she in M) and her worst partner in M' is better. Finally, for $Y_j \in \mathcal{Y}'$, match s_j to v_j^1, v_j^2 and z_j as in M. Then, for example agent a strictly improves. It is straightforward to verify that M' is feasible and each agent weakly improves in M', so M' dominates M.

For the other direction suppose that there is a matching M' Pareto dominating M. We claim that the agents in Q must strictly improve. Indeed, if an agent p_i, p_i^1 or p_i^2 improves, then p_i must drop q_{i+1} (otherwise she can only exchange her best partner to a worse one), so she must strictly improve. If an agent s_j strictly improves, then she must get a q_i agent, so q_i strictly improves. If an agent v_j^1, v_j^2, z_j, a or b strictly improves, then an s_j agent loses a partner so she must strictly improve and therefore a q_i agent also has to improve. Hence, we obtained that there is a q_i agent strictly improving in M'. To see that all of them must strictly improve, observe that for q_i to improve, she cannot keep p_{i-1} (otherwise she can only exchange her best partner to a worse one). Hence, p_{i-1} improves and drops q_{i-1}, so q_{i-1} improves. Iterating the argument we get that all agents in Q strictly improve and no agents in P can remain matched to agents in Q.

Now, for q_i to improve, she must get matched to two s_j agents, since it had two partners in M. Also, for an s_j agent to improve, she cannot keep z_j. However, only $2n$ z_j agents can improve in M', as b has capacity $2n$. Hence, there must be at most $2n$ agents s_j, such that their corresponding sets in $\mathcal{Y}$ cover each element at least twice. This is only possible, if the rest of the sets in $\mathcal{Y}$ form an exact 3-cover, as desired. □

6.3. Finding maximum size Pareto-optimal solutions

While finding a maximum size Pareto-optimal solution is NP-hard in general if the preferences are lexicographic, the problem becomes tractable with RL-preferences, even in the fixtures case.

We present an algorithm that always finds a maximum size matching that is Pareto-optimal (and hence also weakly Pareto-optimal) with RL-preferences in the stable fixtures problem. Let $worst_E(v)$ denote the edge e that is worst for v among the adjacent edges in E. Informally, the algorithm does the following. We go through each agent, and while the deletion of her worst adjacent edge does not decrease the size of the maximum size matching, we delete it. Then, after we processed each agent, we add the edges that are worst for some agent among the remaining edges to M. Then, we also delete these edges from the graph, decrease the capacities by the number of adjacent edges added in M for each agent and iterate this for the remaining graph and the new capacities until M becomes a maximum size matching (of the whole graph).

Algorithm 4 Maximum size Pareto-optimal matching with RL-preferences.

```
Let M := ∅
x := |M| = 0
k := maximum size of a matching in G
while |M| < k do
    for i = 1, ..., n do
        l = deg_E(u_i)
        u_j = l-th choice of u_i
        while l > 0 and there is a matching of size k − x in the graph (V, E \ {u_i u_j}) do
            u_j = l-th choice of u_i
            E := E \ {u_i u_j}
            l = l − 1
        end while
    end for
    E_w := {e | e = worst_E(v) for some v ∈ V}
    M := M ∪ E_w
    x = |M|
    k = k − x
    Decrease the capacities of each vertex by the number of adjacent edges added
    E := E \ E_w
end while
```

Theorem 6.6. *Algorithm 4 finds a maximum size matching that is Pareto optimal (with respect to RL-preferences) in polynomial time in the stable fixtures problem.*

Proof. We start by showing that M is feasible, that is, no capacity is violated. To see this, observe that if an edge e is added to M in some iteration, then it must hold that e is included in all maximum size matchings of the remaining edges (otherwise we would have deleted it). In particular, when we add all edges from $\{e \mid e = worst_E(v) \text{ for some } v \in V\}$, then each of them must be included in all maximum size matchings in the current graph (which have capacities decreased by the number of adjacent edges from the previous iterations), hence they form a feasible matching.

M is clearly maximum size, as we never delete edges that make the size of the maximum size matching smaller.

We prove the Pareto-optimality of the solution by induction on the number of iterations of the outer While loop. Suppose the algorithm terminates in one iteration.

Suppose there is a matching M' that Pareto-dominates M. Then, each agent has as many adjacent edges in M' as in M. As M is maximum size we get that each agent has the same number of adjacent edges in M (so $|M| = |M'|$ too). M' cannot contain deleted edges, because otherwise some agent would receive a worse partner in M', than the worst one she receives in M, a contradiction. If the worst edge in M' and M is the same for each agent, then $M' = M$, because the algorithm ended in one iteration with a maximum size matching, contradiction. Otherwise, let u_i be an agent with smallest index, whose worst partner is better in M'. Then, when the algorithm was processing u_i, it would have deleted the worst adjacent edge to u_i, as there would still have been a maximum size matching M', a contradiction again.

Now suppose the number of iterations is $i > 1$. A similar argument shows, that if $F_1 \subset E$ is the set of edges that we add in the first iteration to M, then $F_1 \subset M'$. Also, M' cannot contain other edges from the ones that were deleted in the first iteration. Hence, M' restricted to the remaining edges still dominates M restricted to the remaining edges (with the decreased capacities after the first iteration). But, from our inductive assumption, we get a contradiction, as the matching $M \setminus F_1$ is exactly the one which the algorithm would have found if we had run it for the graph where we deleted all edges from the first iteration plus the ones in F_1 and decreased the capacities. □

6.4. Relaxed strong core solutions

Just like with lexicographic preferences, there is always a fractional matching in the strong core of the fractional matchings.

Theorem 6.7. *If f^M is a stable fractional matching, then f^M is in the strong fractional core for RL-preferences.*

Proof. Let f^M be a fractional stable matching. Suppose that there is a weakly blocking coalition $\mathcal{P}$ to f^M. Let $f^M_{\mathcal{P}}$ denote the fractional matching of the agents contained in $\mathcal{P}$, that each of them weakly prefers, and at least one strictly prefers to f^M.

It must hold that there is an edge uv such that $f^M_{\mathcal{P}}(uv) > f^M(uv)$, as otherwise no one could strictly improve. If u was saturated in f^M, then she also must be in $f^M_{\mathcal{P}}$. Therefore, there is a vertex v', such that $0 \le f^M_{\mathcal{P}}(uv') < f^M(uv')$. Take the one among such vertices that is the worst for u. Then, $v >_u v'$, as otherwise if $v <_u v'$ for each v'' such that $v'' <_u v$, $f^M_{\mathcal{P}}(uv'') \ge f^M(uv'')$ and $f^M_{\mathcal{P}}(uv) > f^M(uv)$ meaning that u strictly RL-prefers f^M to $f^M_{\mathcal{P}}$, contradiction.

Similar argument shows that if v was saturated in f^M, then there is a vertex u' with $u' <_v u$, such that $0 \le f^M_{\mathcal{P}}(u'v) < f^M(u'v)$.

Therefore, uv is a blocking edge to M, contradiction. □

Since we can always find a half-integral stable matching in any stable fixtures problem in polynomial time (Biró and Fleiner, 2010; Dean and Munshi, 2010) with an extension of Tan's algorithm (Tan, 1991), so we get the following result.

Theorem 6.8. *The strong fractional core is always nonempty with RL-preferences, even for the stable fixtures problem. Also, a half-integer strong core solution can be found in polynomial time.*

We can also find near feasible solutions. First, observe that in a stable fixtures problem, any stable half-integral matching satisfies that there is at most two fractional edges adjacent to any vertex v and these fractional edges form edge disjoint cycles. To see this, observe that a vertex v can only dominate one fractional edge, because among any two adjacent fractional edges, one is worse than the other and cannot be dominated. Furthermore, it can only dominate a half-integral edge, if there is another adjacent half-integral edge, so v is saturated. By rounding the solution up/down alternatively on each cycle, and then adjusting the capacities, we get the following theorem.

Theorem 6.9. *For each stable fixtures problem $G = (V, E)$ with capacities $k(v) \in \mathbb{Z}$ for each $v \in V$, there exists capacities $k'(v)$, such that $|k(v) - k'(v)| \le 1$ for all $v \in V$, $\sum_{v\in V} k(v) - 1 \le \sum_{v\in V} k'(v) \le \sum_{v\in V} k(v) + 1$, and for the capacities $k'(v)$ there exists a stable matching and those capacities and the stable matching can be found in polynomial time.*

As a corollary we get the following result.

Theorem 6.10. *For any stable fixtures problem with RL-preferences, there always exists a near feasible strong core solution, where every capacity is modified by at most 1, and the aggregate capacity is also modified by at most 1 and it can be found in polynomial time.*

Final notes

We studied weak and strong core and Pareto-optimal solutions for multiple partners matching problems under lexicographic preferences. An intriguing open question is whether the weak core is always non-empty for the stable many-to-many matching problem, and what is the computational complexity of finding such a solution.

We worked with strict preferences in this paper, but some of our tractability results might be possible to get extended for weak preferences. Finally, one could consider also to extend our efficient algorithms for finding certain solutions for more general preference domains, such as additive and responsive preferences.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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